# ECE/ENGRD 2100

## Introduction to Circuits for ECE

## Lecture 4

### Node Analysis

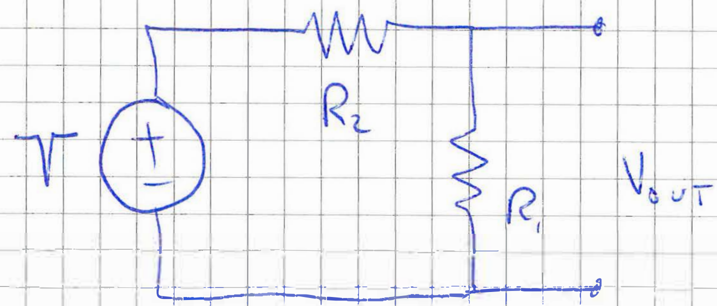
# Announcements

- Recommended Reading:
  - Textbook Chapter ? 4
- Upcoming due dates:
  - Homework 1 due by 11:59 pm on Friday February 1, 2019
  - Lab report 1 due by 11:59 pm on Friday February 8, 2019

# Circuit Analysis Techniques

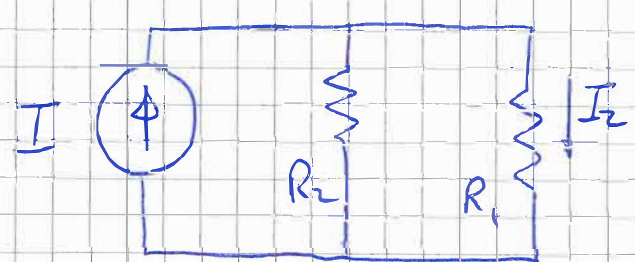
- Brute force method of writing KVL, KCL and element constitutive relationships results in a large number of simultaneous equations to be solved
  - 2B equations have to be solved for 2B branch variables, where B is the number of branches/elements in the circuit
- Circuit simplification techniques/tricks do not always work Review Voltage, Current divider
- Node Analysis always works and requires fewer equations to be solved
  - Node analysis is also based on a combination of KVL, KCL and element constitutive relationships, but the analysis is organized in such a way that fewer equations have to be solved
  - Only solve for node voltages
  - Once node voltages are known, it is easy to determine any branch variable

Voltage Divider - voltage across $R_1$




$$V_{OUT} = \frac{R_1}{R_1+R_2} V$$

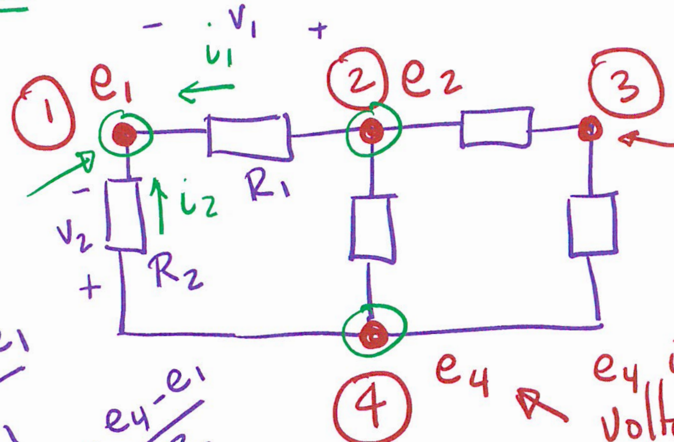
Current Divider - current through $R_1$




$$I_2 = \frac{R_2}{R_1+R_2} \cdot I$$

# Node Analysis – Basic Procedure

1) Select a reference node (also called datum node or ground – 0V)
2) Specify other node voltages with respect to the reference node ⇦ Implicitly imposes KVL
3) Write KCL for nodes with unknown node voltage in terms of node voltages and element constitutive relationships
4) Solve for node voltages ← $e_1, e_2, e_4$
5) Back-solve for any required branch voltages and currents




$e_4$ is the voltage at node 4

KCL

$i_1 + i_2 = 0$

$i_1 = \frac{v_1}{R_1} = \frac{e_2 - e_1}{R_1}$

$i_2 = \frac{v_2}{R_2} = \frac{e_4 - e_1}{R_2}$

KCL at Node ①

$$\boxed{\frac{e_2 - e_1}{R_1} + \frac{e_4 - e_1}{R_2} = 0}$$

KCL at Node ②

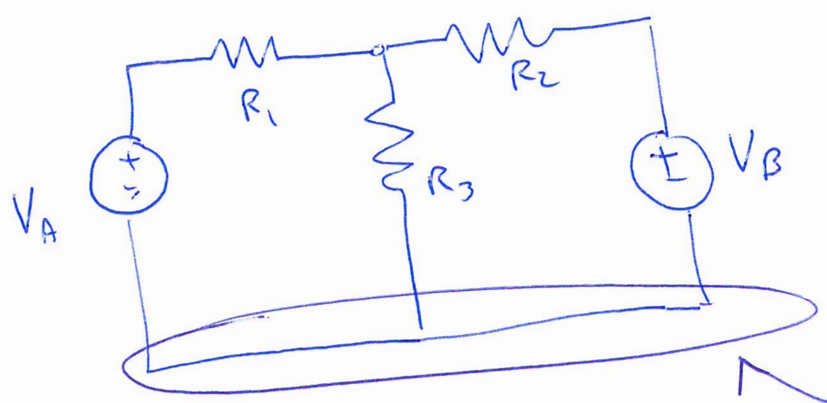
KCL at Node ③

# Node Analysis – Comments

- Like the brute force KVL/KCL method, Node Analysis always works
- There is a dual method called Mesh (or more general loop) Analysis in which we set up equations in terms of the independent loop currents
  - If a circuit has fewer independent loops than nodes, the loop analysis method will be faster
  - We will rarely use Mesh Analysis in this course

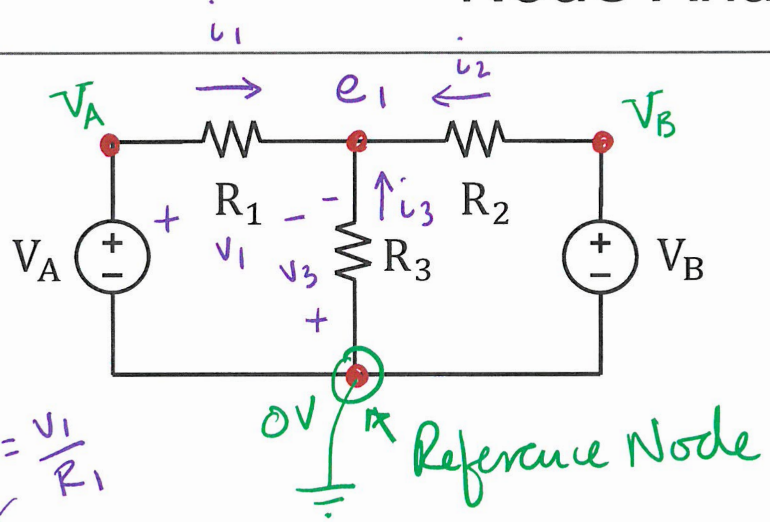
# Selection of Reference Node

- You can pick any node, but some make analysis simpler
  1 – Good to select node connecting most independent voltage sources (reduces number of unknowns)
  2 – Good to select node with many elements connected to it (equations are slightly simplified)




Good choice for reference

# Node Analysis Example 1




Brute force KVL/KCL method requires 10 equations to be solved

4 nodes

$i_1 = \frac{v_1}{R_1}$

$i_3 = \frac{v_3}{R_3} = \frac{0 - e_1}{R_3}$

Conductance

$G \equiv \frac{1}{R}$

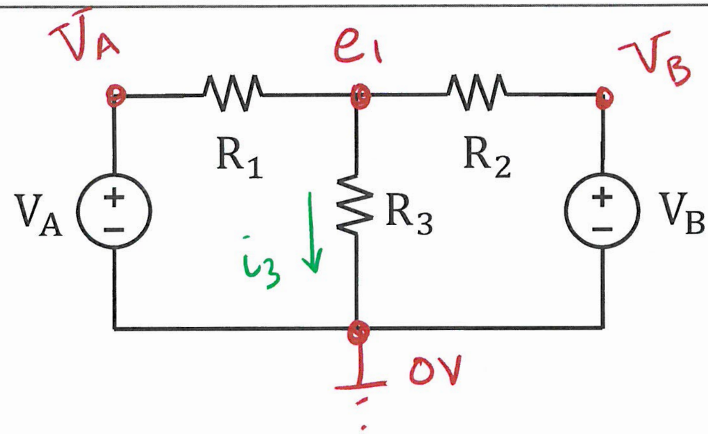
KCL @ $e_1$ : $i_1 + i_2 + i_3 = 0$

$$\Rightarrow \frac{V_A - e_1}{R_1} + \frac{V_B - e_1}{R_2} + \frac{0 - e_1}{R_3} = 0$$

$$\Rightarrow G_1 V_A - G_1 e_1 + G_2 V_B - G_2 e_1 - G_3 e_1 = 0$$

$$e_1 (G_1 + G_2 + G_3) = G_1 V_A + G_2 V_B \Rightarrow \boxed{e_1 = \frac{G_1 V_A + G_2 V_B}{G_1 + G_2 + G_3}}$$

# Node Analysis Example 1 (Cont.)

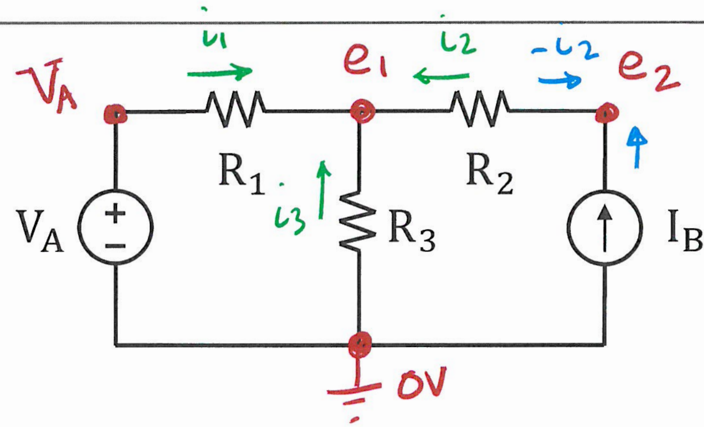



$$\boxed{e_1 = \frac{G_1 V_A + G_2 V_B}{G_1 + G_2 + G_3}}$$

Branch variables (currents and voltages) are now easy to determine

$$i_3 = \frac{e_1 - 0}{R_3} = G_3\left(\frac{G_1 V_A + G_2 V_B}{G_1 + G_2 + G_3}\right)$$

# Node Analysis Example 2




Have replaced one voltage source with a current source

KCL @ $e_1$

$$i_1 + i_2 + i_3 = 0 \Rightarrow \frac{V_A - e_1}{R_1} + \frac{e_2 - e_1}{R_2} + \frac{0 - e_1}{R_3} = 0 \quad \text{— } (1)$$

KCL @ $e_2$

$$-i_2 + I_B = 0 \Rightarrow \frac{e_1 - e_2}{R_2} + I_B = 0 \quad \text{— } (2)$$

$$(1) \Rightarrow G_1 V_A - G_1 e_1 + G_2 e_2 - G_2 e_1 - G_3 e_1 = 0 \Rightarrow \boxed{e_1 (G_1 + G_2 + G_3) - G_2 e_2 = G_1 V_A}$$

$$(2) \Rightarrow \boxed{-G_2 e_1 + G_2 e_2 = +I_B}$$

$$e_1 \left( \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \right) - \frac{e_2}{R_2} = \frac{V_A}{R_1}$$

$$\frac{e_1}{R_2} - \frac{e_2}{R_2} = -I_B$$

$$-\frac{e_1}{R_2} + \frac{e_2}{R_2}$$

# Node Analysis Example 2 (Cont.)

$$① \Rightarrow \quad (G_1 + G_2 + G_3)e_1 - G_2 e_2 = G_1 V_A$$

$$② \Rightarrow \quad -G_2 e_1 + G_2 e_2 = I_B$$

Matrix

$$\left[\begin{array}{c|c} (G_1 + G_2 + G_3) & -G_2 \\ \hline -G_2 & G_2 \end{array}\right] \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} G_1 V_A \\ I_B \end{bmatrix}$$

# Solving 2x2 Matrix Equations

known  unknowns  known

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{array}\right\} \Rightarrow \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\bar{x}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}_{b} \Rightarrow \mathbf{A}\bar{x} = \bar{\mathrm{b}}$$

$$\mathbf{A}\bar{x} = \bar{\mathrm{b}} \Rightarrow \bar{x} = \mathbf{A}^{-1}\bar{\mathrm{b}} \quad \text{where} \quad \boxed{\mathbf{A}^{-1} = \frac{\mathrm{adj}(\mathbf{A})}{\det(\mathbf{A})}}$$

diagnol

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\det(\mathbf{A})} \underbrace{\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}}_{\mathrm{adj}(A)} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

where $\det(\mathbf{A}) = a_{11}a_{22} - a_{12}a_{21}$

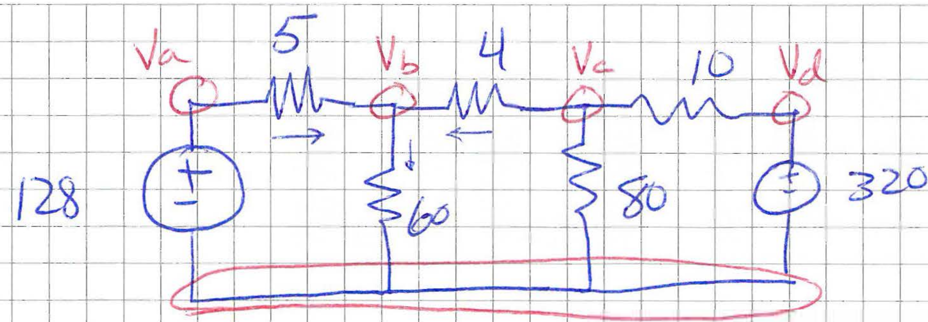
Adjoint of 2x2

swap $a_{11}$ and $a_{22}$

change sign of $a_{12}$, $a_{21}$

$$x_1 = \frac{a_{22}b_1 - a_{12}b_2}{\det(A)}$$

$$x_2 = \frac{-a_{21}b_1 + a_{11}b_2}{\det(A)}$$




4 nodes plus reference

We can see $V_a = 128V$

$V_d = 320$

So we need to write the node equations for $V_b$ and $V_c$

Node b $$\frac{128 - V_b}{5} - \frac{V_b}{60} + \frac{V_c - V_b}{4} = 0$$

Node c $$\frac{V_b - V_c}{4} - \frac{V_c}{80} + \frac{320 - V_c}{10} = 0$$

Consolidate (b) $$-V_b\left(\frac{1}{5} + \frac{1}{60} + \frac{1}{4}\right) + V_c\left(\frac{1}{4}\right) = -\frac{128}{5}$$

(c) $$V_b\left(\frac{1}{4}\right) - V_c\left(\frac{1}{4} + \frac{1}{80} + \frac{1}{10}\right) = -\frac{320}{10}$$

There are clearly some common factors here that can be used to simplify the calculation. Nevertheless I'll just use my calculator to compute values

(b) $-0.4667 V_b + 0.25 V_c = -25.6$

2 eq, 2 unknowns

(c) $0.25 V_b - 0.362 \cdot V_c = -32$

We can solve this easily by hand, but for larger systems one usually uses linear algebra

$$\begin{bmatrix} -0.4667 & 0.25 \\ 0.25 & -0.362 \end{bmatrix} \begin{bmatrix} V_b \\ V_c \end{bmatrix} = \begin{bmatrix} -25.6 \\ -32 \end{bmatrix}$$

$V_b = 162.24 \qquad V_c = 200.44$

(If I cleared fractions, I get 162 and 200).

Does this make sense?

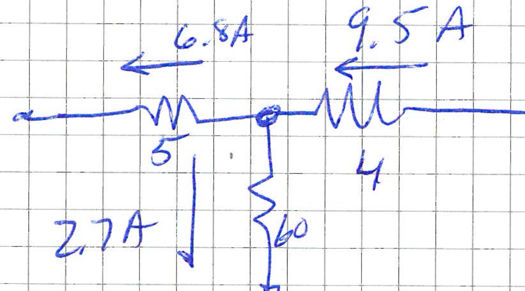
Test $V_b$, determine current in each resistor.

$5\Omega \qquad \dfrac{128-162}{5} = -6.8\,A$

$60\Omega \qquad -\dfrac{162}{60} = -2.7\,A$

$\left.\right\} = -9.5$

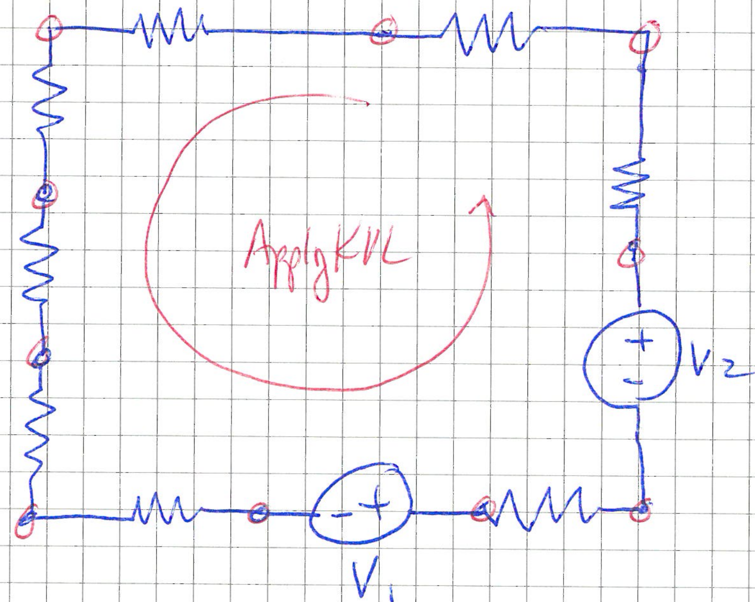
$4\Omega \qquad \dfrac{200-162}{4} = +9.5\,A$




$\Sigma i = 0$

# Mesh Analysis

What if we had a circuit like this?

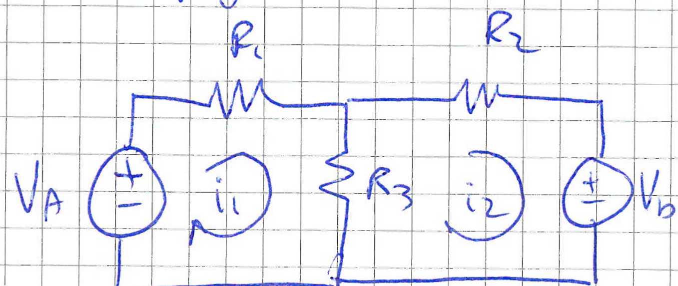



10 nodes
9 independent equations
(not to mention 2 super nodes)

In cases like this node analysis is more cumbersome that other methods, notably loop or mesh

KVL will yield one equation, giving us $i$. From that we can find all voltages

We could apply it to the earlier example




$-V_a + i_1 R_1 + (i_1 - i_2) R_3 = 0$

$+V_b + (i_2 - i_1) R_3 + i_2 R_2 = 0$

2 eq, 2 unknowns