ECE/ENGRD 2100

Introduction to Circuits for ECE

Lecture 35

Circuit Analysis using the Laplace Transform

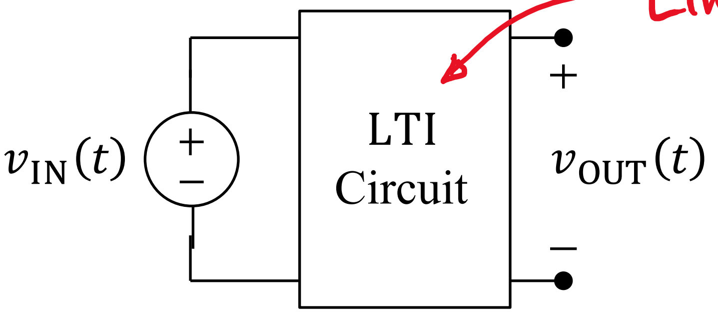
Announcements

- Recommended Reading:
  - Textbook Chapter 12 and Chapter 13

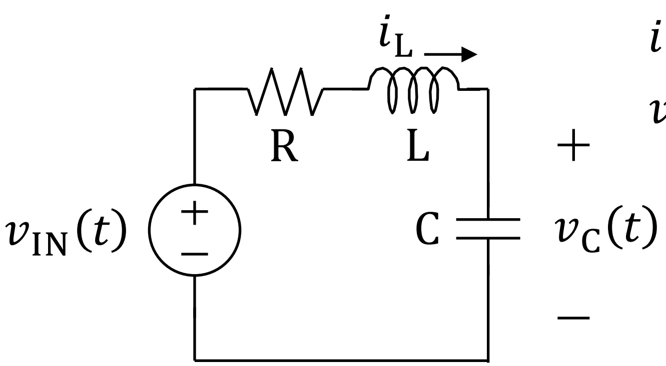
- Upcoming due dates:
  - Homework 5 due by 11:59 pm on Friday April 26, 2019
  - Prelab 6 due by 11:59 pm on Friday April 26, 2019
  - Lab report 6 due by 11:59 pm on Friday May 3, 2019
  - Homework 6 due by 11:59 pm on Friday May 3, 2019

Complete Solution for Arbitrary Drives

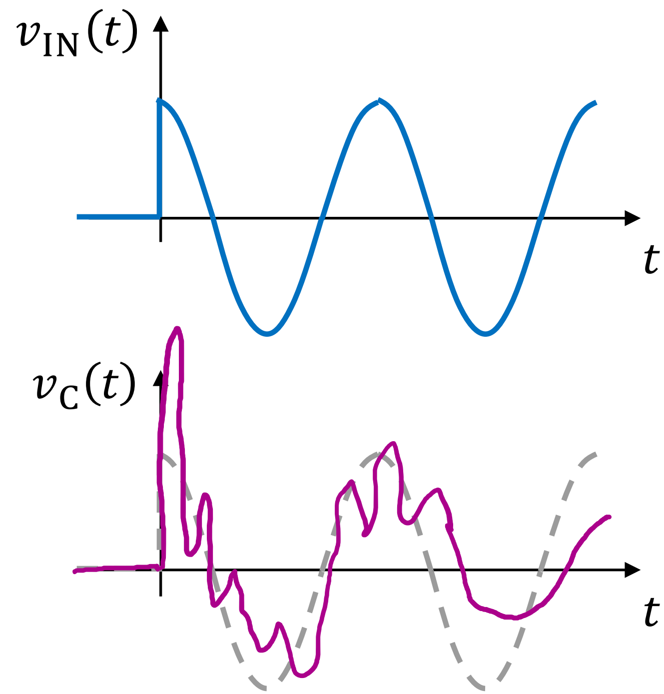
Linear Time-Invariant



Example



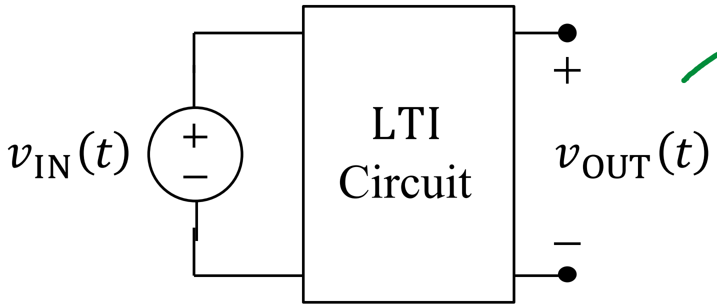
$$i_L(0^-) = 0$$
$$v_C(0^-) = V_0$$

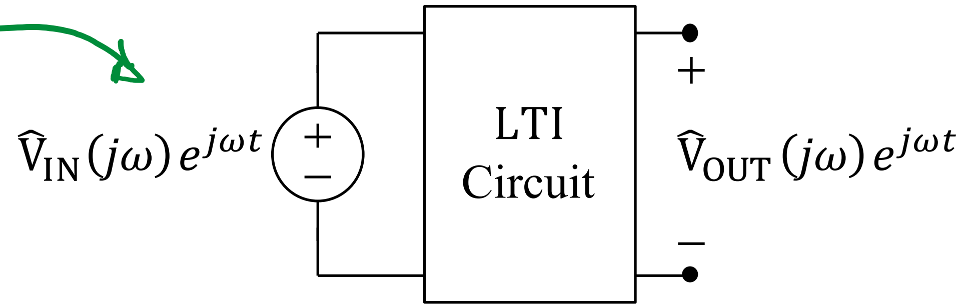


Complete Solution in Frequency Domain

Time Domain | **Frequency Domain**

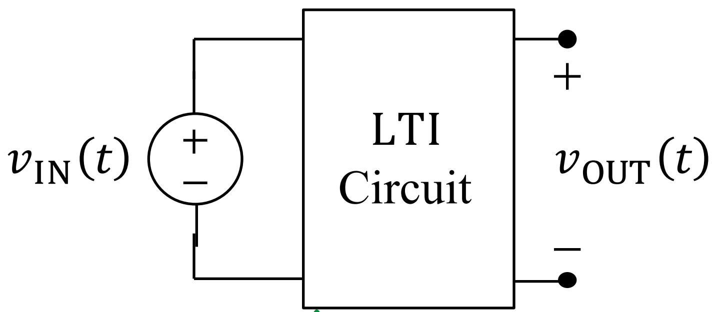
ω

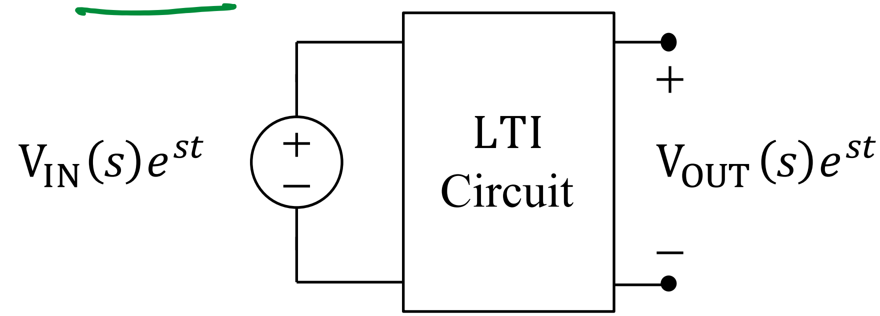




$j\omega$

$s = \underline{\sigma + j\omega}$





σ	ω	Ae^{st}
0	0	A
σ_1	0	$Ae^{\sigma_1 t}$
σ_1	ω_1	$Ae^{\sigma_1 t}e^{j\omega_1 t}$

$$\int V_{\text{IN}}(s)e^{st}\,ds \Rightarrow \int V_{\text{OUT}}(s)e^{st}\,ds$$

Laplace Transform

Unilateral Laplace Transform

$$\boxed{\mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} f(t)\,e^{-st}\,dt = F(s)}$$

where $s = \sigma + j\omega$

$s = \sigma_0 + j\omega \Rightarrow ds = j\,d\omega$

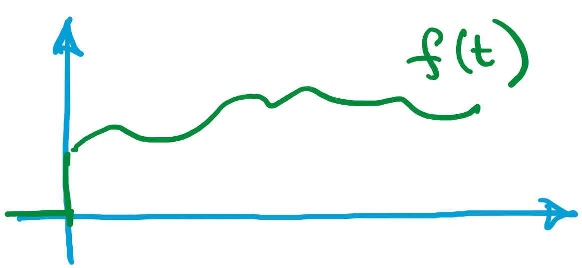
Inverse Laplace Transform

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j}\int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} F(s)\,e^{st}\,ds = f(t)$$

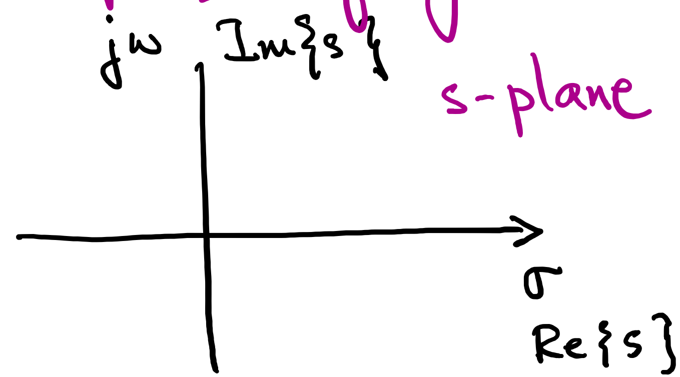
s-Domain

(Complex) Frequency Domain

Time Domain



s-plane



Laplace Transform – Linearity Property

$$\mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} f(t)e^{-st}dt = F(s)$$

$$f(t) = K_1 f_1(t) + K_2 f_2(t)$$

$$\mathcal{L}\{K_1 f_1(t) + K_2 f_2(t)\} = \int_{0^-}^{\infty} (K_1 f_1(t) + K_2 f_2(t))e^{-st}dt$$

$$= K_1 \int_{0^-}^{\infty} f_1(t)e^{-st}dt + K_2 \int_{0^-}^{\infty} f_2(t)e^{-st}dt$$

$$= K_1 \mathcal{L}\{f_1(t)\} + K_2 \mathcal{L}\{f_2(t)\}$$

$$\mathcal{L}\{K_1 f_1(t) + K_2 f_2(t)\} = K_1 \mathcal{L}\{f_1(t)\} + K_2 \mathcal{L}\{f_2(t)\}$$

Laplace Transform – Differentiation Property

$$\boxed{v' = \frac{dv}{dt}}$$

$$\mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} f(t)\, e^{-st}\, dt = F(s)$$

$$(uv)' = u'v + uv'$$

$$\Rightarrow \int_{0^-}^{\infty} u'v\, dt = [uv]_0^{\infty} - \int_{0^-}^{\infty} uv'\, dt$$

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = \int_{0^-}^{\infty} \frac{df}{dt} e^{-st}\, dt$$

$$u' = \frac{df}{dt} \Rightarrow u = f$$

$$v = e^{-st} \Rightarrow v' = -se^{-st}$$

$$= \left[f(t)e^{-st}\right]_{0^-}^{\infty} + \int_{0^-}^{\infty} f(t)\, s\, e^{-st}\, dt$$

$$= 0 - f(0^-) + s\underbrace{\int_{0^-}^{\infty} f(t) e^{-st}\, dt}_{F(s)} = sF(s) - f(0^-)$$

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = sF(s) - f(0^-)$$

Laplace Transform Properties

Time Domain		Frequency Domain
Linear $K_1 f_1(t) + K_2 f_2(t)$	$\Longleftrightarrow$	$K_1 F_1(s) + K_2 F_2(s)$
Differentiation $\dfrac{df(t)}{dt}$	$\Longleftrightarrow$	$sF(s) - f(0^-)$
$\dfrac{d^2 f(t)}{dt^2}$	$\Longleftrightarrow$	$s^2 F(s) - sf(0^-) - f'(0^-)$
$\displaystyle\int_{-\infty}^{t} f(t)dt$	$\Longleftrightarrow$	$\displaystyle\frac{F(s)}{s} + \frac{1}{s}\int_{-\infty}^{0^-} f(t)\,dt$
$f(t-t_0)u(t-t_0)$	$\Longleftrightarrow$	$e^{-t_0 s}F(s)$
$e^{-\alpha t} f(t)$	$\Longleftrightarrow$	$F(s+\alpha)$
$f(0^+)$	$\Longleftrightarrow$	$\lim_{s\to\infty} sF(s)$
$f(\infty)$	$\Longleftrightarrow$	$\lim_{s\to 0} sF(s)$

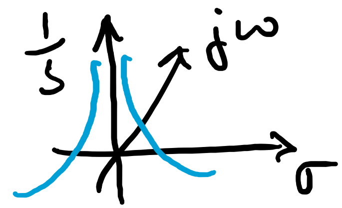
Laplace Transform of $u(t)$

$f(t) = u(t)$

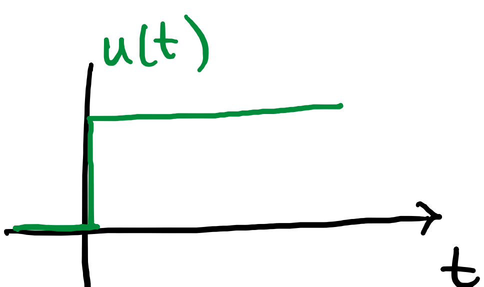
$$F(s) = \int_{0^-}^{\infty} f(t)\, e^{-st}\, dt = \int_{0^-}^{\infty} u(t)\, e^{-st}\, dt = \int_{0^-}^{\infty} e^{-st}\, dt$$

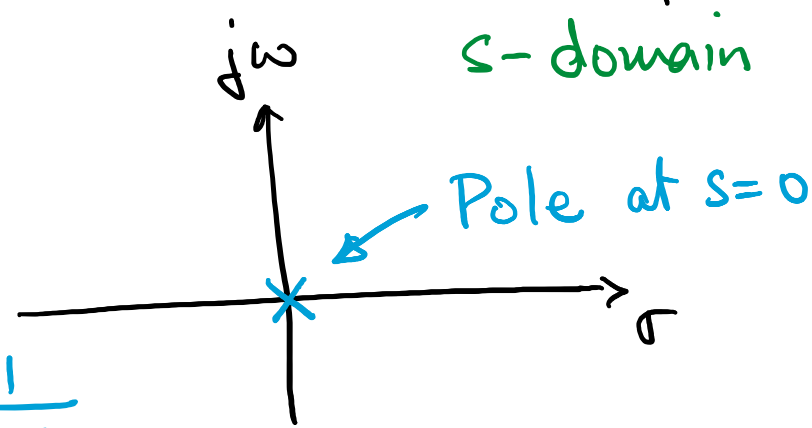
$$= \left[\frac{e^{-st}}{-s}\right]_{0^-}^{\infty} = \frac{0-1}{-s} = \frac{1}{s}$$

for $\mathrm{Re}\{s\} > 0$



$u(t) \Longleftrightarrow \frac{1}{s}$





$\frac{1}{s} = \frac{1}{\sigma + j\omega}$

Laplace Transform of $e^{-\alpha t}$

$f(t) = e^{-\alpha t}$

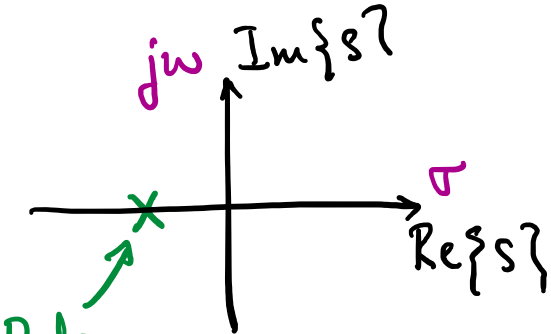
$$F(s) = \int_{0^-}^{\infty} f(t) e^{-st} dt = \int_{0^-}^{\infty} e^{-\alpha t} e^{-st} dt = \int_{0^-}^{\infty} e^{-(s+\alpha)t} dt$$

$$= \left[\frac{e^{-(s+\alpha)t}}{-(s+\alpha)} \right]_{0^-}^{\infty} = \frac{0-1}{-(s+\alpha)} = \frac{1}{s+\alpha} \qquad Re\{s\} > -\alpha$$

$$e^{-\alpha t} u(t) \Longleftrightarrow \frac{1}{s+\alpha}$$

$$\frac{1}{\sigma + j\omega + \alpha}$$

$\Rightarrow \sigma = -\alpha \;\&\; \omega = 0$

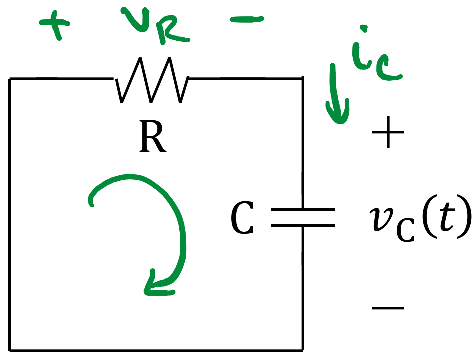


Pole at $s = -\alpha$

Laplace Transform Pairs

Time Domain		Frequency Domain
$\delta(t)$	$\Longleftrightarrow$	1
$u(t)$	$\Longleftrightarrow$	$\frac{1}{s}$
$tu(t)$	$\Longleftrightarrow$	$\frac{1}{s^2}$
$e^{-\alpha t}u(t)$	$\Longleftrightarrow$	$\frac{1}{s+\alpha}$
$\sin(\omega t)\,u(t)$	$\Longleftrightarrow$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)\,u(t)$	$\Longleftrightarrow$	$\frac{s}{s^2+\omega^2}$
$te^{-\alpha t}u(t)$	$\Longleftrightarrow$	$\frac{1}{(s+\alpha)^2}$
$e^{-\alpha t}\sin(\omega t)\,u(t)$	$\Longleftrightarrow$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$
$e^{-\alpha t}\cos(\omega t)\,u(t)$	$\Longleftrightarrow$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$

Laplace Transform based Analysis - Example 1



$v_C(0^-) = V_0$

$$v_R + v_c = 0$$

$$R\,C\frac{dv_c}{dt} + v_c = 0 \qquad \text{Diff Eqn}$$

$$\mathcal{L}\left\{RC\frac{dv_c}{dt} + v_c\right\} = 0$$

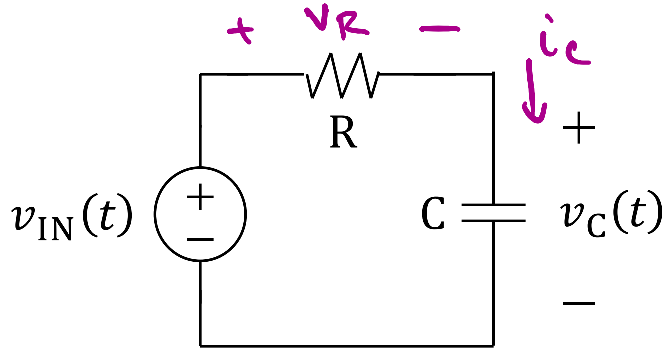
$$\Rightarrow RC\left(sV_c(s) - v_c(0^-)\right) + V_c(s) = 0$$

$$\Rightarrow (sRC + 1)V_c(s) = RCv_c(0^-) \Rightarrow V_c(s) = \frac{RCv_c(0^-)}{sRC + 1}$$

$$\Rightarrow V_c(s) = \frac{v_c(0^-)}{s + 1/RC} = v_c(0^-) \cdot \frac{1}{s + 1/RC}$$

$$\Rightarrow \boxed{v_c(t) = v_c(0^-)\, e^{-t/RC}\, u(t)}$$

Example 2



$v_C(0^-) = V_0$

$v_{IN}(t) = V_I u(t)$

$$v_R + v_C = v_{IN}$$

$$RC\frac{dv_C}{dt} + v_C = V_I u(t)$$

$$\Downarrow \mathcal{L}$$

$$RC\left(sV_C(s) - v_C(0^-)\right) + V_C(s) = \frac{V_I}{s}$$

$$\Rightarrow (RCs+1)V_C(s) = \frac{V_I}{s} + RC\,v_C(0^-)$$

$$V_C(s) = \frac{V_I/RC}{s(s+1/RC)} + \frac{v_C(0^-)}{s+1/RC}$$

$$V_C(s) = \frac{A_1}{s} + \frac{A_2}{s+1/RC} + \frac{v_C(0^-)}{s+1/RC}$$