ECE/ENGRD 2100

Introduction to Circuits for ECE

Lecture 26

Sinusoidal Steady State Analysis Using Phasors

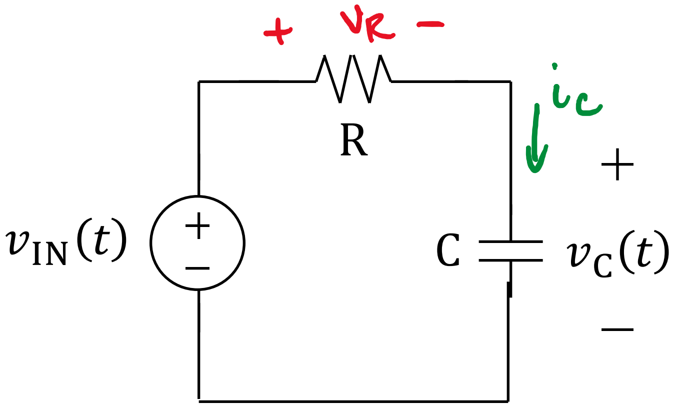
Announcements

- Recommended Reading:
  - Textbook Chapter 9

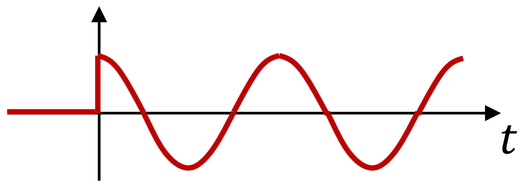
- Upcoming due dates:
  - Homework 4 due by 11:59 pm on Monday March 25, 2019
  - Lab report 4 due by 11:59 pm on Monday April 8, 2019

- Prelim 2 on Thursday March 28, 2019 from 7:30 – 9 pm in 203 Phillips
  - Email afridi@cornell.edu if have conflict
  - Will cover material through Lecture 24
  - Prelim is closed-book and closed-notes
  - Two double-sided page formula sheet is allowed
  - Bring a calculator

Dynamic Circuit with Sinusoidal Drive



$$v_{IN}(t) = V_I \cos(\omega t + \phi_I)\, u(t)$$



$v_{IN} = v_R + v_c = R i_c + v_c$

$i_c = C \dfrac{dv_c}{dt}$ $\Rightarrow$ $\boxed{RC \dfrac{dv_c}{dt} + v_c = v_{IN}}$ Diff. Eqn.

$t < 0 \Rightarrow v_c(t) = 0$

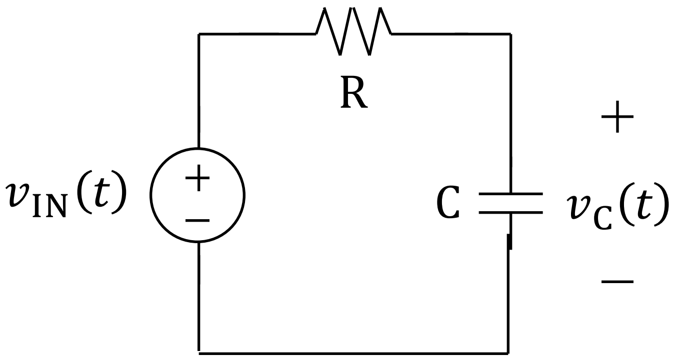
$\underline{t > 0} \Rightarrow v_c(t) = v_{c,h}(t) + v_{c,p}(t) = \underline{\underline{A}}\, e^{-t/\tau} + \underbrace{\underline{\underline{B}} \cos(\omega t + \underline{\underline{\gamma}})}_{}$, with $v_{c,p}(t) = B\cos(\omega t + \gamma)$

unknowns

where $\tau = RC$

Initial Condition: $v_c(0^+) = v_c(0^-) = 0$

Particular Solution



For $t > 0$:

$$\mathrm{RC}\frac{dv_{\mathrm{C}}}{dt} + v_{\mathrm{C}} = \mathrm{V_I}\cos(\omega t + \phi_{\mathrm{I}})$$ ← Diff. Eqn

Guess:

$$v_{\mathrm{C,p}}(t) = B\cos(\omega t + \gamma)$$ ← Particular Soln

Substitute guess into differential equation:

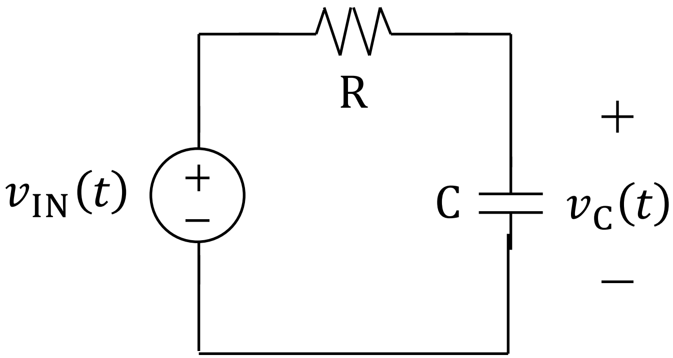
$$-\mathrm{RC}\omega B\sin(\omega t + \gamma) + B\cos(\omega t + \gamma) = \mathrm{V_I}\cos(\omega t + \phi_{\mathrm{I}})$$ ← True for all time

$$-\mathrm{RC}\omega B[\sin(\omega t)\cos(\gamma) + \cos(\omega t)\sin(\gamma)] + B[\cos(\omega t)\cos(\gamma) - \sin(\omega t)\sin(\gamma)]$$
$$= \mathrm{V_I}[\cos(\omega t)\cos(\phi_{\mathrm{I}}) - \sin(\omega t)\sin(\phi_{\mathrm{I}})]$$

Match $\sin(\omega t)$ and $\cos(\omega t)$ terms:

$$\begin{cases} -\mathrm{RC}\omega B\cos(\gamma) - B\sin(\gamma) = -\mathrm{V_I}\sin(\phi_{\mathrm{I}}) & \text{— A} \\ -\mathrm{RC}\omega B\sin(\gamma) + B\cos(\gamma) = \mathrm{V_I}\cos(\phi_{\mathrm{I}}) & \text{— B} \end{cases}$$

Particular Solution (Cont.)



$$
\begin{cases}
\mathrm{RC}\omega B\cos(\gamma) + B\sin(\gamma) = \mathrm{V_I}\sin(\phi_\mathrm{I}) & \text{— A} \\
-\mathrm{RC}\omega B\sin(\gamma) + B\cos(\gamma) = \mathrm{V_I}\cos(\phi_\mathrm{I}) & \text{— B}
\end{cases}
$$

Square and add:

$$(\mathrm{RC}\omega)^2 B^2\cos^2(\gamma) + B^2\sin^2(\gamma) + 2\mathrm{RC}\omega B^2\cos(\gamma)\sin(\gamma)$$

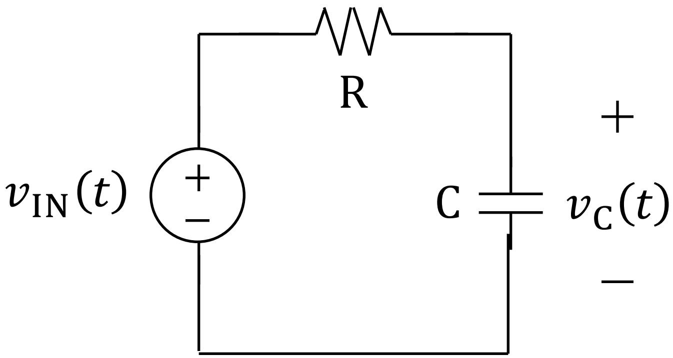
$$+(\mathrm{RC}\omega)^2 B^2\sin^2(\gamma) + B^2\cos^2(\gamma) - 2\mathrm{RC}\omega B^2\cos(\gamma)\sin(\gamma)$$

$$= \mathrm{V_I^2}\sin^2(\phi_\mathrm{I}) + \mathrm{V_I^2}\cos^2(\phi_\mathrm{I})$$

$$\Rightarrow (\mathrm{RC}\omega)^2 B^2\underbrace{[\cos^2(\gamma) + \sin^2(\gamma)]}_{=1} + B^2\overbrace{[\cos^2(\gamma) + \sin^2(\gamma)]}^{=1} = \mathrm{V_I^2}\underbrace{[\sin^2(\phi_\mathrm{I}) + \cos^2(\phi_\mathrm{I})]}_{=1}$$

$$\Rightarrow \underline{(\mathrm{RC}\omega)^2 B^2 + B^2 = \mathrm{V_I^2}} \quad \Rightarrow \quad \boxed{B = \frac{\mathrm{V_I}}{\sqrt{1 + (\mathrm{RC}\omega)^2}}}$$

Particular Solution (Cont.)

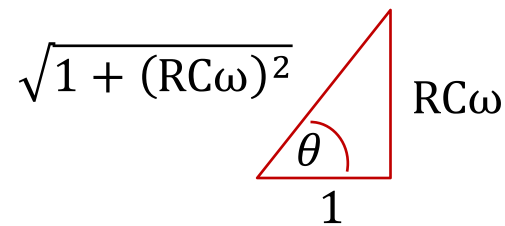


$$\mathrm{RC}\omega B\cos(\gamma) + B\sin(\gamma) = \mathrm{V_I}\sin(\phi_\mathrm{I})$$

$$\boxed{B = \frac{\mathrm{V_I}}{\sqrt{1+(\mathrm{RC}\omega)^2}}}$$

$$\Rightarrow \frac{\mathrm{RC}\omega B}{\sqrt{1+(\mathrm{RC}\omega)^2}}\cos(\gamma) + \frac{B}{\sqrt{1+(\mathrm{RC}\omega)^2}}\sin(\gamma) = \frac{\mathrm{V_I}}{\sqrt{1+(\mathrm{RC}\omega)^2}}\sin(\phi_\mathrm{I})$$

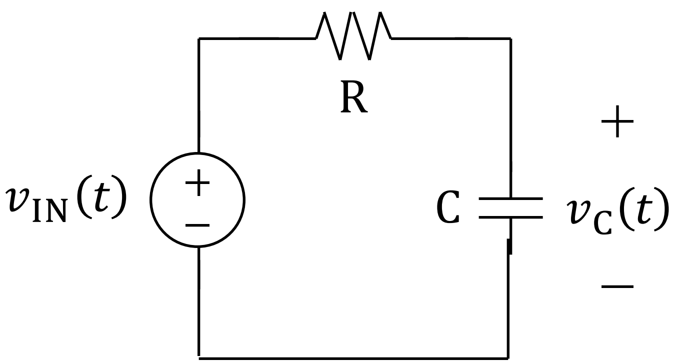
$$\Rightarrow B\sin(\theta)\cos(\gamma) + B\cos(\theta)\sin(\gamma) = B\sin(\phi_\mathrm{I})$$



$$\Rightarrow \sin(\theta+\gamma) = \sin(\phi_\mathrm{I}) \qquad \text{where} \quad \theta = \tan^{-1}(\mathrm{RC}\omega)$$

$$\Rightarrow \theta + \gamma = \phi_\mathrm{I} \qquad \Rightarrow \boxed{\gamma = \phi_\mathrm{I} - \tan^{-1}(\mathrm{RC}\omega)}$$

Total Solution



$$v_C(t) = v_{C,h}(t) + v_{C,p}(t)$$

$$v_{C,h}(t) = Ae^{-t/\tau} \qquad \text{where} \quad \tau = RC$$

$$v_{C,p}(t) = \frac{V_I}{\sqrt{1+(RC\omega)^2}}\cos(\omega t + \phi_I - \tan^{-1}(RC\omega))$$

$$v_C(t) = Ae^{-t/\tau} + \frac{V_I}{\sqrt{1+(RC\omega)^2}}\cos(\omega t + \phi_I - \tan^{-1}(RC\omega))$$

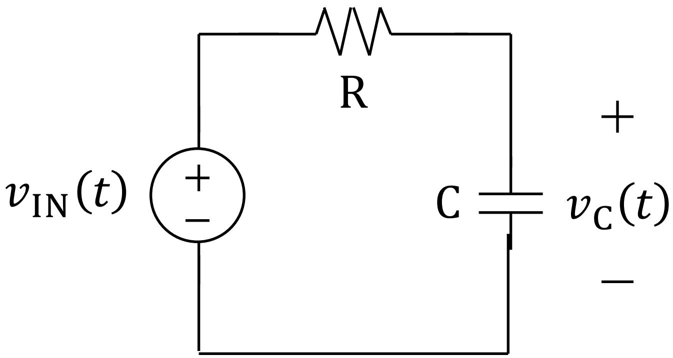
Apply initial condition:

$$v_C(0^+) = A + \frac{V_I}{\sqrt{1+(RC\omega)^2}}\cos(\phi_I - \tan^{-1}(RC\omega)) = 0$$

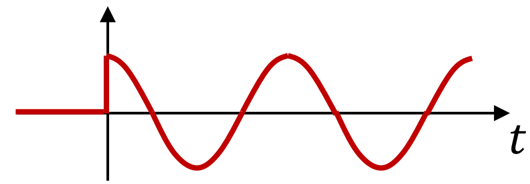
$$A = \frac{-V_I}{\sqrt{1+(RC\omega)^2}}\cos(\phi_I - \tan^{-1}(RC\omega))$$

$$v_C(t) = \frac{-V_I}{\sqrt{1+(RC\omega)^2}}\cos(\phi_I - \tan^{-1}(RC\omega))\, e^{-t/\tau} + \frac{V_I}{\sqrt{1+(RC\omega)^2}}\cos(\omega t + \phi_I - \tan^{-1}(RC\omega))$$

Sinusoidal Steady State



$$v_{IN}(t) = V_I \cos(\omega t + \phi_I)\, u(t)$$



For $t > 0$:

Focus on this

$$v_C(t) = \underbrace{\frac{-V_I}{\sqrt{1 + (RC\omega)^2}} \cos(\phi_I - \tan^{-1}(RC\omega))\, e^{-t/\tau}}_{\text{goes to zero as } t \to \infty \text{ (in } 5\tau\text{'s)}}$$

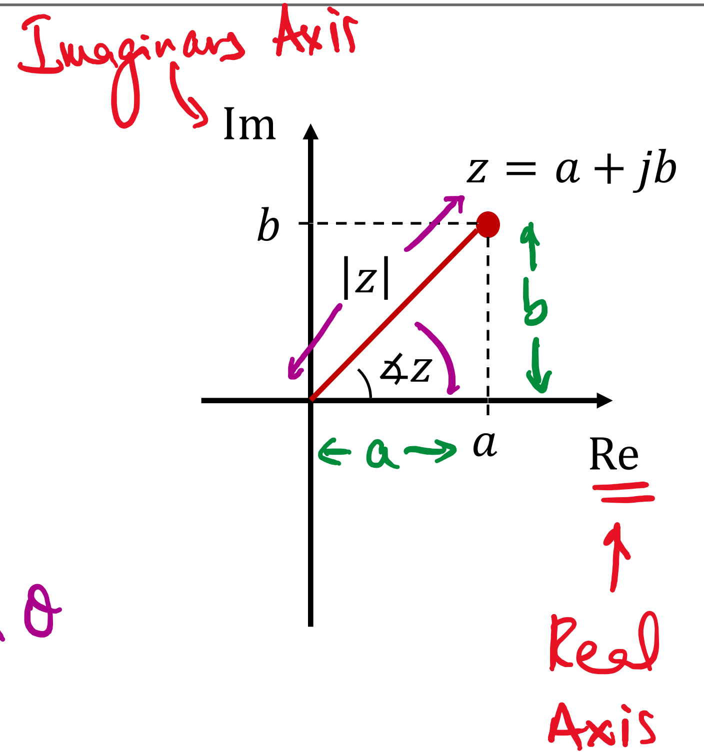
$$+ \frac{V_I}{\sqrt{1 + (RC\omega)^2}} \cos(\omega t + \phi_I - \tan^{-1}(RC\omega))$$

Stays for $t \to \infty$

Sinusoidal Steady State

Review of Complex Numbers in Polar Form

$z = a + jb$ $\qquad$ $j \equiv \sqrt{-1}$



$a = |z| \cos(\measuredangle z)$

$b = |z| \sin(\measuredangle z)$

$z = |z| \left(\cos(\measuredangle z) + j \sin(\measuredangle z) \right)$

Euler's Formula: $e^{j\theta} = \cos\theta + j\sin\theta$

$z = |z| e^{j\measuredangle z}$

$|z| = \sqrt{a^2 + b^2}$

$\measuredangle z = \tan^{-1}\left(\frac{b}{a}\right)$

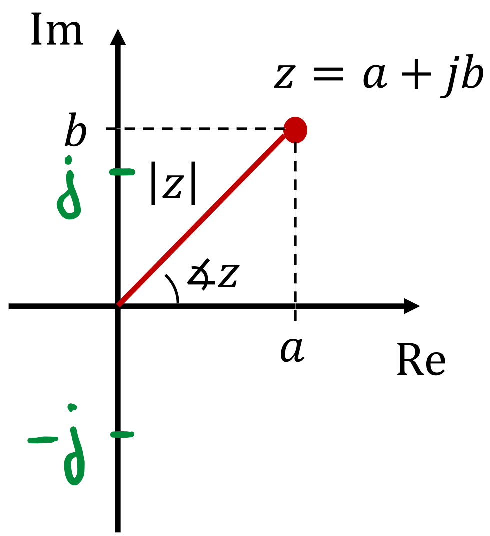
$$\boxed{z = \sqrt{a^2+b^2}\, e^{j\tan^{-1}(b/a)}}$$

Polar Form Complex Number Useful Relationships

$$z = a + jb = \sqrt{a^2 + b^2}\, e^{j \tan^{-1}(b/a)}$$

$$\mathrm{Re}\{z\} = \mathrm{Re}\{a + jb\} = a$$ $\leftarrow$ Real part of z

$$\mathrm{Im}\{z\} = \mathrm{Im}\{a + jb\} = b$$ $\leftarrow$ Imaginary part of z



$$z_1 = |z_1| e^{j\measuredangle z_1}$$

$$z_2 = |z_2| e^{j\measuredangle z_2}$$

$$z_1 z_2 = |z_1||z_2| e^{j(\measuredangle z_1 + \measuredangle z_2)}$$

$$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|} e^{j(\measuredangle z_1 - \measuredangle z_2)}$$

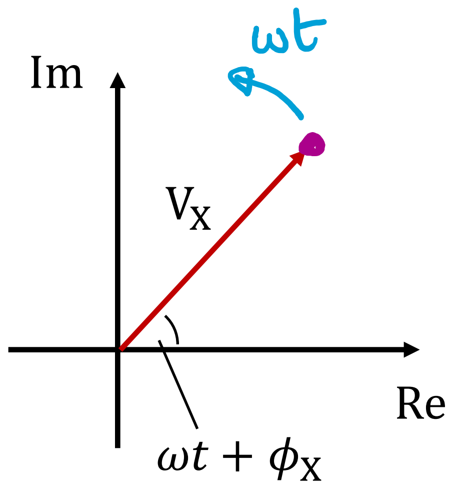
$$(z_1)^n = |z_1|^n e^{jn\measuredangle z_1}$$

$$e^{j\frac{\pi}{2}} = j$$

$$e^{-j\frac{\pi}{2}} = -j$$

Sinusoid as Complex Number in Polar Form

$$v_X(t) = V_X \cos(\omega t + \phi_X)$$



Using Euler's Formula: $e^{j\theta} = \cos(\theta) + j\sin(\theta)$

$$V_X e^{j(\omega t+\phi_X)} = \overbrace{V_X \cos(\omega t + \phi_X)}^{\text{Real part}} + \overbrace{jV_X \sin(\omega t + \phi_X)}^{\text{Imaginary part}}$$

$$\mathrm{Re}\{V_x e^{j(\omega t+\phi_x)}\} = V_x \cos(\omega t + \phi_x)$$

$$V_X \cos(\omega t + \phi_X) = \mathrm{Re}\{V_X e^{j(\omega t+\phi_X)}\} = \mathrm{Re}\{\overbrace{\underbrace{V_X e^{j\phi_X}}_{=\,\hat{V}_x} e^{j\omega t}}^{\text{Rotating Phasor}}\} = \mathrm{Re}\{\hat{V}_X e^{j\omega t}\}$$

$\hat{V}_x \leftarrow$ Phasor

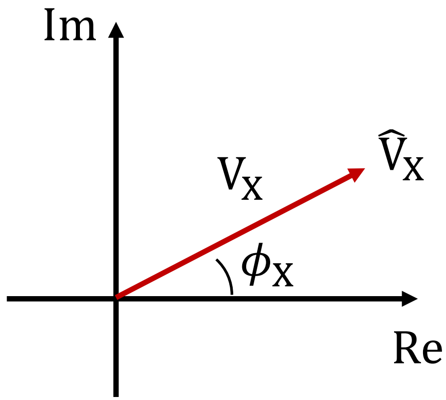
Phasor

Phasor

$$\hat{V}_X = V_X e^{j\phi_X}$$

Phasor has magnitude and phase

Alternate notation $\hat{V}_X = V_X \angle \phi_X$



Rotating Phasor (Complex Exponential)

$$\hat{V}_X e^{j\omega t} = V_X e^{j\phi_X} e^{j\omega t}$$

Using Phasors to Solve Circuits in Sinusoidal Steady State

Drive: $v_{\mathrm{IN}}(t) = \mathrm{V_I}\cos(\omega t + \phi_\mathrm{I})$

$$\mathrm{V_I} e^{j(\omega t+\phi_\mathrm{I})} = \underbrace{\mathrm{V_I}\cos(\omega t + \phi_\mathrm{I})}_{=a} + j\underbrace{\mathrm{V_I}\sin(\omega t + \phi_\mathrm{I})}_{=b} = a + jb$$

$$a = \mathrm{V_I}\cos(\omega t + \phi_\mathrm{I}) = \mathrm{Re}\left\{\mathrm{V_I} e^{j(\omega t+\phi_\mathrm{I})}\right\}$$

$$b = \mathrm{V_I}\sin(\omega t + \phi_\mathrm{I}) = \mathrm{Im}\left\{\mathrm{V_I} e^{j(\omega t+\phi_\mathrm{I})}\right\}$$

$$a + jb = \mathrm{V_I} e^{j(\omega t+\phi_\mathrm{I})}$$

$$a \longrightarrow \boxed{\text{LTI Circuit}} \longrightarrow x$$

$$b \longrightarrow \boxed{\text{LTI Circuit}} \longrightarrow y$$

$$a + jb \longrightarrow \boxed{\text{LTI Circuit}} \longrightarrow x + jy$$

$x = \mathrm{Re}\{x + jy\} =$

LTI ≡ Linear Time-Invariant

Phasor Analysis Method

1. Express drive as real part of a rotating phasor (phasor x $e^{j\omega t}$)

$$v_{\mathrm{IN}}(t) = \mathrm{V_I}\cos(\omega t + \phi_\mathrm{I}) = \mathrm{Re}\left\{\mathrm{V_I} e^{j(\omega t + \phi_\mathrm{I})}\right\} = \mathrm{Re}\left\{\mathrm{V_I} e^{j\phi_\mathrm{I}} e^{j\omega t}\right\} = \mathrm{Re}\left\{\hat{\mathrm{V}}_\mathrm{I} e^{j\omega t}\right\}$$

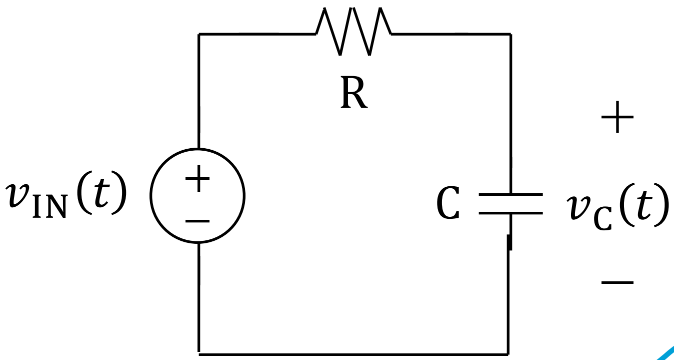
2. Calculate response to the rotating phasor

$$\hat{\mathrm{V}}_\mathrm{I} e^{j\omega t} \longrightarrow \boxed{\text{LTI Circuit}} \longrightarrow \hat{\mathrm{V}}_\mathrm{C} e^{j\omega t}$$

3. Take real part of the response to the rotating phasor

$$v_\mathrm{C}(t) = \mathrm{Re}\left\{\hat{\mathrm{V}}_\mathrm{C} e^{j\omega t}\right\} = \mathrm{Re}\left\{\mathrm{V_C} e^{j\phi_\mathrm{C}} e^{j\omega t}\right\} = \mathrm{V_C}\cos(\omega t + \phi_\mathrm{C})$$

Phasor Analysis Example



$$v_{\mathrm{IN}}(t) = \mathrm{V_I}\cos(\omega t + \phi_{\mathrm{I}})\,u(t)$$

$\underline{t > 0}$

$$\tilde{v}_{IN}(t) = V_I e^{j(\omega t + \phi_I)}$$

Particular Soln: $\tilde{v}_{c,p}(t) = V_c e^{j(\omega t + \angle v_c)}$

$$= \hat{V}_c e^{j\omega t}$$

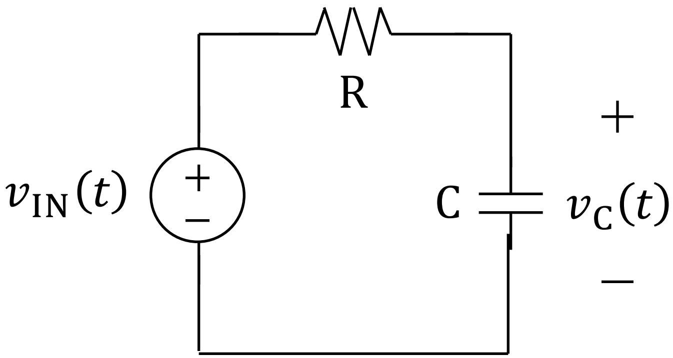
$$\hat{V}_c = V_c e^{j\angle v_c}$$

$$\mathrm{RC}\frac{dv_{\mathrm{C}}}{dt} + v_{\mathrm{C}} = v_{\mathrm{IN}}$$

$$RC\,\hat{V}_c\, j\omega \cancel{e^{j\omega t}} + \hat{V}_c \cancel{e^{j\omega t}} = V_I e^{j\phi_I} \cancel{e^{j\omega t}}$$

$$\hat{V}_c = \frac{V_I e^{j\phi_I}}{1 + jRC\omega}$$

Phasor Analysis Example (Cont.)



$$\hat{V}_C = \frac{V_I e^{j\phi_I}}{1 + jRC\omega}$$

$$\hat{V}_c = \frac{V_I e^{j\phi_I}}{\sqrt{1+(RC\omega)^2}\; e^{j\tan^{-1}(RC\omega)}}$$

$$\boxed{\hat{V}_c = \frac{V_I}{\sqrt{1+(RC\omega)^2}} e^{j(\phi_I - \tan^{-1}(RC\omega))}}$$

Particular Soln

$$v_c(t) = \mathrm{Re}\{\hat{V}_c e^{j\omega t}\} = \frac{V_I}{\sqrt{1+(RC\omega)^2}} \cos(\omega t + \phi_I - \tan^{-1}(RC\omega))$$