ECE/ENGRD 2100

Introduction to Circuits for ECE

Lecture 22

Driven Damped Second Order LC Circuits

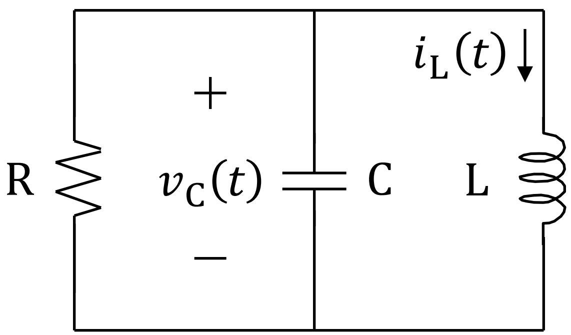
Announcements

- Recommended Reading:
  - Textbook Chapter 8

- Upcoming due dates:
  - Lab report 3 due by 11:59 pm on Friday March 15, 2019
  - Prelab 4 due by 12:20 pm on Tuesday March 19, 2019
  - Homework 4 due by 11:59 pm on Monday March 25, 2019
  - Lab report 4 due by 11:59 pm on Monday April 8, 2019

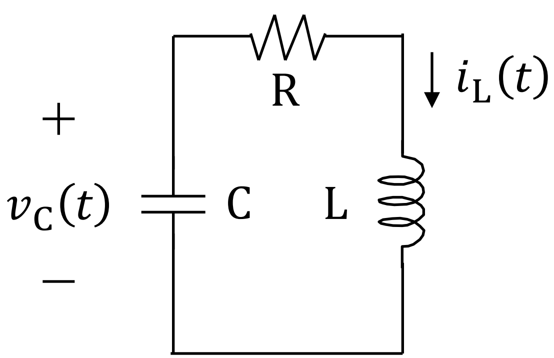
- Prelim 2 on Thursday March 28, 2019 from 7:30 – 9 pm in 203 Phillips
  - Email afridi@cornell.edu if have conflict
  - Will cover material through Lecture 24
  - Prelim is closed-book and closed-notes
  - Two double-sided page formula sheet is allowed
  - Bring a calculator

RLC Circuit – Natural Response – Summary



Parallel RLC Circuit

$$\alpha = \frac{1}{2RC} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$



Series RLC Circuit

$$\alpha = \frac{R}{2L} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

Characteristic equation: $s^2 + 2\alpha s + \omega_0^2 = 0$

$$s = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$\alpha > \omega_0$ Overdamped

$$A_1 e^{-\left(\alpha - \sqrt{\alpha^2 - \omega_0^2}\right)t} + A_2 e^{-\left(\alpha + \sqrt{\alpha^2 - \omega_0^2}\right)t}$$

$\alpha = \omega_0$ Critically damped

$$A_1 e^{-\alpha t} + A_2 t e^{-\alpha t}$$

$\alpha < \omega_0$ Underdamped

$$\omega_d \equiv \sqrt{\omega_0^2 - \alpha^2}$$

$$A_1 e^{-\alpha t}\cos(\omega_d t) + A_2 e^{-\alpha t}\sin(\omega_d t)$$

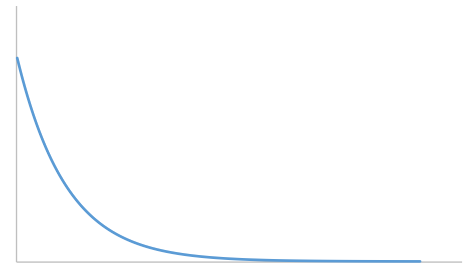
$$= B e^{-\alpha t}\cos(\omega_d t + \phi)$$

Quality Factor

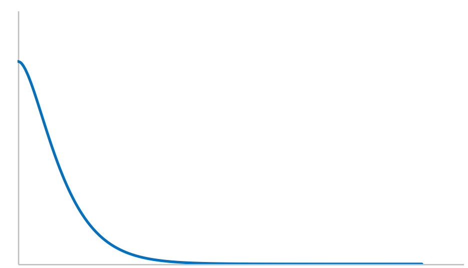
Characteristic equation: $s^2 + 2\alpha s + \omega_0^2 = 0$

Quality factor: $Q \equiv \dfrac{\omega_0}{2\alpha}$

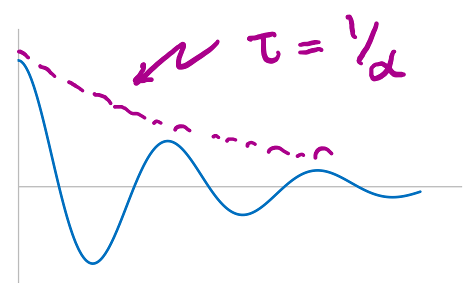
Overdamped: $\alpha > \omega_0$ ⇨ $Q < \dfrac{1}{2}$



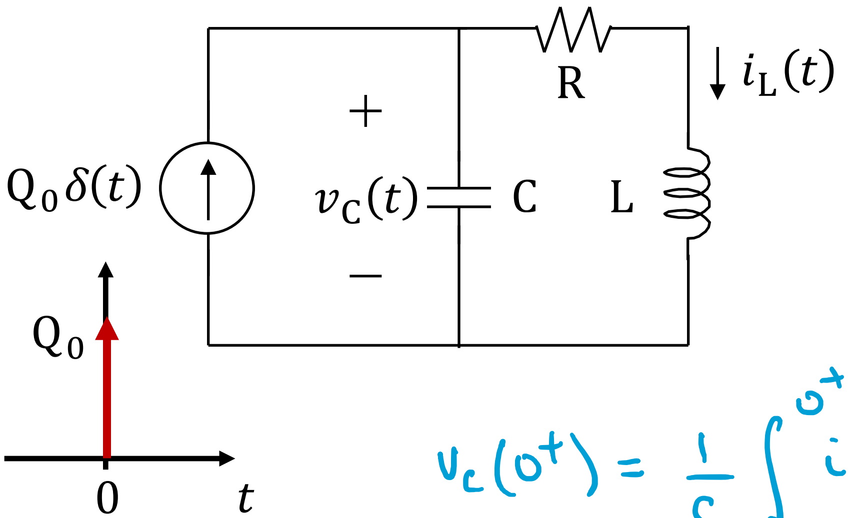
Critically damped: $\alpha = \omega_0$ ⇨ $Q = \dfrac{1}{2}$



Underdamped: $\alpha < \omega_0$ ⇨ $Q > \dfrac{1}{2}$



Underdamped Series RLC Circuit - Impulse Response



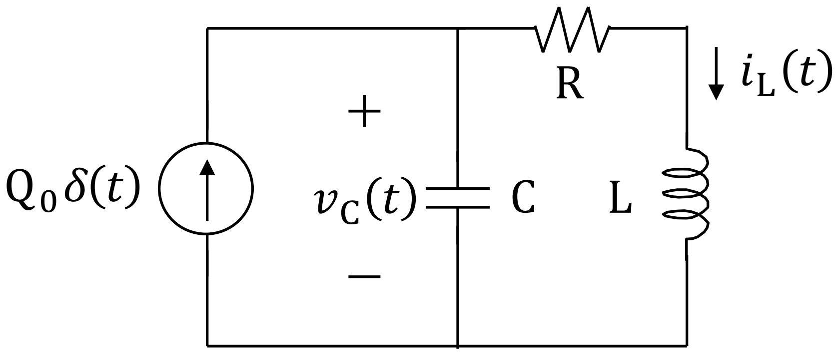
$v_C(0^+) = ?$

$i_L(0^+) = ?$

$$v_c(0^+) = \frac{1}{C}\int_{-\infty}^{0^+} i_c(t')\,dt' = \frac{1}{C}\int_{0^-}^{0^+} Q_0\delta(t')\,dt' = \frac{Q_0}{C}$$

$$i_L(0^+) = \frac{1}{L}\int_{0^-}^{0^+} v_L(t')\,dt' = 0$$

Underdamped RLC – Impulse Response (Cont.)



$$v_C(0^+) = \frac{Q_0}{C} \qquad i_L(0^+) = 0$$

Assuming Underdamped

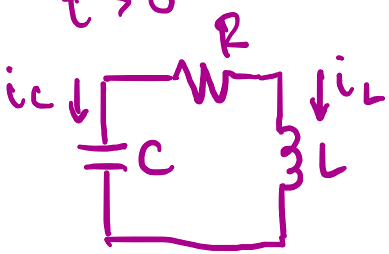
t > 0

$$v_c(t) = B\,e^{-\alpha t}\cos(\omega_d t + \phi)$$

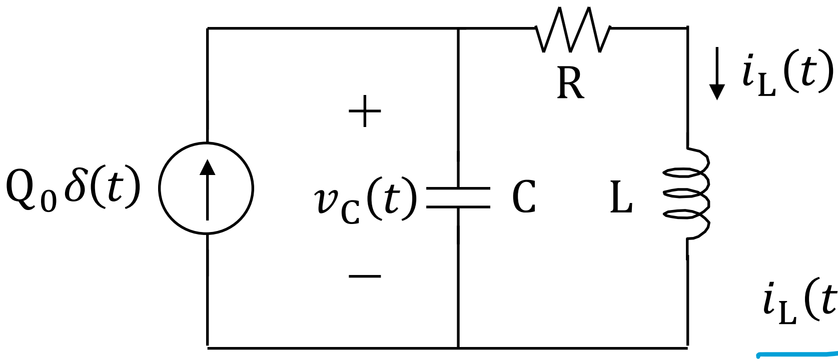
$$i_L = -i_c = -C\frac{dv_c}{dt}$$

$$i_L = -CB\left[-\alpha\, e^{-\alpha t}\cos(\omega_d t+\phi) - \omega_d\, e^{-\alpha t}\sin(\omega_d t+\phi)\right]$$

t > 0



Underdamped RLC – Impulse Response (Cont.)



$$v_C(t) = e^{-\alpha t}B\cos(\omega_d t + \phi)$$

$$i_L(t) = e^{-\alpha t}BC[\omega_d \sin(\omega_d t + \phi) + \alpha\cos(\omega_d t + \phi)]$$

Use Trigonometric Identity:

$$a\sin(x) + b\cos(x) = \sqrt{a^2 + b^2}\sin\left(x + \tan^{-1}\left({}^{b}/_{a}\right)\right)$$

$$\Rightarrow \quad i_L(t) = e^{-\alpha t}BC\underbrace{\sqrt{\omega_d^2 + \alpha^2}}_{\omega_0}\sin(\omega_d t + \phi + \tan^{-1}(\alpha/\omega_d))$$

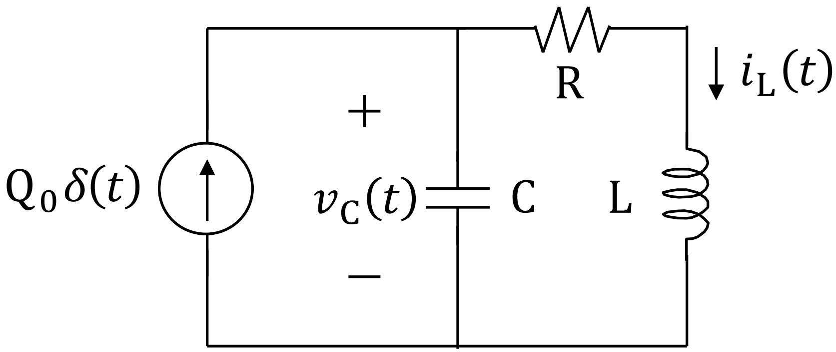
$$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

$$\sqrt{\omega_d^2 + \alpha^2} = \omega_0$$

$$C\omega_0 = C\frac{1}{\sqrt{LC}} = \sqrt{\frac{C}{L}} = \frac{1}{Z_0}$$

$$i_L(t) = e^{-\alpha t}\frac{B}{Z_0}\sin(\omega_d t + \phi + \tan^{-1}(\alpha/\omega_d))$$

Underdamped RLC – Impulse Response (Cont.)



$$v_C(t) = e^{-\alpha t} B\cos(\omega_d t + \phi)$$

$$i_L(t) = e^{-\alpha t}\frac{B}{Z_0}\sin(\omega_d t + \phi + \tan^{-1}(\alpha/\omega_d))$$

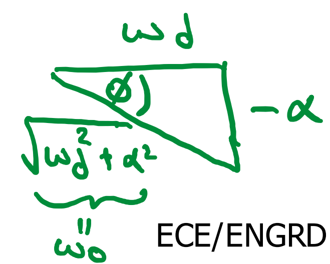
$$v_C(0^+) = \frac{Q_0}{C} \qquad i_L(0^+) = 0$$

$$v_c(0^+) = B\cos\phi = \frac{Q_0}{C}$$

$$i_L(0^+) = \frac{B}{Z_0}\sin(\underbrace{\phi + \tan^{-1}(\alpha/\omega_d)}_{=0}) = 0 \Rightarrow \phi = -\tan^{-1}(\alpha/\omega_d)$$

$$\Downarrow$$

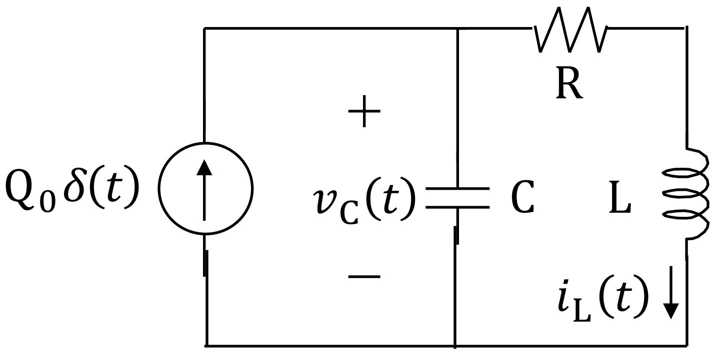
$$\tan\phi = -\alpha/\omega_d$$

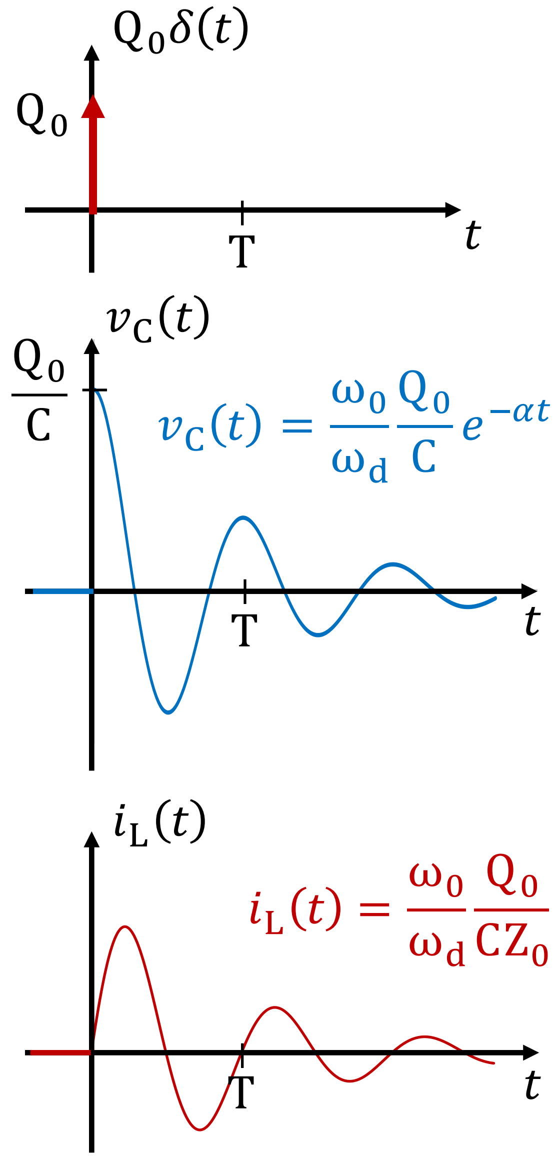


$$\cos\phi = \frac{\omega_d}{\omega_0} \Leftarrow$$

$$\Rightarrow B = \frac{Q_0}{C}\frac{1}{\cos\phi} = \frac{\omega_0}{\omega_d}\frac{Q_0}{C}$$

Underdamped RLC – Impulse Response - Waveforms

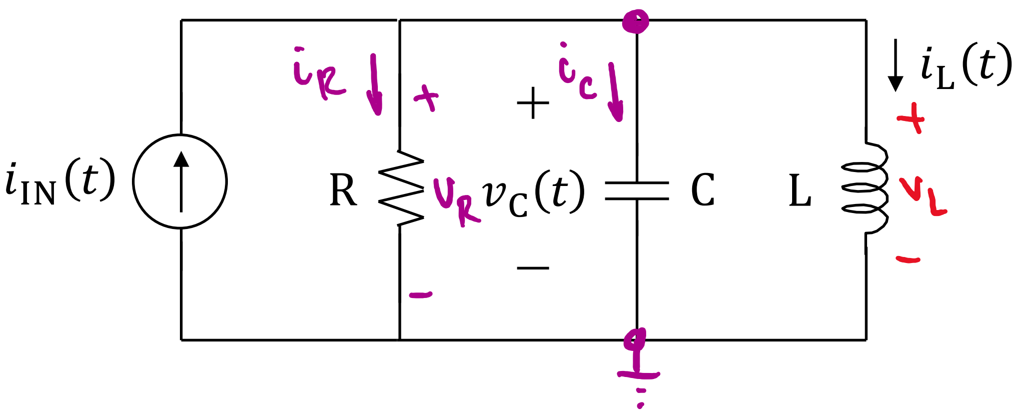




$$v_C(t) = \frac{\omega_0}{\omega_d}\frac{Q_0}{C}e^{-\alpha t}\cos(\omega_d t - \tan^{-1}(\alpha/\omega_d))$$

$$i_L(t) = \frac{\omega_0}{\omega_d}\frac{Q_0}{CZ_0}e^{-\alpha t}\sin(\omega_d t)$$

Driven Parallel RLC Circuit – Differential Equation



$$i_{IN} = i_R + i_C + i_L$$

$$v_L = L\frac{di_L}{dt}$$

$$i_C = C\frac{dv_C}{dt} = C\frac{dv_L}{dt} = LC\frac{d^2 i_L}{dt^2}$$

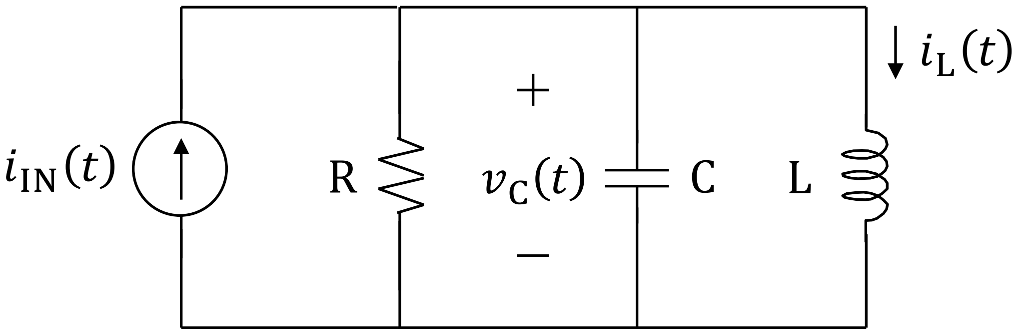
$$i_R = \frac{v_R}{R} = \frac{v_L}{R} = \frac{L}{R}\frac{di_L}{dt}$$

$$\Rightarrow \quad LC\frac{d^2 i_L}{dt^2} + \frac{L}{R}\frac{di_L}{dt} + i_L = i_{IN}$$

$$\Rightarrow \quad \boxed{\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{i_L}{LC} = \frac{i_{IN}}{LC}}$$

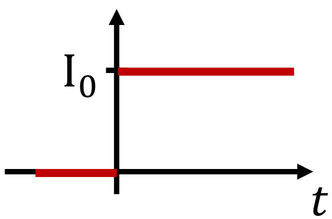
Diff. Eqn

Driven Parallel RLC Circuit – Step Response



$$\frac{d^2 i_{\mathrm{L}}}{dt^2} + \frac{1}{\mathrm{RC}}\frac{di_{\mathrm{L}}}{dt} + \frac{i_{\mathrm{L}}}{\mathrm{LC}} = \frac{i_{\mathrm{IN}}}{\mathrm{LC}}$$

$$i_{\mathrm{IN}}(t) = \mathrm{I}_0 u(t)$$



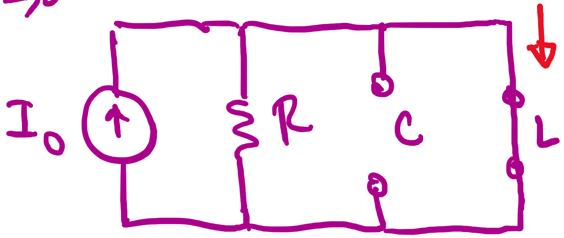
Natural Response

$$i_{\mathrm{L}}(t) = i_{\mathrm{L},h} + i_{\mathrm{L},p}$$

Driven Response (Particular Solution)

$t > 0$

$t \to \infty$



$i_{L,p} = I_0$

Initial Conditions

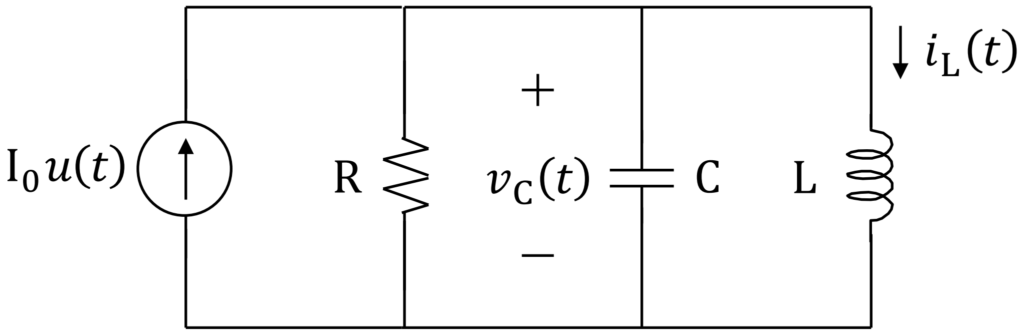
$i_L(0^-) = 0$

$v_c(0^-) = 0$

$i_L(0^+) = i_L(0^-) = 0$

$v_c(0^+) = v_c(0^-) = 0$

Parallel RLC Circuit – Step Response (Cont.)



$C = 1\ \mu F \qquad L = 1\ \mu H \qquad R = 5\ \Omega$

$$\omega_0 = \frac{1}{\sqrt{LC}} = 10^6 \text{ rad/s}$$

$$\alpha = \frac{1}{2RC} = 10^5 \text{ rad/s}$$

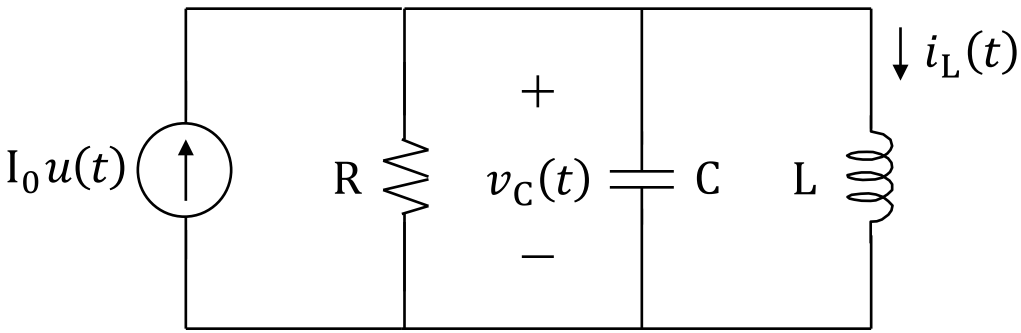
$$\omega_d \equiv \sqrt{\omega_0^2 - \alpha^2} = 9.95 \times 10^5 \text{ rad/s}$$

$\omega_0 > \alpha \Rightarrow$ Underdamped

$$i_L(t) = \underline{\underline{B}}\, e^{-\alpha t} \cos(\omega_d t + \underline{\underline{\phi}})$$

$$v_C(t) = v_L(t) = L\frac{di_L}{dt} = BL\left[-\alpha e^{-\alpha t}\cos(\omega_d t + \phi) - \omega_d e^{-\alpha t}\sin(\omega_d t + \phi)\right]$$

Parallel RLC Circuit – Step Response (Cont.)



$v_C(0^+) = 0$ $\qquad i_L(0^+) = 0$

$$i_L(t) = e^{-\alpha t} B \cos(\omega_d t + \phi) + I_0$$

$$v_C(t) = -e^{-\alpha t} BL[\omega_d \sin(\omega_d t + \phi) + \alpha \cos(\omega_d t + \phi)]$$

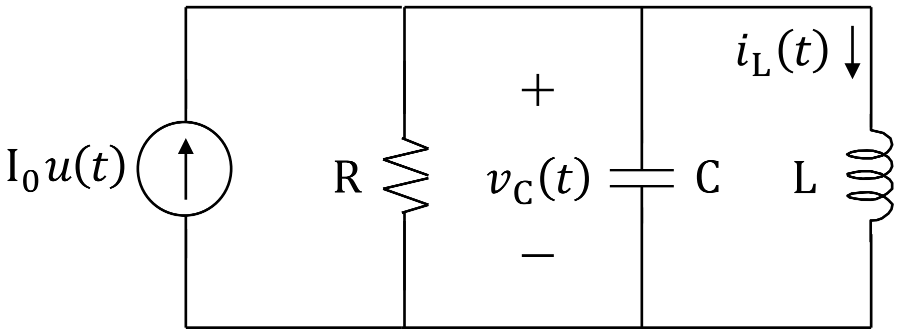
$$i_L(0^+) = B \cos\phi + I_0 = 0$$

$$v_C(0^+) = -BL[\underbrace{\omega_d \sin\phi + \alpha \cos\phi}] = 0$$

$$\Rightarrow \phi = -\tan^{-1}(\alpha/\omega_d) \Rightarrow \cos\phi = \frac{\omega_d}{\omega_0}$$

$$\Rightarrow B = -I_0 \frac{\omega_0}{\omega_d}$$

Parallel RLC Circuit – Step Response (Cont.)



$$B = -\frac{I_0 \omega_0}{\omega_d} \qquad \phi = -\tan^{-1}(\alpha/\omega_d)$$

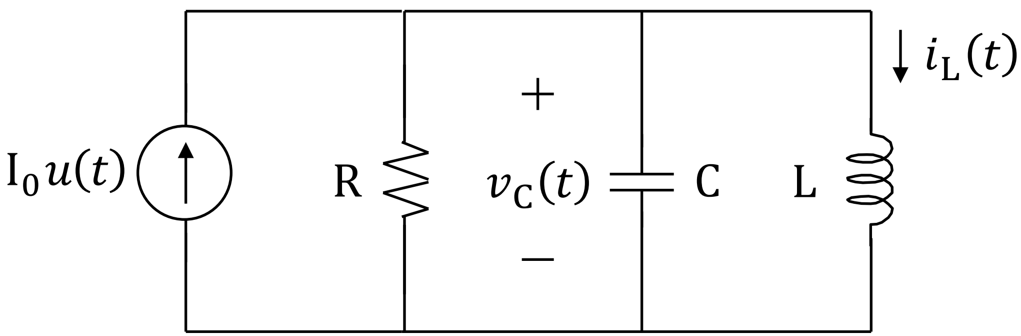
$$i_L(t) = e^{-\alpha t} B \cos(\omega_d t + \phi) + I_0$$

$$v_C(t) = -e^{-\alpha t} BL[\omega_d \sin(\omega_d t + \phi) + \alpha \cos(\omega_d t + \phi)]$$

$$i_L(t) = I_0 \left[1 - \frac{\omega_0}{\omega_d} e^{-\alpha t} \cos\left(\omega_d t - \tan^{-1}\left(\frac{\alpha}{\omega_d}\right)\right)\right]$$

$$v_C(t) = I_0 Z_0 \frac{\omega_0}{\omega_d} e^{-\alpha t} \sin(\omega_d t)$$

Parallel RLC Circuit – Step Response – Waveforms



$C = 1\ \mu F \qquad L = 1\ \mu H \qquad R = 5\ \Omega$

$$\omega_0 = 10^6 \text{ rad/sec}$$

$$\alpha = 10^5 \text{ rad/sec}$$

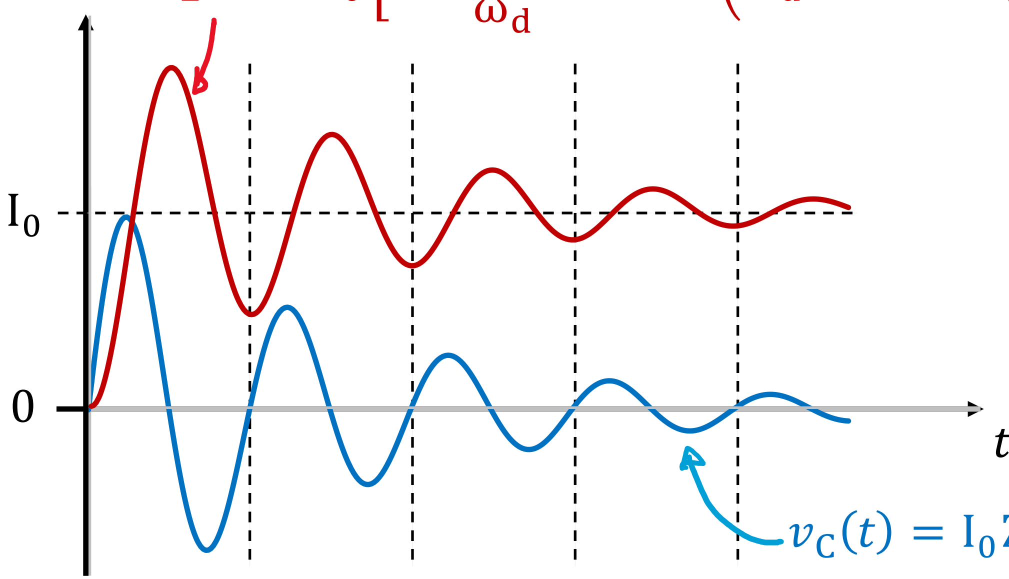
$$\omega_d = 9.95 \times 10^5 \text{ rad/s}$$

$$f_d = \frac{\omega_d}{2\pi} = 158 \text{ kHz}$$

$$Z_0 = \sqrt{\frac{L}{C}} = 1\ \Omega$$

$$Q_0 = \frac{\omega_0}{2\alpha} = 5$$

$$i_L(t) = I_0 \left[1 - \frac{\omega_0}{\omega_d} e^{-\alpha t} \cos\left(\omega_d t - \tan^{-1}\left(\frac{\alpha}{\omega_d} \right) \right) \right]$$



$$v_C(t) = I_0 Z_0 \frac{\omega_0}{\omega_d} e^{-\alpha t} \sin(\omega_d t)$$