ECE/ENGRD 2100

Introduction to Circuits for ECE

Lecture 27

Sinusoidal Steady State Analysis Using Impedances

Announcements

- Recommended Reading:
  - Textbook Chapter 9

- Upcoming due dates:
  - Lab report 4 due by 11:59 pm on Monday April 8, 2019
  - Prelab 5 due by 12:20 pm on Tuesday April 9, 2019
  - Homework 5 due by 11:59 pm on Friday April 12, 2019
  - Lab report 5 due by 11:59 pm on Friday April 19, 2019

- Prelim 2 on Thursday March 28, 2019 from 7:30 – 9 pm in 203 Phillips
  - Make up exams today (Wednesday March 27):
    - 3:30 – 5 pm in 354 Duffield
    - 5 – 6:30 pm in 354 Duffield
  - Will cover material through Lecture 24
  - Prelim is closed-book and closed-notes
  - Two double-sided page formula sheet is allowed
  - Bring a calculator

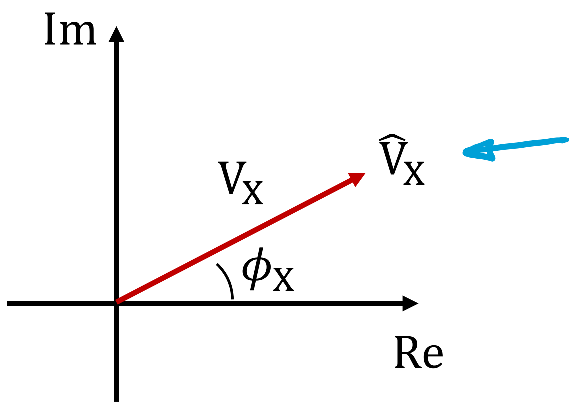
Phasors

Phasor

$$\hat{V}_X = V_X e^{j\phi_X}$$

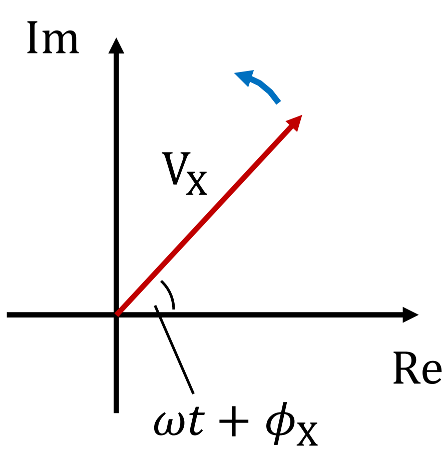
Phasor has magnitude and phase

Alternate notation: $\hat{V}_X = V_X \angle \phi_X$



Rotating Phasor (Complex Exponential)

$$\hat{V}_X e^{j\omega t} = V_X e^{j\phi_X} e^{j\omega t}$$



Sinusoidal Steady State Analysis using Phasors

1. Express drive as real part of a rotating phasor (phasor x $e^{j\omega t}$)

$$v_{\mathrm{IN}}(t) = \mathrm{V_I}\cos(\omega t + \phi_{\mathrm{I}}) = \mathrm{Re}\left\{\mathrm{V_I}e^{j(\omega t+\phi_{\mathrm{I}})}\right\} = \mathrm{Re}\left\{\mathrm{V_I}e^{j\phi_{\mathrm{I}}}e^{j\omega t}\right\} = \mathrm{Re}\left\{\hat{\mathrm{V}}_{\mathrm{I}}e^{j\omega t}\right\}$$

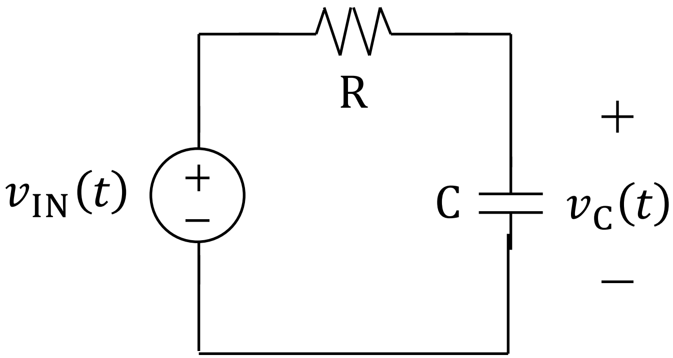
2. Calculate response to the rotating phasor

$$\hat{\mathrm{V}}_{\mathrm{I}}e^{j\omega t} \longrightarrow \boxed{\text{LTI Circuit}} \longrightarrow \hat{\mathrm{V}}_{\mathrm{C}}e^{j\omega t}$$

3. Take real part of the response to the rotating phasor

$$v_{\mathrm{C}}(t) = \mathrm{Re}\left\{\hat{\mathrm{V}}_{\mathrm{C}}e^{j\omega t}\right\} = \mathrm{Re}\left\{\mathrm{V_C}e^{j\phi_{\mathrm{C}}}e^{j\omega t}\right\} = \mathrm{V_C}\cos(\omega t + \phi_{\mathrm{C}})$$

Phasor Analysis using Circuit's Differential Equation



$$\boxed{RC\frac{dv_C}{dt} + v_C = v_{IN}}$$ ← Diff. Eqn.

$$v_{IN}(t) = V_I\cos(\omega t + \phi_I)\,u(t)$$

For $t > 0$: $\tilde{v}_{IN}(t) = V_I e^{j(\omega t + \phi_I)}$

Sinusoidal Steady State Solution (Particular Solution): $\tilde{v}_{C,p}(t) = \hat{V}_C e^{j\omega t}$

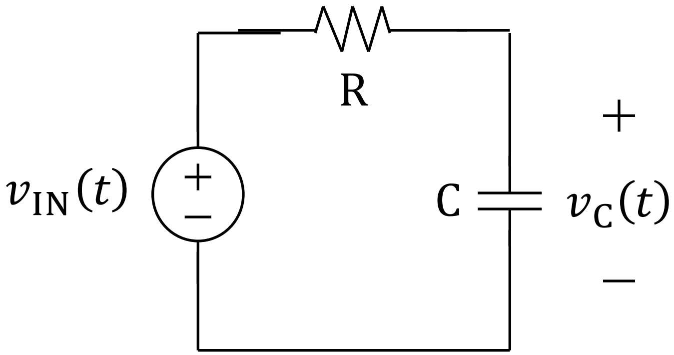
$$RC\frac{d\tilde{v}_{C,p}}{dt} + \tilde{v}_{C,p} = \tilde{v}_{IN} \Rightarrow RCj\omega\hat{V}_C e^{j\omega t} + \hat{V}_C e^{j\omega t} = V_I e^{j\phi_I} e^{j\omega t} \Rightarrow \boxed{\hat{V}_C = \frac{V_I e^{j\phi_I}}{1 + jRC\omega}}$$

$$\Rightarrow \hat{V}_C = \frac{V_I e^{j\phi_I}}{\sqrt{1 + (RC\omega)^2}\, e^{j\tan^{-1}(RC\omega)}} \Rightarrow \hat{V}_C = \frac{V_I}{\sqrt{1 + (RC\omega)^2}} e^{j(\phi_I - \tan^{-1}(RC\omega))}$$

$$\Rightarrow v_{C,p}(t) = \mathrm{Re}\{\hat{V}_C e^{j\omega t}\} = \mathrm{Re}\left\{\frac{V_I}{\sqrt{1 + (RC\omega)^2}} e^{j(\omega t + \phi_I - \tan^{-1}(RC\omega))}\right\}$$

$$\Rightarrow v_{C,p}(t) = \frac{V_I}{\sqrt{1 + (RC\omega)^2}}\cos(\omega t + \phi_I - \tan^{-1}(RC\omega))$$

Sinusoidal Steady State Analysis – Alternate Approach



Diff. Eqn.

$$RC\frac{dv_C}{dt} + v_C = v_{IN}$$

Phasor World

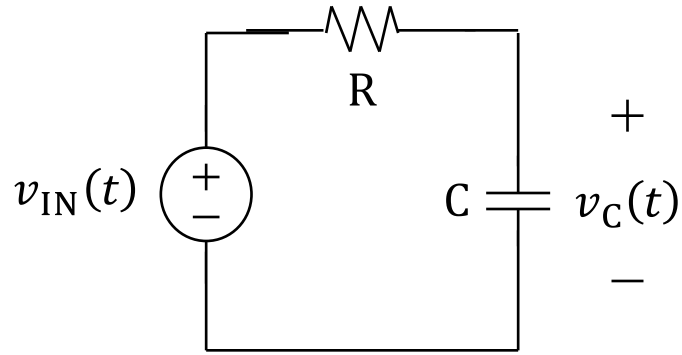
Applied Phasor

$$\tilde{v}_{IN}(t) = V_I e^{j(\omega t + \phi_I)}$$

Phasor Equation

Phasor Response

Real World



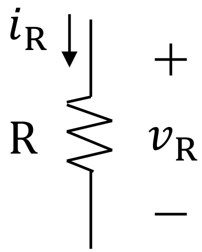
Phasor World

Eqn

Solve

Phasor Response of Components – Impedance

Resistor



$$v_R = R i_R$$

$$i_R(t) \rightarrow \hat{I}_R e^{j\omega t} = I_R e^{j\phi_{I_R}} e^{j\omega t}$$

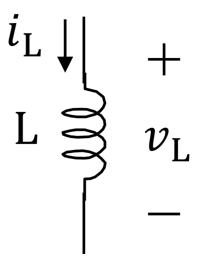
$$v_R(t) \rightarrow \hat{V}_R e^{j\omega t} = V_R e^{j\phi_{V_R}} e^{j\omega t}$$

$$\hat{V}_R \cancel{e^{j\omega t}} = R \hat{I}_R \cancel{e^{j\omega t}} \Rightarrow \hat{V}_R = R \hat{I}_R \Rightarrow$$

Define Impedance

$$\underline{\underline{Z_R}} \equiv \frac{\hat{V}_R}{\hat{I}_R} = R$$

Inductor



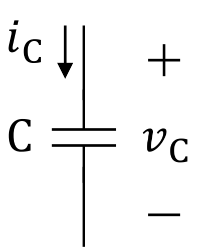
$$v_L = L \frac{di_L}{dt}$$

$$i_L(t) \rightarrow \hat{I}_L e^{j\omega t}$$

$$v_L(t) \rightarrow \hat{V}_L e^{j\omega t}$$

$$\hat{V}_L \cancel{e^{j\omega t}} = L j\omega \hat{I}_L \cancel{e^{j\omega t}} \Rightarrow \hat{V}_L = j\omega L \hat{I}_L \Rightarrow Z_L \equiv \frac{\hat{V}_L}{\hat{I}_L} = j\omega L$$

Capacitor



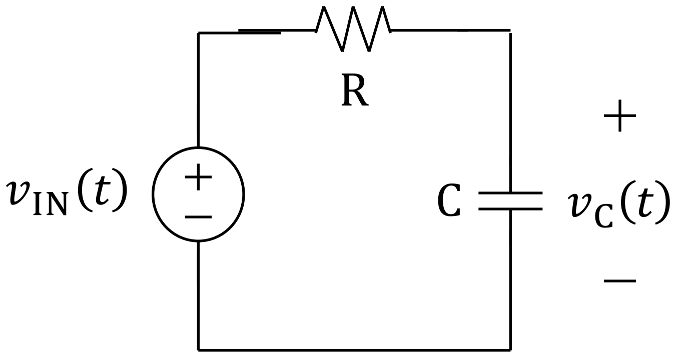
$$i_C = C \frac{dv_C}{dt}$$

$$v_C(t) \rightarrow \hat{V}_C e^{j\omega t}$$

$$i_C(t) \rightarrow \hat{I}_C e^{j\omega t}$$

$$\hat{I}_C \cancel{e^{j\omega t}} = C j\omega \hat{V}_C \cancel{e^{j\omega t}} \Rightarrow \hat{V}_C = \frac{1}{j\omega C} \hat{I}_C \Rightarrow Z_C \equiv \frac{\hat{V}_C}{\hat{I}_C} = \frac{1}{j\omega C}$$

Sinusoidal Steady State Analysis – Alternate Approach



⇓ Diff. Eqn.

$$RC\frac{dv_C}{dt} + v_C = v_{IN}$$

Phasor World

⇓ Applied Phasor

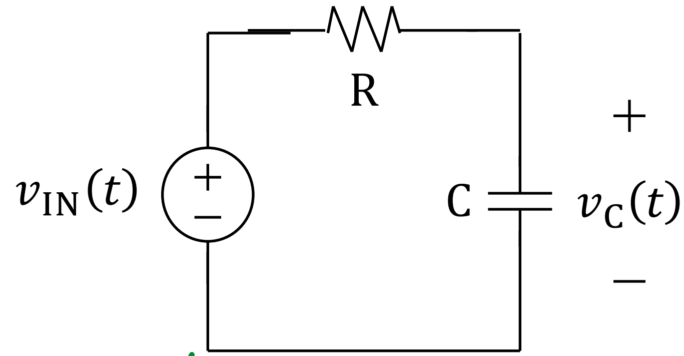
$$\tilde{v}_{IN}(t) = V_I e^{j(\omega t + \phi_I)}$$

⇓ Phasor Equation

Phasor Response

⇓

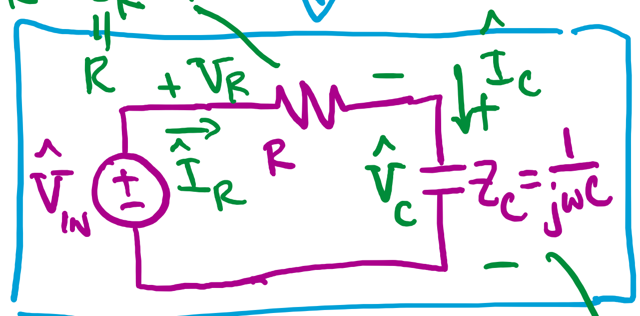
Real World



$\hat{V}_R = Z_R \hat{I}_R$

$Z_R = R$

⇓



Phasor World

⇓ Eqn

Solve

$\hat{V}_C = Z_C \hat{I}_C$

Impedance Summary

- In phasor (frequency) domain, for linear R, L and C, phasor voltage $\hat{\mathrm{V}}$ and phasor current $\hat{\mathrm{I}}$ are related by algebraic expressions
  - In time domain, for linear L and C, voltage $v(t)$ and current $i(t)$ related by differentials

- Impedance $\boxed{\mathrm{Z} \equiv \dfrac{\hat{\mathrm{V}}}{\hat{\mathrm{I}}}}$
  - Impedance has units of ohms $[\Omega]$
  - In general, Z is a complex number: $\mathrm{Z} = \mathrm{R} + j\mathrm{X} = |\mathrm{Z}|e^{j\measuredangle \mathrm{Z}}$

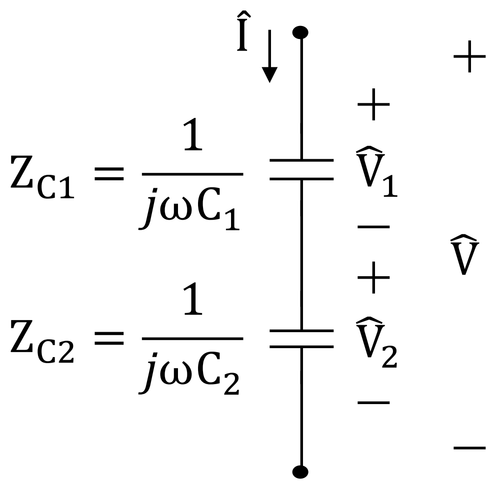
    R is "resistance"; X is called "reactance"
  - Inverse of Impedance is called "admittance" $\mathrm{Y} = \frac{1}{\mathrm{Z}} = \mathrm{G} + j\mathrm{B}$

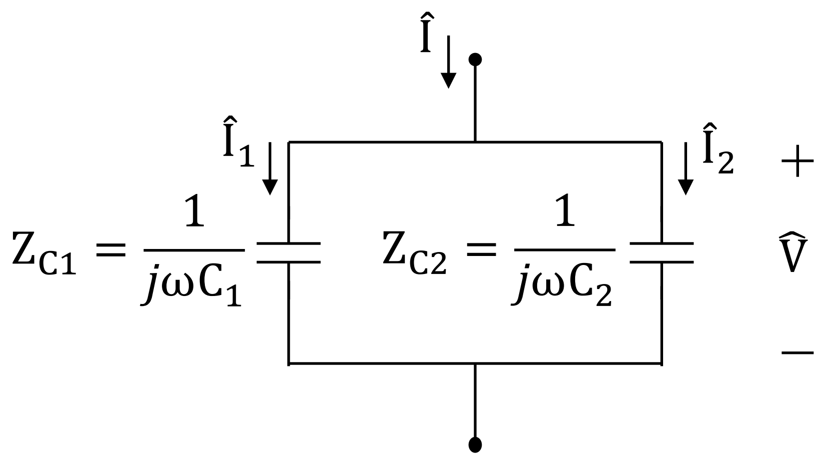
    G is "conductance"; B is called "susceptance"

- Impedances can be treated the same way resistances are treated

Combining Impedance

- Impedances follow the same combination rules as resistors
  - Impedances in series add
  - Admittances in parallel add





$$\hat{V} = \hat{V}_1 + \hat{V}_2 = Z_{c_1}\hat{I} + Z_{c_2}\hat{I}$$

$$Z_{eff} = Z_{TH} \equiv \frac{\hat{V}}{\hat{I}} = Z_{c_1} + Z_{c_2}$$

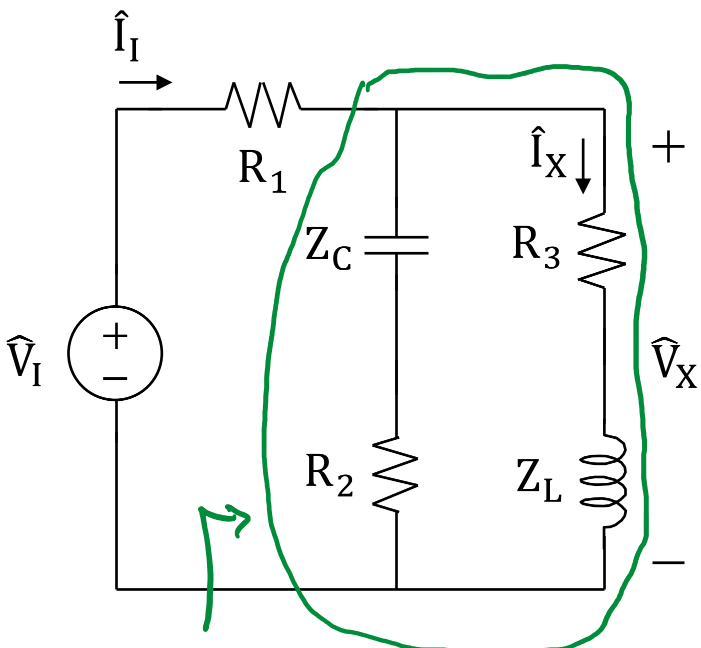
$$\hat{I} = \hat{I}_1 + \hat{I}_2 = \frac{\hat{V}}{Z_{c_1}} + \frac{\hat{V}}{Z_{c_2}}$$

$$\Rightarrow \frac{1}{Z_{eff}} = \frac{\hat{I}}{\hat{V}} = \frac{1}{Z_{c_1}} + \frac{1}{Z_{c_2}}$$

Voltage and Current Division with Impedances

- Impedances follow the same voltage and current division rules as resistors

$$\hat{V}_x = \frac{\overbrace{(Z_c + R_2) \| (R_3 + Z_L)}^{Z_{eff}}}{R_1 + (Z_c + R_2) \| (R_3 + Z_L)} \cdot \hat{V}_1$$



$$v_x(t) = Re\{\hat{V}_x e^{j\omega t}\}$$

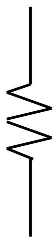
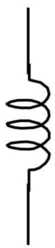
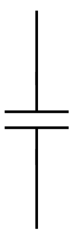
Z_{eff}

$\parallel$

$Z_{TH} \leftarrow$ Thevenin Impedance

Can also create Thevenin and Norton Equivalents in frequency domain

Impedance – Dependence on Frequency

	$\omega = 0$	$\omega \to \infty$
Resistor R  $Z_R = R$	$Z_R = R$	$Z_R = R$
Inductor L  $Z_L = j\omega L$	No change $Z_L = 0$	Fast Change $Z_L \to \infty$
Capacitor C  $Z_C = \dfrac{1}{j\omega C}$	No change $Z_C \to \infty$	Fast Change $Z_C = 0$

Sinusoidal Steady State Analysis using Impedances

1. Create impedance (frequency domain) model of the circuit
   - Replace sinusoidal sources by the equivalent phasor

$$v_{\mathrm{IN}}(t) = \mathrm{V_I}\cos(\omega t + \phi_{\mathrm{I}}) \rightarrow \mathrm{V_I} e^{j\phi_{\mathrm{I}}}$$

   - Replace circuit elements by their impedances models

$$\mathrm{R} \rightarrow \mathrm{R} \qquad \mathrm{L} \rightarrow j\omega\mathrm{L} \qquad \mathrm{C} \rightarrow \frac{1}{j\omega\mathrm{C}}$$

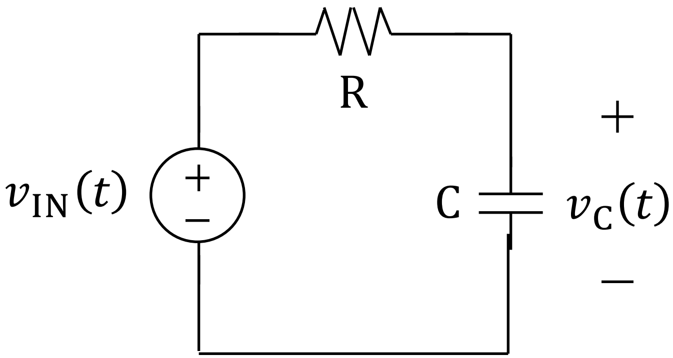
2. Solve frequency domain circuit (with algebraic constitutive relationships) for phasors of interest

Solve it like a resistive circuit

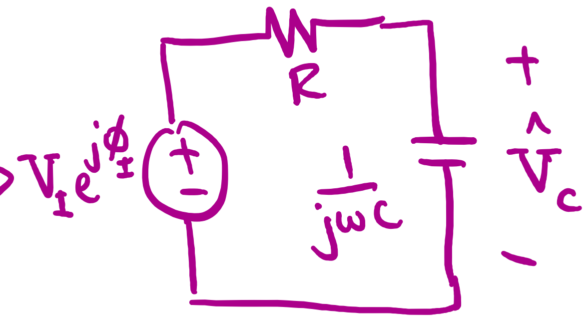
3. Convert phasors of interest into time domain by multiplying by $e^{j\omega t}$ and taking the real part

$$v_{\mathrm{C}}(t) = \mathrm{Re}\left\{\widehat{\mathrm{V}}_{\mathrm{C}} e^{j\omega t}\right\} = \mathrm{Re}\left\{\mathrm{V_C} e^{j\phi_{\mathrm{C}}} e^{j\omega t}\right\} = \mathrm{V_C}\cos(\omega t + \phi_{\mathrm{C}})$$

Circuit Analysis using Impedances – Example 1



$$v_{IN}(t) = V_I \cos(\omega t + \phi_I)\, u(t)$$

$\Rightarrow$ 

$$\hat{V}_c = \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + R} \cdot V_I e^{j\phi_I}$$

$$\Rightarrow \hat{V}_c = \frac{V_I e^{j\phi_I}}{1 + jRC\omega} = \frac{V_I e^{j\phi_I}}{\sqrt{1+(RC\omega)^2}\, e^{j\tan^{-1}(RC\omega)}} = \frac{V_I}{\sqrt{1+(RC\omega)^2}} e^{j(\phi_I - \tan^{-1}(RC\omega))}$$

$$v_c(t) = \mathrm{Re}\{\hat{V}_c e^{j\omega t}\} = \frac{V_I}{\sqrt{1+(RC\omega)^2}} \cos\left(\omega t + \phi_I - \tan^{-1}(RC\omega)\right)$$