ECE/ENGRD 2100

Introduction to Circuits for ECE

Lecture 36

Transfer Function in the s-Domain

and Poles and Zeros

Announcements

- Recommended Reading:
  - Textbook Chapter 13 and Appendix E

- Upcoming due dates:
  - Homework 5 due by 11:59 pm on Monday April 29, 2019
  - Prelab 6 due by 11:59 pm on Monday April 29, 2019
  - Lab report 6 due by 11:59 pm on Friday May 3, 2019
  - Homework 6 due by 11:59 pm on Tuesday May 7, 2019

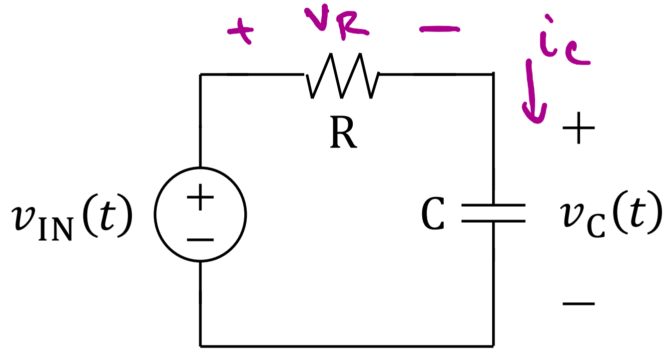
Laplace Transform Properties

Time Domain		Frequency Domain
$K_1 f_1(t) + K_2 f_2(t)$	$\Longleftrightarrow$	$K_1 F_1(s) + K_2 F_2(s)$
$\dfrac{df(t)}{dt}$	$\Longleftrightarrow$	$sF(s) - f(0^-)$
$\dfrac{d^2 f(t)}{dt^2}$	$\Longleftrightarrow$	$s^2 F(s) - sf(0^-) - f'(0^-)$
$\int_{-\infty}^{t} f(t)dt$	$\Longleftrightarrow$	$\dfrac{F(s)}{s} + \dfrac{1}{s}\int_{-\infty}^{0^-} f(t)\,dt$
$f(t-t_0)u(t-t_0)$	$\Longleftrightarrow$	$e^{-t_0 s}F(s)$
$e^{-\alpha t}f(t)$	$\Longleftrightarrow$	$F(s+\alpha)$
$f(0^+)$	$\Longleftrightarrow$	$\lim_{s\to\infty} sF(s)$
$f(\infty)$	$\Longleftrightarrow$	$\lim_{s\to 0} sF(s)$

Laplace Transform Pairs

Time Domain		Frequency Domain
$\delta(t)$	$\Longleftrightarrow$	1
$u(t)$	$\Longleftrightarrow$	$\frac{1}{s}$
$tu(t)$	$\Longleftrightarrow$	$\frac{1}{s^2}$
$e^{-\alpha t}u(t)$	$\Longleftrightarrow$	$\frac{1}{s+\alpha}$
$\sin(\omega t)\, u(t)$	$\Longleftrightarrow$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)\, u(t)$	$\Longleftrightarrow$	$\frac{s}{s^2+\omega^2}$
$te^{-\alpha t}u(t)$	$\Longleftrightarrow$	$\frac{1}{(s+\alpha)^2}$
$e^{-\alpha t}\sin(\omega t)\, u(t)$	$\Longleftrightarrow$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$
$e^{-\alpha t}\cos(\omega t)\, u(t)$	$\Longleftrightarrow$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$

Example 2 – Step Response with Initial Condition



$v_C(0^-) = V_0$

$v_{IN}(t) = V_I u(t)$

$$v_R + v_c = v_{IN}$$

$$RC\frac{dv_c}{dt} + v_c = V_I u(t)$$

$$\Downarrow \mathcal{L}$$

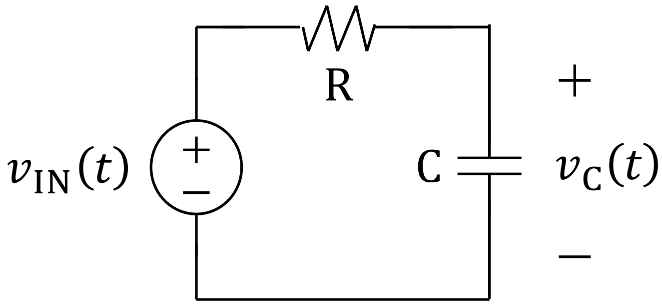
$$RC\left(sV_c(s) - v_c(0^-)\right) + V_c(s) = \frac{V_I}{s}$$

$$\Rightarrow (RCs+1)V_c(s) = \frac{V_I}{s} + RCv_c(0^-)$$

$$V_c(s) = \frac{V_I/RC}{s(s+1/RC)} + \frac{v_c(0^-)}{s+1/RC}$$

$$V_c(s) = \frac{A_1}{s} + \frac{A_2}{s+1/RC} + \frac{v_c(0^-)}{s+1/RC}$$

Example 2 (Cont.)



$v_C(0^-) = V_0$

$$V_C(s) = \frac{V_I/RC}{s(s+1/RC)} + \frac{V_0}{s+1/RC}$$

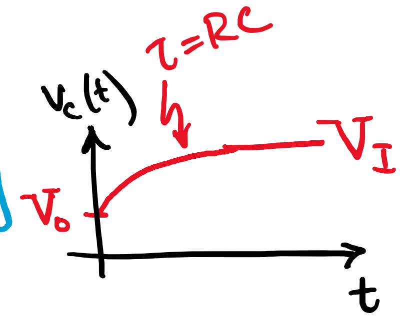
$$V_C(s) = \frac{A_1}{s} + \frac{A_2}{s+1/RC} + \frac{V_0}{s+1/RC}$$

$v_{IN}(t) = V_I u(t)$

$$\frac{A_1 s + A_1/RC + A_2 s}{s(s+1/RC)} = \frac{V_I/RC}{s(s+1/RC)}$$

$A_1 + A_2 = 0$ & $A_1/\cancel{RC} = V_I/\cancel{RC} \Rightarrow \boxed{A_1 = V_I}$

$\Rightarrow A_2 = -A_1 \Rightarrow \boxed{A_2 = -V_I}$



$$V_c(s) = \frac{V_I}{s} - \frac{V_I}{s+1/RC} + \frac{V_0}{s+1/RC} \Longleftrightarrow v_c(t) = V_I u(t) - V_I e^{-t/RC} u(t) + V_0 e^{-t/RC} u(t)$$

$$\Rightarrow \boxed{v_c(t) = V_I\left(1 - e^{-t/RC}\right)u(t) + V_0 e^{-t/RC} u(t)}$$

Partial Fractions – Faster Method

$$\frac{V_I/RC}{s(s+1/RC)} \equiv \frac{A_1}{s} + \frac{A_2}{s+1/RC}$$

To find A_1, multiply by s & set $s=0$

$$\Rightarrow \frac{V_I/RC\,\cancel{s}}{\cancel{s}(s+1/RC)} = \frac{A_1}{\cancel{s}}\cdot\cancel{s} + \cancel{\frac{A_2}{s+1/RC}\cdot s}$$

$s = 0$

$$\Rightarrow \underline{A_1 = V_I}$$

To find A_2, multiply by $(s+1/RC)$ & set $s = -1/RC$

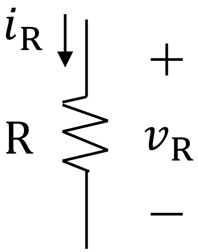
$$\Rightarrow \frac{V_I/RC}{s} = \cancel{\frac{A_1(s+1/RC)}{s}} + A_2$$

$s = -1/RC$

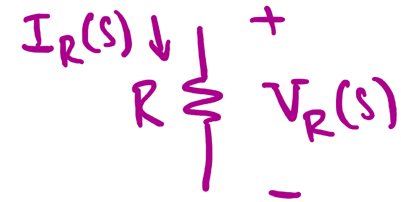
$$\Rightarrow \underline{A_2 = -V_I}$$

Laplace Transform of Components

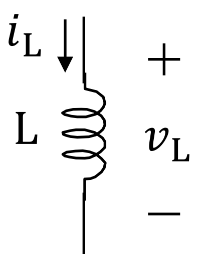
Resistor



$v_R = Ri_R \Longleftrightarrow \mathcal{L}\{v_R\} = \mathcal{L}\{Ri_R\} \Rightarrow V_R(s) = RI_R(s)$



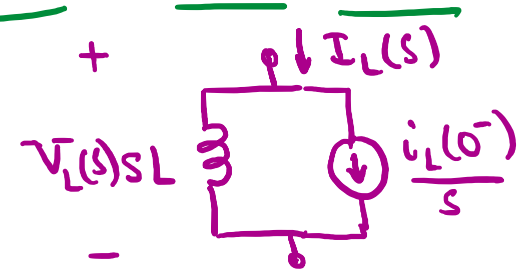
Inductor



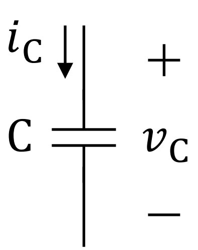
$v_L = L\dfrac{di_L}{dt} \Longleftrightarrow \mathcal{L}\{v_L\} = \mathcal{L}\left\{L\dfrac{di_L}{dt}\right\} \Rightarrow V_L(s) = L\big(sI_L(s) - i_L(0^-)\big)$

$i_L(0^-) = I_0$

$$I_L(s) = \frac{V_L(s)}{sL} + \frac{i_L(0^-)}{s}$$



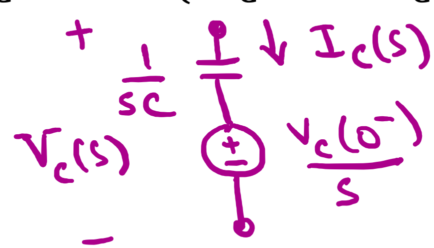
Capacitor



$i_C = C\dfrac{dv_C}{dt} \Longleftrightarrow \mathcal{L}\{i_C\} = \mathcal{L}\left\{C\dfrac{dv_C}{dt}\right\} \Rightarrow I_C(s) = C\big(sV_C(s) - v_C(0^-)\big)$

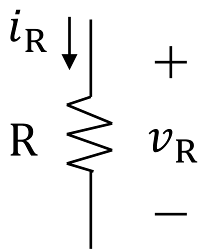
$v_C(0^-) = V_0$

$$V_c(s) = \frac{I_c(s)}{sC} + \frac{v_c(0^-)}{s}$$

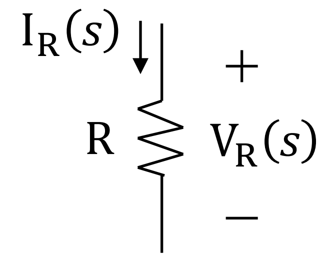


Laplace Transform of Components – Summary

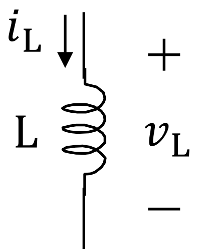
Resistor



$$v_R = R i_R \quad \Longleftrightarrow \quad V_R(s) = R I_R(s)$$

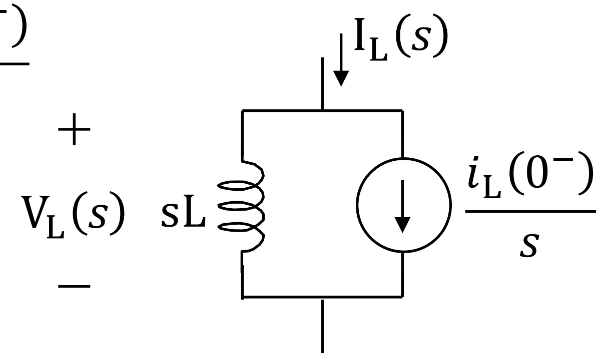


Inductor

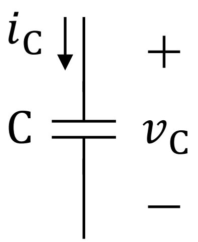


$$v_L = L\frac{di_L}{dt} \quad \Longleftrightarrow \quad I_L(s) = \frac{V_L(s)}{sL} + \frac{i_L(0^-)}{s}$$

$$i_L(0^-) = I_0$$

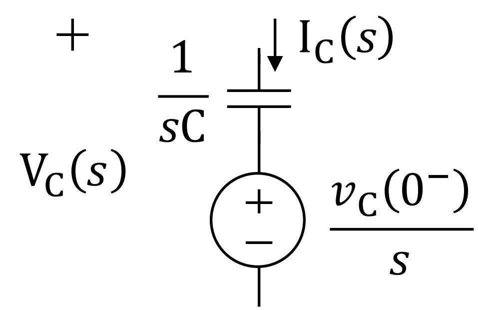


Capacitor



$$i_C = C\frac{dv_C}{dt} \quad \Longleftrightarrow \quad V_C(s) = \frac{I_C(s)}{sC} + \frac{v_C(0^-)}{s}$$

$$v_C(0^-) = V_0$$



Laplace Transform of Circuit Laws

Kirchhoff's Voltage Law

$$\sum_{k=1}^{N} v_k(t) = 0 \quad \Longleftrightarrow \quad \mathcal{L}\left\{\sum_{k=1}^{N} v_k(t)\right\} = 0 \quad \Rightarrow \quad \sum_{k=1}^{N} \mathcal{L}\{v_k(t)\} = 0$$

$$\Rightarrow \quad \sum_{k=1}^{N} V_k(s) = 0$$

Kirchhoff's Current Law

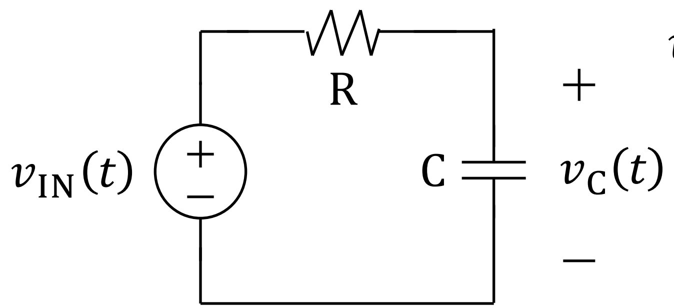
$$\sum_{k=1}^{M} i_k(t) = 0 \quad \Longleftrightarrow \quad \mathcal{L}\left\{\sum_{k=1}^{M} i_k(t)\right\} = 0 \quad \Rightarrow \quad \sum_{k=1}^{M} \mathcal{L}\{i_k(t)\} = 0$$

$$\Rightarrow \quad \sum_{k=1}^{M} I_k(s) = 0$$

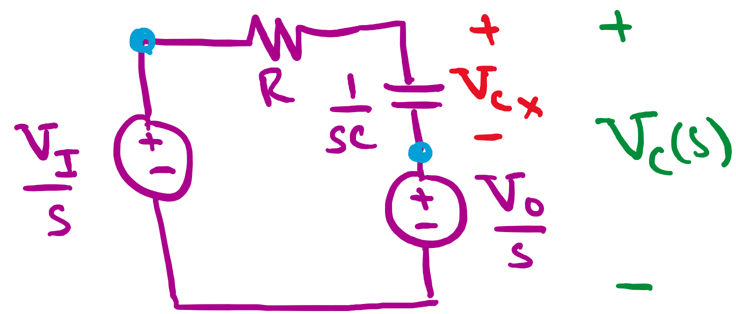
Circuit Analysis using Laplace Transform

1. Create s-domain (frequency domain) model of the circuit
   - Replace sources by their Laplace transform
   - Replace circuit elements by their s-domain impedance models
2. Solve s-domain circuit for s-domain variables of interest
3. Convert s-domain variables of interest into time domain by taking inverse Laplace transform
   - Use partial fraction expansion to convert s-domain result into form with known inverse Laplace transform

Example 2 (Again)



$v_C(0^-) = V_0$



$v_{IN}(t) = V_I u(t)$

$$V_{cx} = \frac{\frac{1}{sC}}{\frac{1}{sC}+R}\left(\frac{V_I}{s}-\frac{V_o}{s}\right) = \frac{1}{(1+sRC)s}(V_I - V_o)$$

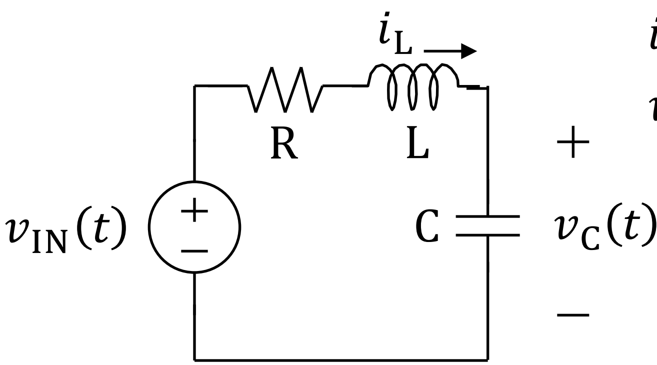
$$V_c(s) = V_{cx} + \frac{V_o}{s} = \frac{V_I}{s(1+sRC)} - \frac{V_o}{s(1+sRC)} + \frac{V_o}{s}$$

$$\frac{\cancel{-V_o}+\cancel{V_o}+\cancel{s}RCV_o}{\cancel{s}(1+sRC)}$$

$$V_c(s) = \frac{V_I/RC}{s(s+1/RC)} + \frac{V_o}{s+1/RC} \longrightarrow$$

$$V_C(s) = \frac{V_I/RC}{s(s+1/RC)} + \frac{V_0}{s+1/RC}$$

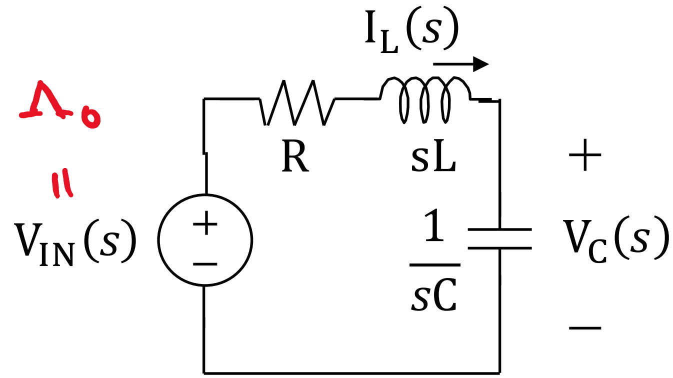
Example 3 – Impulse Response in s-Domain



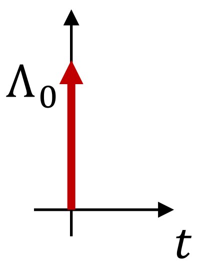
$i_L(0^-) = 0$

$v_C(0^-) = 0$

$\Lambda_0 = V_{IN}(s)$



$v_{IN}(t) = \Lambda_0 \delta(t)$



Λ_0

$$\mathcal{L}\{\delta(t)\} = \int_{0^-}^{\infty} \delta(t) e^{-st} dt = \int_{0^-}^{0^+} \delta(t) e^{0} dt = 1$$

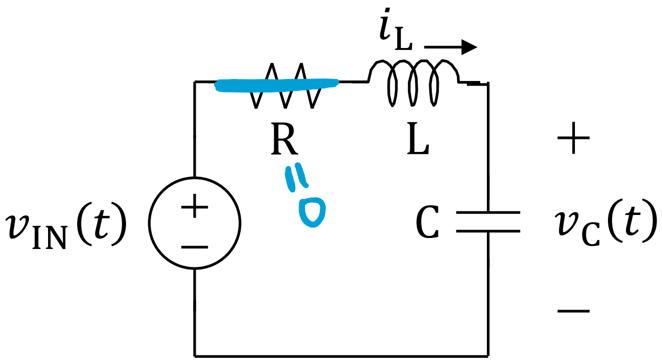
$$I_L(s) = \frac{\Lambda_0}{sL + R + 1/sC}$$

$$I_L(s) = \frac{s\Lambda_0 C}{s^2 LC + sRC + 1}$$

Example: $R = 0$ & define $\frac{1}{\sqrt{LC}} = \omega_0$

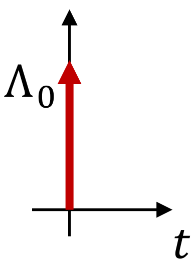
$$I_L(s) = \frac{s\Lambda_0 C}{s^2 LC + 1} = \frac{s\Lambda_0/L}{s^2 + 1/LC} = \frac{s\Lambda_0/L}{s^2 + \omega_0^2}$$

Example 3 (Cont.)



$\cos(\omega t)\, u(t)$	⇔	$\dfrac{s}{s^2+\omega^2}$

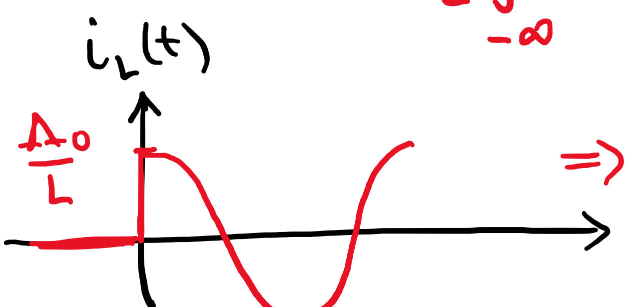
$v_{IN}(t) = \Lambda_0 \delta(t)$



$$I_L(s) = \frac{s\,\Lambda_0 / L}{s^2 + \omega_0^2}$$

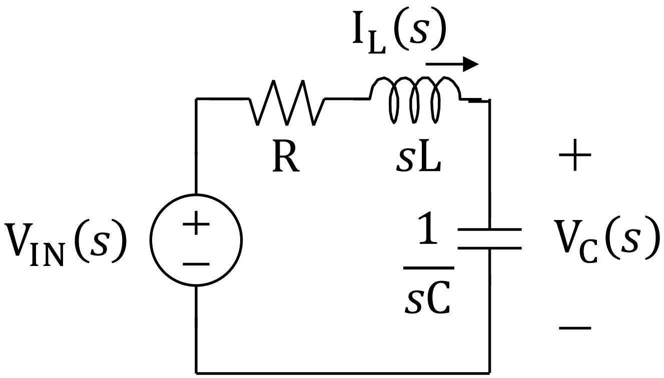
$$\Rightarrow \boxed{i_L(t) = \frac{\Lambda_0}{L} \cos(\omega_0 t)\, u(t)}$$

$$i_L(0^+) = \frac{1}{L}\int_{-\infty}^{0^+} v_L(t)\,dt = \frac{1}{L}\int_{0^-}^{0^+} \Lambda_0 \delta(t)\,dt = \frac{\Lambda_0}{L}$$



$$\Rightarrow \underline{\underline{\frac{\Lambda_0}{L} \cos(\omega_0 t)\, u(t)}}$$

Transfer Function in s-Domain



$$H(s) \equiv \frac{Y(s)}{X(s)}$$ with initial conditions = 0

output (any variable of interest)

input (drive)

$$H_{IL}(s) \equiv \frac{\mathrm{I_L}(s)}{\mathrm{V_{IN}}(s)} = \frac{\frac{s}{\mathrm{L}}}{s^2 + \frac{R}{L}s + \frac{1}{\mathrm{LC}}}$$

$$H_{VC}(s) \equiv \frac{\mathrm{V_C}(s)}{\mathrm{V_{IN}}(s)}$$

Poles and Zeros of the Transfer Function

$$H(s) \equiv \frac{Y(s)}{X(s)} = \frac{N(s)}{D(s)}$$

Poles are the roots of the denominator polynomial $- D(s)$

Zeros are the roots of the numerator polynomial $- N(s)$

$$Y(s) = H(s)X(s) = \frac{N(s)}{D(s)}X(s)$$

Impulse Response: $X(s) = 1$ $\quad Y(s) = H(s) = \dfrac{N(s)}{D(s)}$