ECE/ENGRD 2100

Introduction to Circuits for ECE

Lecture 6

Linearity and Superposition

Announcements

- Recommended Reading:
  - Textbook Chapter 4

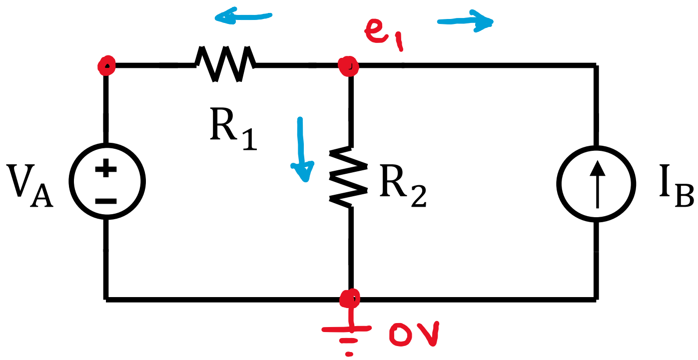
- Upcoming due dates:
  - Lab report 1 due by 11:59 pm on Friday February 8, 2019

- Homework 2, Prelab 2 and Lab 2 are out
  - Prelab 2 due by 12:20 pm on Tuesday February 12, 2019
  - Homework 2 due by 11:59 pm on Friday February 15, 2019
  - Lab report 2 due by 11:59 pm on Friday February 22, 2019

- Lab 2 is next week (starting Tuesday February 12, 2019)

- Prelim 1 on Thursday February 21, 2019 from 7:30 – 9 pm in 203 Phillips

Node Analysis Example



KCL at Node e_1

$$\frac{e_1 - V_A}{R_1} + \frac{e_1}{R_2} - I_B = 0$$

$G_1 \equiv \frac{1}{R_1}$ $\quad$ $G_2 \equiv \frac{1}{R_2}$

Conductance [S or ℧]

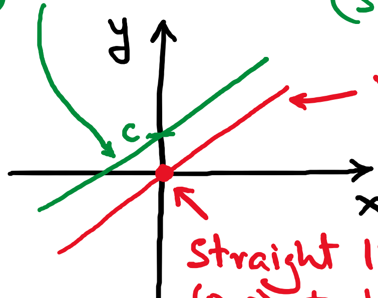
$$\Rightarrow G_1 e_1 - G_1 V_A \quad G_2 e_1 = I_B$$

$$\Rightarrow (G_1 + G_2) e_1 = V_A + I_B \quad \Longrightarrow$$

$$\boxed{e_1 = \frac{G_1 V_A}{G_1 + G_2} + \frac{I_B}{G_1 + G_2}}$$

Value of e_1 is a linear combination of independent sources V_A and I_B

$y = mx + c$ ← is not linear; it is Affine (straight line)

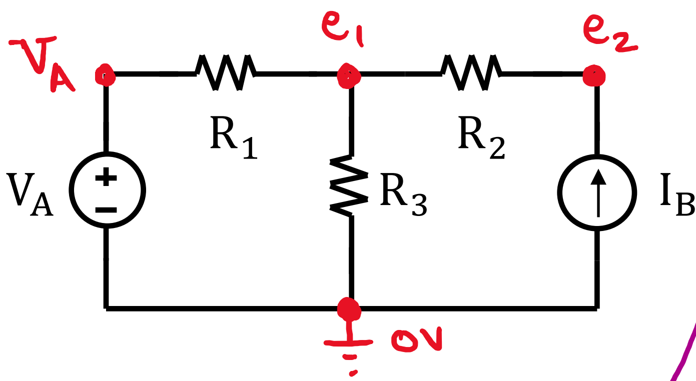


$y = mx$ ← Linear equation

Straight line must pass through (0,0) to be linear

$y = m_1 x_1 + m_2 x_2$ is a linear equation with two inputs: x_1 and x_2

Node Analysis – Circuit with More Nodes



KCL at Node e_1

$$\frac{e_1 - V_A}{R_1} + \frac{e_1}{R_3} + \frac{e_1 - e_2}{R_2} = 0 \quad — ①$$

KCL at Node e_2

$$\frac{e_2 - e_1}{R_2} - I_B = 0 \quad — ②$$

$$(G_1+G_2+G_3)e_1 - G_2 e_2 = G_1 V_A$$

$$-G_2 e_1 + G_2 e_2 = I_B$$

$$\Rightarrow \underbrace{\begin{bmatrix} (G_1+G_2+G_3) & -G_2 \\ -G_2 & G_2 \end{bmatrix}}_{\mathbf{G}} \underbrace{\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}}_{\bar{e}} = \underbrace{\begin{bmatrix} G_1 V_A \\ I_B \end{bmatrix}}_{\bar{b}}$$

$$\Rightarrow \mathbf{G}\,\bar{e} = \bar{b} \quad \Rightarrow \quad \bar{e} = \mathbf{G}^{-1}\,\bar{b}$$

$$\Rightarrow \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \frac{1}{\det(\mathbf{G})} \begin{bmatrix} G_2 & G_2 \\ G_2 & (G_1+G_2+G_3) \end{bmatrix} \begin{bmatrix} G_1 V_A \\ I_B \end{bmatrix}$$

where

$$\det(\mathbf{G}) = G_2(G_1+G_2+G_3) - G_2^2$$
$$= G_1 G_2 + G_2 G_3$$

$$\Rightarrow e_1 = \frac{G_1 \cancel{G_2} V_A + \cancel{G_2} I_B}{G_1 \cancel{G_2} + \cancel{G_2} G_3} = \frac{G_1 V_A}{G_1 + G_3} + \frac{I_B}{G_1 + G_3}$$

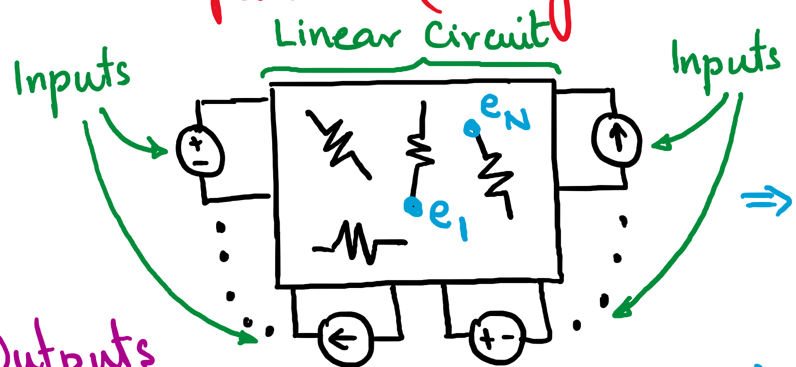
← Linear combination of V_A & I_B

Same true for e_2

Node Analysis – Circuit with More Nodes (Cont.)

In general, for a circuit comprising linear elements (linear resistors and other linear elements we will see later in this course) and independent (voltage and current) sources

e.g., $b_1 = G_1 V_1 + I_1$



$$\Rightarrow \begin{bmatrix} G_{11} & \cdots & G_{1N} \\ \vdots & \ddots & \vdots \\ G_{N1} & \cdots & G_{NN} \end{bmatrix} \begin{bmatrix} e_1 \\ \vdots \\ e_N \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_N \end{bmatrix}$$

$$\Rightarrow \mathbf{G}\,\bar{e} = \bar{b} \quad \Rightarrow \quad \bar{e} = \mathbf{G}^{-1}\bar{b}$$

Outputs

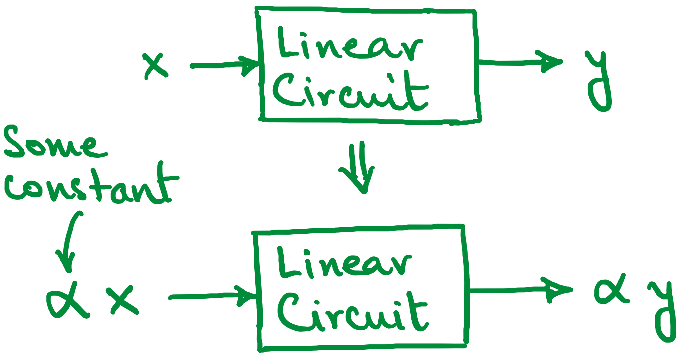
Inputs

$$\Rightarrow \begin{bmatrix} e_1 \\ \vdots \\ e_N \end{bmatrix} = \begin{bmatrix} r_{11} & \cdots & r_{1N} \\ \vdots & \ddots & \vdots \\ r_{N1} & \cdots & r_{NN} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_N \end{bmatrix} \Rightarrow \begin{matrix} e_1 = r_{11} b_1 + r_{12} b_2 + \cdots + r_{1N} b_1 \\ \vdots \\ e_N = r_{N1} b_1 + r_{N2} b_2 + \cdots + r_{NN} b_N \end{matrix}$$

In a circuit constructed using linear resistors, the circuit voltages and currents ("outputs") are linearly related to the voltages and currents of the independent sources ("inputs")

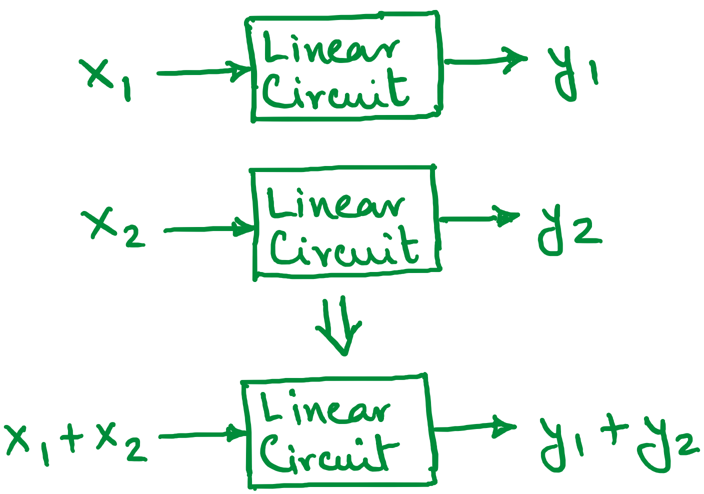
Properties of Linear Circuits

Homogeneity (Scaling inputs yields an equally scaled output)



Must be true if circuit is linear

Superposition (Sum of inputs yields response that is sum of responses to individual inputs)

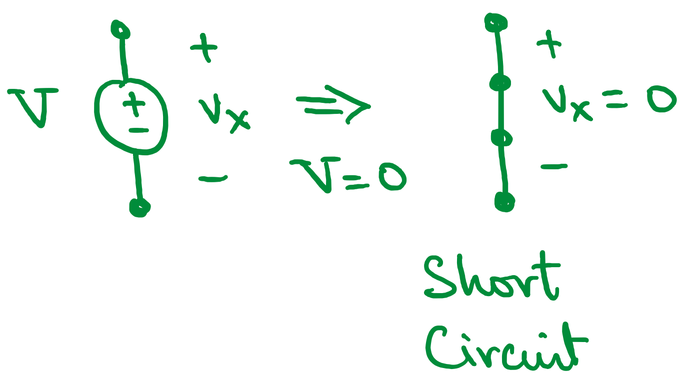


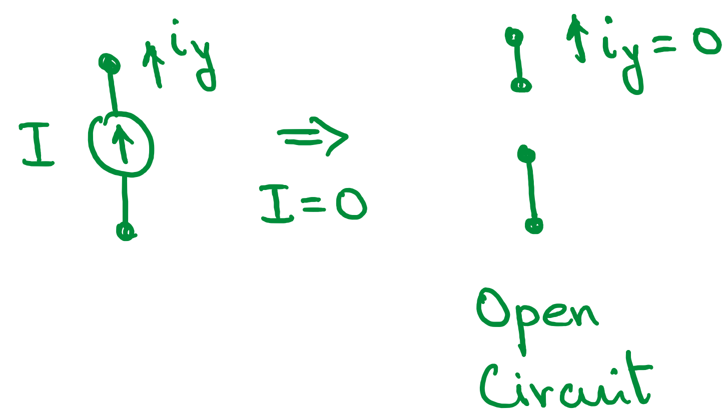
Must be true if circuit is linear

Circuit Inputs

Circuit inputs are independent voltage and current sources

Setting an input to zero

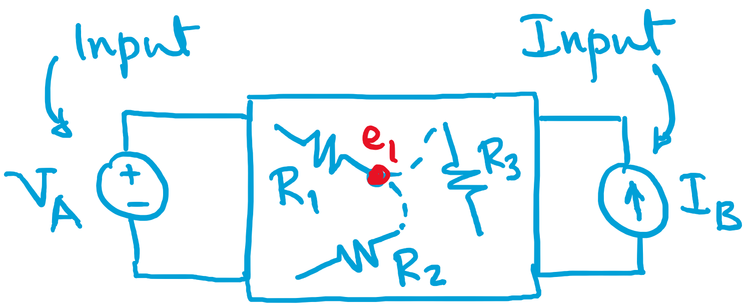




Circuit Analysis Using Superposition

- Can find actual response by adding individual responses to each of the inputs (independent sources)
  - Solve circuit with only one independent source ON at a time

- Circuit analysis using superposition only works for linear circuits



Linear Circuit

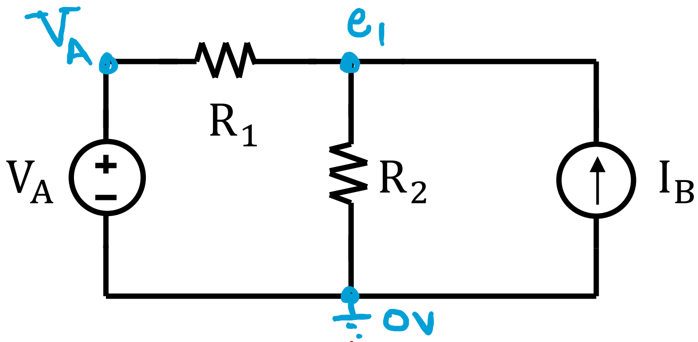
Can use Superposition

$$e_1\Big|_{I_B=0} \equiv e_{1_A}$$

$$e_1\Big|_{V_A=0} \equiv e_{1_B}$$

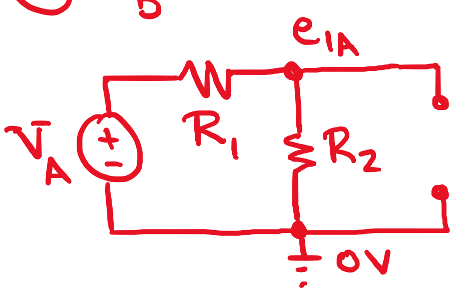
$$\boxed{e_1 = e_{1_A} + e_{1_B}}$$

Circuit Analysis Using Superposition – Example 1



Use Superposition

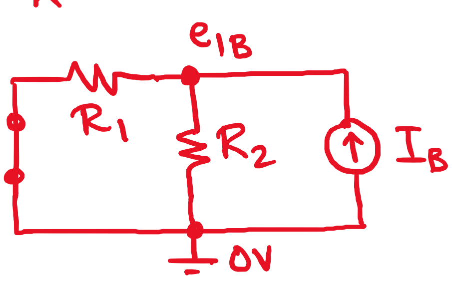
Ⓐ $I_B = 0$



R_1 & R_2 are now in series so can apply voltage divider

$$\Rightarrow e_{1A} = \frac{R_2}{R_1 + R_2} V_A$$

Ⓑ $V_A = 0$



R_1 & R_2 are now in parallel so can combine them into one equivalent resistor $R_1 \| R_2$

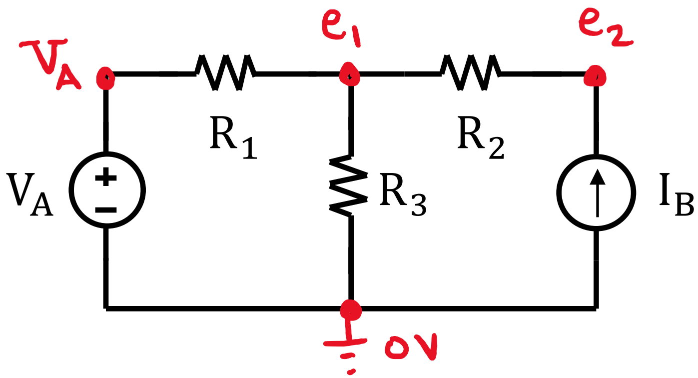
$$\Rightarrow e_{1B} = (R_1 \| R_2) I_B = \frac{R_1 R_2}{R_1 + R_2} I_B$$

By Superposition

$$e_1 = e_{1A} + e_{1B}$$

$$\Rightarrow \boxed{e_1 = \frac{R_2 V_A}{R_1 + R_2} + \frac{R_1 R_2 I_B}{R_1 + R_2}}$$

Circuit Analysis Using Superposition – Example 2

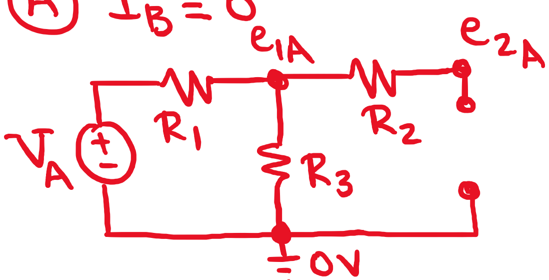


By Superposition

$$e_1 = e_{1A} + e_{1B} = \frac{R_3 V_A}{R_1 + R_3} + \frac{R_1 R_3 I_B}{R_1 + R_3}$$

$$e_2 = e_{2A} + e_{2B} = \frac{R_3 V_A}{R_1 + R_3} + \frac{R_1 R_3 I_B}{R_1 + R_3} + R_2 I_B$$

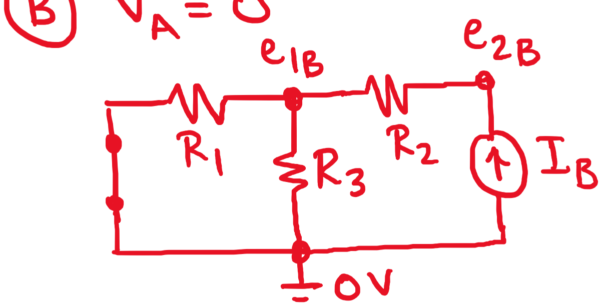
Ⓐ $I_B = 0$



$$e_{1A} = \frac{R_3}{R_1 + R_3} \cdot V_A$$

$$e_{2A} = e_{1A} = \frac{R_3}{R_1 + R_3} V_A$$

Ⓑ $V_A = 0$



$$e_{1B} = \frac{R_1 R_3}{R_1 + R_3} \cdot I_B$$

$$e_{2B} = e_{1B} + R_2 I_B = \frac{R_1 R_3}{R_1 + R_3} I_B + R_2 I_B$$

Summary of Circuit Analysis Using Superposition

- A system (or circuit) is linear if and only if it satisfies:
  - Homogeneity condition
  - Superposition condition

- In a circuit:
  - Inputs are independent voltage and current sources
  - System is the network of resistors (and dependent sources, etc.)
  - Outputs are voltages and currents of interest