ECE/ENGRD 2100

Introduction to Circuits for ECE

Lecture 21

Undamped and Damped Second Order Circuits

Natural and Driven Response

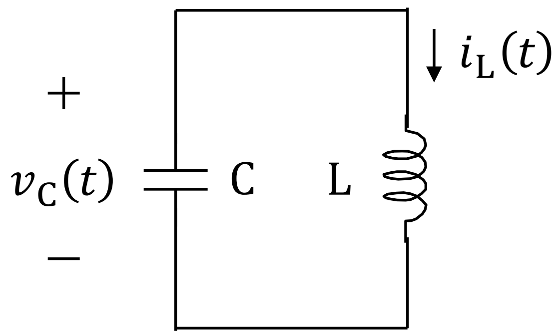
Announcements

- Recommended Reading:
  - Textbook Chapter 8

- Upcoming due dates:
  - Lab report 3 due by 11:59 pm on Friday March 15, 2019
  - Prelab 4 due by 12:20 pm on Tuesday March 19, 2019
  - Homework 4 due by 11:59 pm on Friday March 22, 2019
  - Lab report 4 due by 11:59 pm on Friday March 29, 2019

- Prelim 2 on Thursday March 28, 2019 from 7:30 – 9 pm in 203 Phillips
  - Email afridi@cornell.edu if have conflict
  - Will cover material through Lecture 24
  - Prelim is closed-book and closed-notes
  - Two double-sided page formula sheet is allowed
  - Bring a calculator

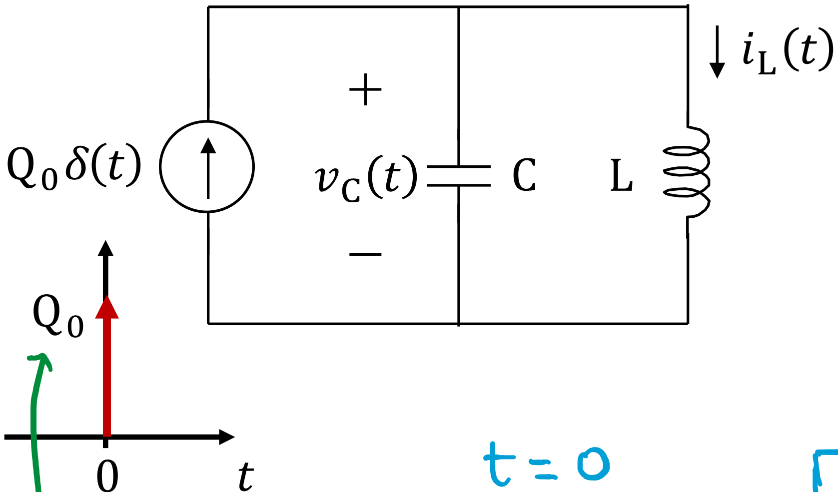
Undriven LC Circuit – Intutive Approach



- Since there is no energy dissipating element in the circuit, $v_C(t)$ and $i_L(t)$ will oscillate without damping
- For total energy to remain constant, $v_C(t)$ and $i_L(t)$ oscillations must remain $90°$ out of phase
- ELI the ICE man: Voltage of inductor leads its current, v_L $(= v_C)$ leads i_L, i.e., if v_C oscillation is $\cos$ then i_L oscillation is $\sin$ (as $\cos$ leads $\sin$ by $90°$)
- To determine dc value $v_C(t)$ and $i_L(t)$ will oscillate around: V_C, I_L, assume circuit has a little damping, then ask what values of $v_C(t)$ and $i_L(t)$ would the circuit settle down to – $v_C(t)$ and $i_L(t)$ will oscillate around these values.
- Frequency of oscillation, $\omega_0 = 1/\sqrt{LC}$
- Relationship between amplitudes of oscillations of v_C and i_L is given by $\Delta V_C = Z_0 \Delta I_L$ where $Z_0 = \sqrt{L/C}$

$$\boxed{\begin{aligned} v_C(t) &= B\cos(\omega_0 t + \phi) \\ i_L(t) &= \frac{B}{Z_0}\sin(\omega_0 t + \phi) \end{aligned}}$$

Driven LC Circuit – Impulse Input

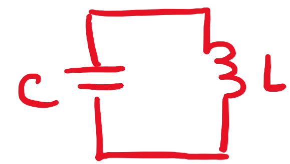


charge [A·s]

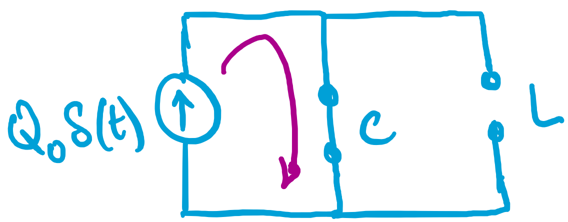
$t<0$

$v_c(t)=0$, $i_L(t)=0$ $\Rightarrow$ $v_c(0^-)=0$, $i_L(0^-)=0$

$t>0$



$t=0$

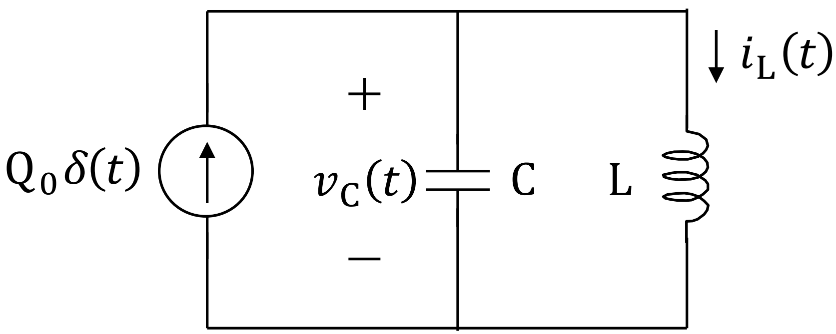


$v_c(0^+)$ & $i_L(0^+)$

$$v_c(t) = \frac{1}{C}\int_{-\infty}^{t} i_c(t')\,dt' \Rightarrow \underline{\underline{v_c(0^+)}} = \frac{1}{C}\int_{-\infty}^{0^+} i_c(t')\,dt' = \frac{1}{C}\int_{0^-}^{0^+} Q_0\delta(t')\,dt' = \underline{\underline{\frac{Q_0}{C}}}$$

$$i_L(t) = \frac{1}{L}\int_{-\infty}^{t} v_L(t')\,dt' \Rightarrow \underline{\underline{i_L(0^+)}} = \frac{1}{L}\int_{-\infty}^{0^+} v_L(t')\,dt' = \frac{1}{L}\int_{0^-}^{0^+} \frac{Q_0}{2C}\,dt' = \underline{\underline{0}}$$

Driven LC Circuit – Impulse Input (Cont.)



Initial Conditions

$$v_C(0^+) = \frac{Q_0}{C}$$

$$i_L(0^+) = 0$$

$$v_C(t) = B\cos(\omega_0 t + \phi) \quad \Rightarrow \quad v_c(0^+) = B\cos\phi = \frac{Q_0}{C}$$

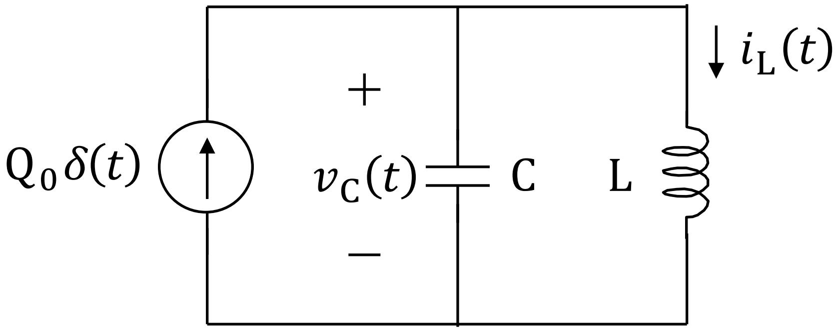
$$i_L(t) = \frac{B}{Z_0}\sin(\omega_0 t + \phi) \quad \Rightarrow \quad i_L(0^+) = \frac{B}{Z_0}\sin\phi = 0 \quad \Rightarrow \quad \boxed{\phi = 0}$$

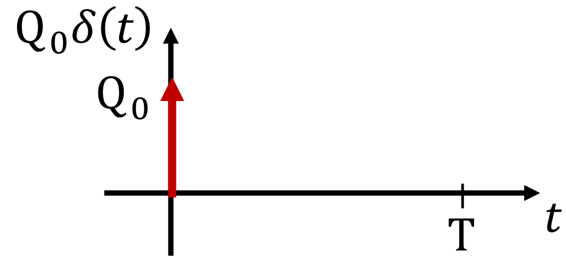
$$\Rightarrow \quad B\underbrace{\cos(0)}_{=1} = \frac{Q_0}{C} \quad \Rightarrow \quad B = \frac{Q_0}{C}$$

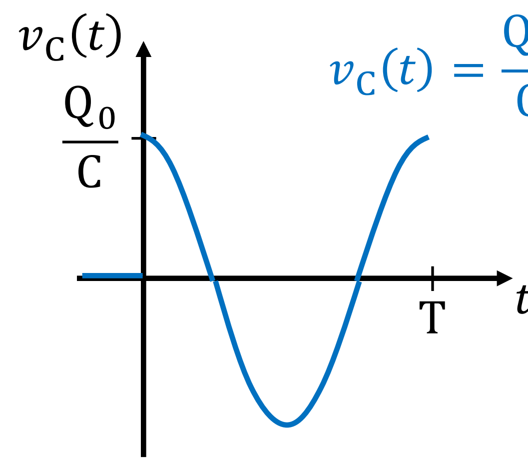
$$v_c(t) = \frac{Q_0}{C}\cos\omega_0 t$$

$$i_L(t) = \frac{Q_0}{CZ_0}\sin\omega_0 t$$

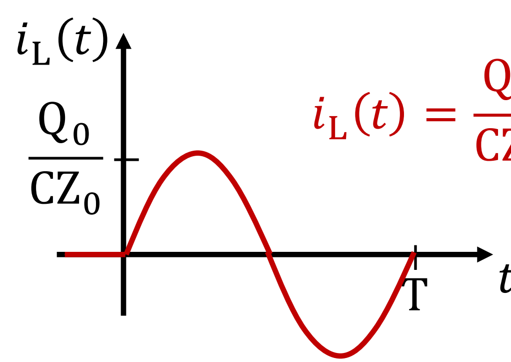
Driven LC Circuit – Impulse Input – Waveforms





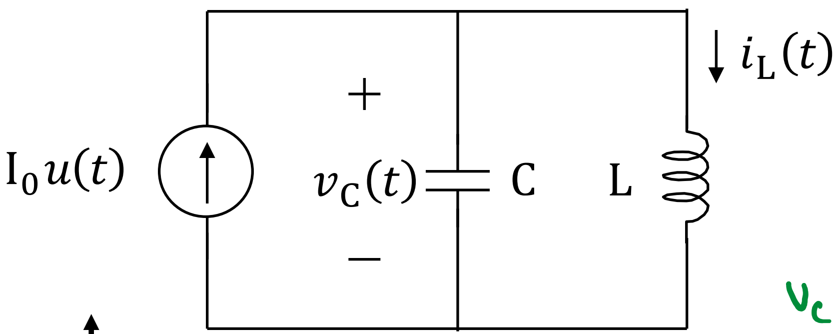


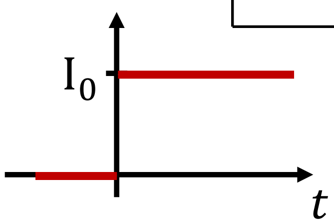
$$v_C(t) = \frac{Q_0}{C}\cos(\omega_0 t)$$



$$i_L(t) = \frac{Q_0}{CZ_0}\sin(\omega_0 t)$$

Driven LC Circuit – Step Input





Total Response
= Natural Response + Driven Response

$$v_C(t) = v_{C,h} + v_{C,p}$$

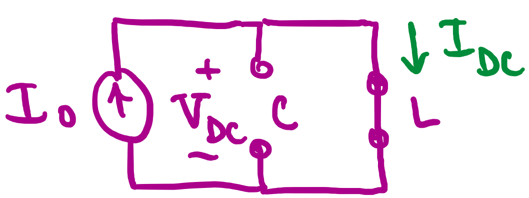
$$v_c(t) = B\cos(\omega_0 t + \phi) + V_{DC}$$

$$i_L(t) = i_{L,h} + i_{L,p}$$

$$i_L(t) = \frac{B}{Z_0}\sin(\omega_0 t + \phi) + I_{DC}$$

" constant since drive is constant for $t > 0$

DC Circuit ($t \to \infty$)



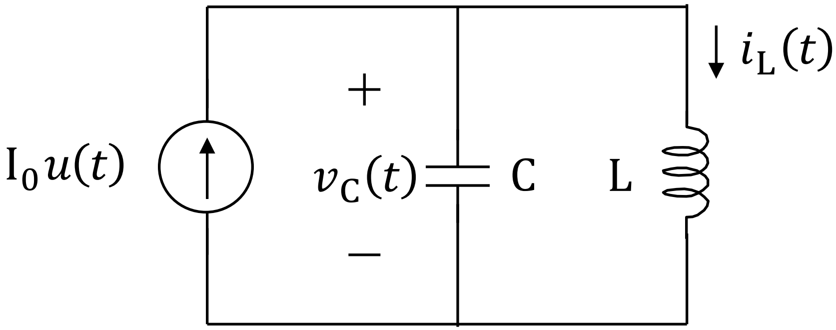
$$V_{DC} = 0$$

$$I_{DC} = I_0$$

$$v_c(t) = B\cos(\omega_0 t + \phi) + 0$$

$$i_L(t) = \frac{B}{Z_0}\sin(\omega_0 t + \phi) + I_0$$

Driven LC Circuit – Step Input (Cont.)



$$v_C(t) = B\cos(\omega_0 t + \phi)$$

$$i_L(t) = \frac{B}{Z_0}\sin(\omega_0 t + \phi) + I_0$$

for $t > 0$ $\quad v_c(0^+) \quad i_L(0^+)$

$v_c(0^-) = 0 \qquad i_L(0^-) = 0 \qquad \Rightarrow \quad v_c(0^+) = v_c(0^-) = 0$

$i_L(0^+) = i_L(0^-) = 0$

No impulse so continuity applies

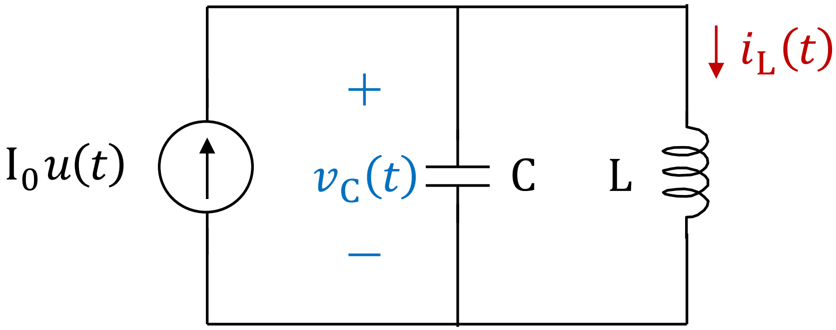
$v_c(0^+) = B\cos\phi = 0 \quad \Rightarrow \quad \phi = \pi/2$

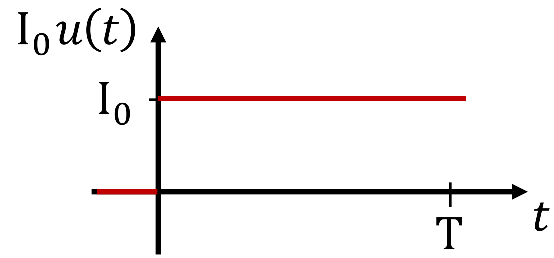
$i_L(0^+) = \frac{B}{Z_0}\sin\phi + I_0 = 0 \quad \Rightarrow \quad \frac{B}{Z_0} = -I_0 \quad \Rightarrow \quad B = -I_0 Z_0$

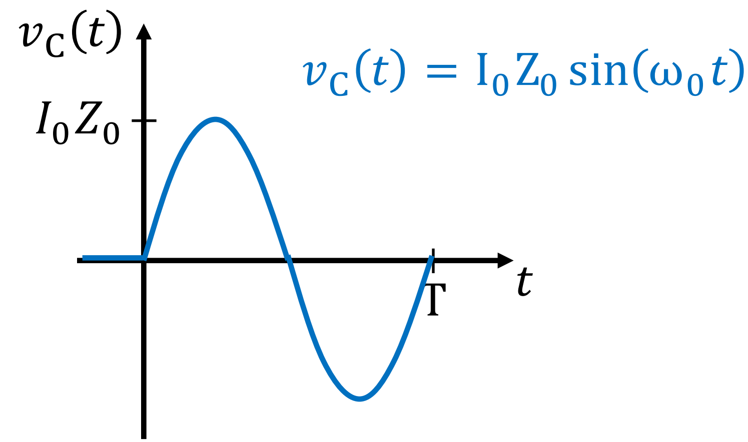
$v_c(t) = -I_0 Z_0 \cos(\omega_0 t + \pi/2) = I_0 Z_0 \sin(\omega_0 t)$

$i_L(t) = -I_0 \sin(\omega_0 t + \pi/2) + I_0 = I_0(1 - \cos(\omega_0 t))$

Driven LC Circuit – Step Input – Waveforms

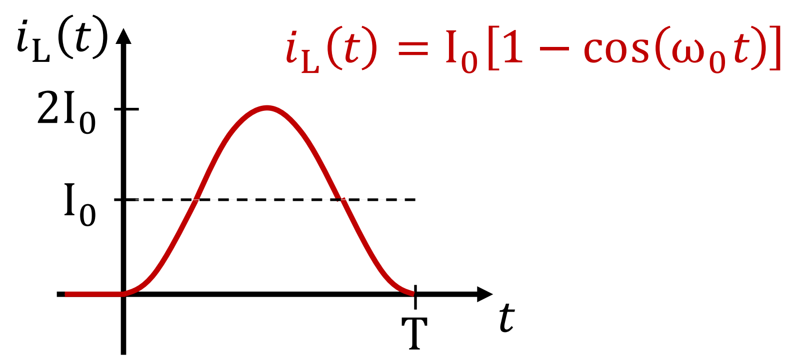






$$v_C(t) = -I_0 Z_0 \cos\left(\omega_0 t + \frac{\pi}{2}\right)$$

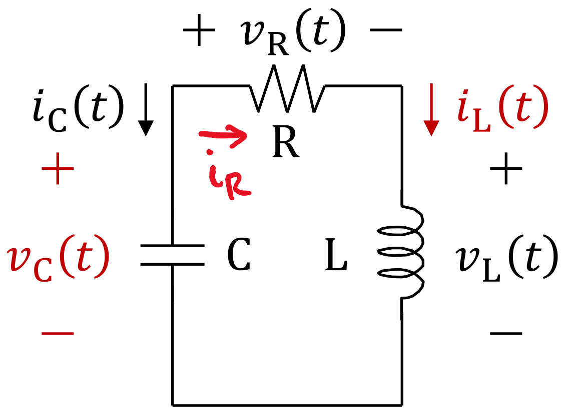
$$= I_0 Z_0 \sin(\omega_0 t)$$



$$i_L(t) = I_0\left[1 - \sin\left(\omega_0 t + \frac{\pi}{2}\right)\right]$$

$$= I_0[1 - \cos(\omega_0 t)]$$

Damped LC (RLC) Circuit – Natural Response



Series RLC Circuit

KVL

$$v_C = v_R + v_L$$

$$i_C = C\frac{dv_C}{dt} = -i_L$$

$$v_L = L\frac{di_L}{dt} = -LC\frac{d^2v_C}{dt^2}$$

$$v_R = Ri_R = Ri_L = -RC\frac{dv_C}{dt}$$

$$\Rightarrow v_C = -RC\frac{dv_C}{dt} - LC\frac{d^2v_C}{dt^2}$$

$$\Rightarrow \boxed{\frac{d^2v_C}{dt^2} + \frac{R}{L}\frac{dv_C}{dt} + \frac{v_C}{LC} = 0}$$

Diff. Eq.

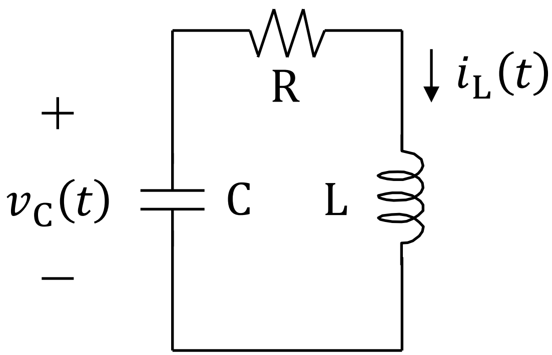
$$\Rightarrow \frac{d^2v_C}{dt^2} + 2\alpha\frac{dv_C}{dt} + \omega_0^2 v_C = 0$$

Damping Factor, $\alpha = \frac{R}{2L}$

Undamped Natural Radial Freq, $\omega_0 = \frac{1}{\sqrt{LC}}$

Series RLC Circuit – Differential Equation

Particular Soln
$=$
0



Series RLC Circuit

$$\frac{d^2 v_C}{dt^2} + 2\alpha \frac{dv_C}{dt} + \omega_0^2 v_C = 0$$

$\alpha = \frac{R}{2L}$ Damping Factor

$\omega_0 = \frac{1}{\sqrt{LC}}$ Undamped Natural Radial Frequency

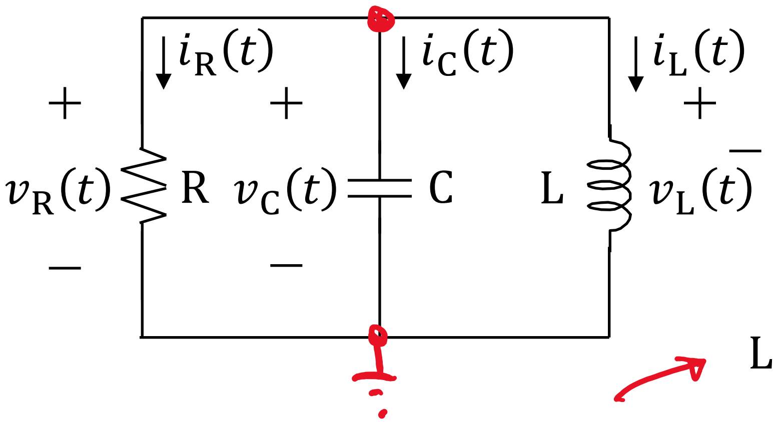
Homogeneous Soln $v_{c,h} = A e^{st}$

$$\Rightarrow \cancel{A} s^2 \cancel{e^{st}} + 2\alpha \cancel{A} s \cancel{e^{st}} + \omega_0^2 \cancel{A} \cancel{e^{st}} = 0$$

$$\Rightarrow \boxed{s^2 + 2\alpha s + \omega_0^2 = 0}$$

Characteristic Eqn.

Parallel RLC Circuit – Differential Equation



KCL

$$i_C + i_R + i_L = 0$$

$$v_L = L\frac{di_L}{dt} = v_C$$

$$i_C = C\frac{dv_C}{dt} = LC\frac{d^2 i_L}{dt^2}$$

$$i_R = \frac{v_R}{R} = \frac{v_C}{R} = \frac{L}{R}\frac{di_L}{dt}$$

$$LC\frac{d^2 i_L}{dt^2} + \frac{L}{R}\frac{di_L}{dt} + i_L = 0$$

Diff. Eqn.

$$\boxed{\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{i_L}{LC} = 0}$$

$$\alpha = \frac{1}{2RC}$$

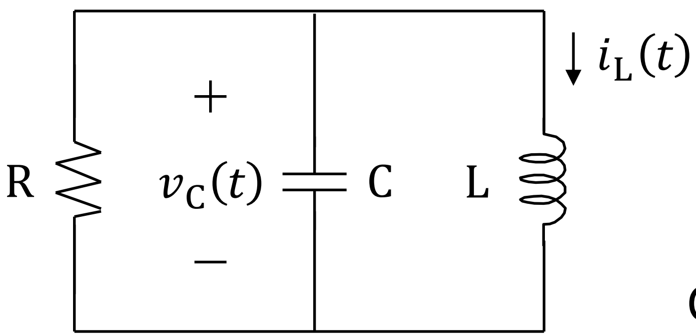
$$\frac{d^2 i_L}{dt^2} + 2\alpha\frac{di_L}{dt} + \omega_0^2 i_L = 0$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$i_{L,h} = Ae^{st} \Rightarrow \cancel{A}s^2\cancel{e^{st}} + 2\alpha\cancel{A}s\cancel{e^{st}} + \omega_0^2\cancel{Ae^{st}} = 0$$

$$\Rightarrow \boxed{s^2 + 2\alpha s + \omega_0^2 = 0}$$

RLC Circuit: Characteristic Equation - General Form



Parallel RLC Circuit

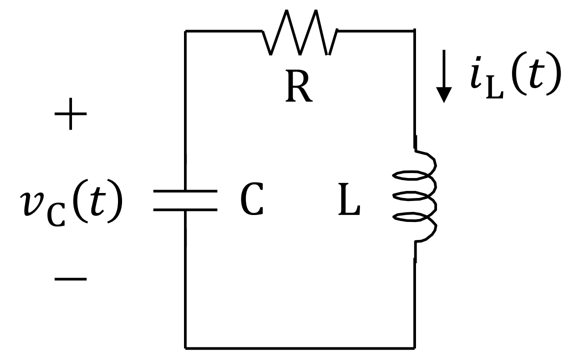
$$\alpha = \frac{1}{2RC}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Ae^{st}

Characteristic equation

$$s^2 + 2\alpha s + \omega_0^2 = 0$$



Series RLC Circuit

$$\alpha = \frac{R}{2L}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$s = \frac{-\cancel{2}\alpha \pm \sqrt{\cancel{4}\alpha^2 - \cancel{4}\omega_0^2}}{\cancel{2}}$$

$$\Rightarrow \boxed{s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}} \quad \text{or} \quad \boxed{s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}}$$

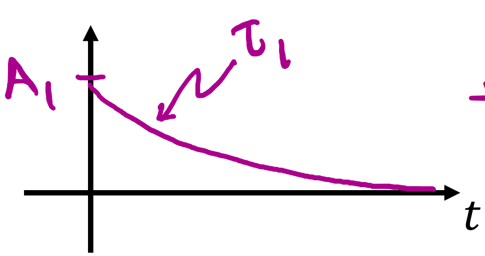
Characteristic Equation – Possible Solution – Case 1

$s = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$ Case 1: $\alpha > \omega_0$

$\Rightarrow \quad s_1 = -\alpha + \underbrace{\sqrt{\alpha^2 - \omega_0^2}}_{+ve} \qquad s_2 = -\alpha - \underbrace{\sqrt{\alpha^2 - \omega_0^2}}_{+ve}$

Two real solutions

$v_c(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} = A_1 e^{-t/\tau_1} + A_2 e^{-t/\tau_2}$



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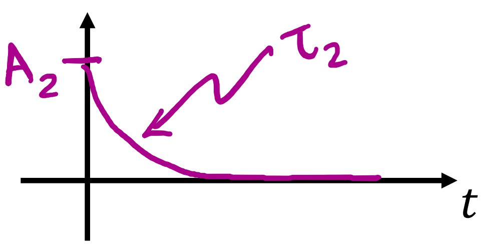


Characteristic Equation – Possible Solution – Case 2

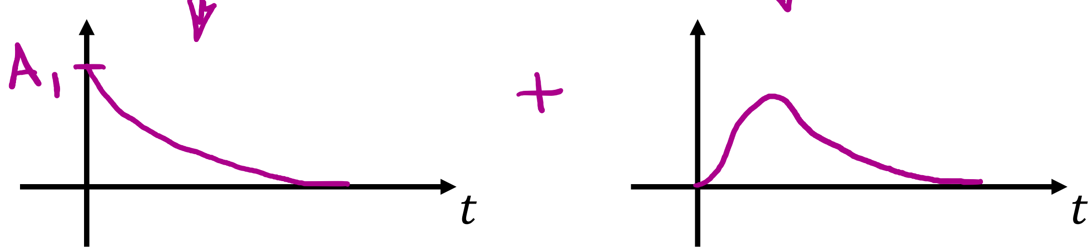
$s = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$

Case 2: $\alpha = \omega_0$

$s_1 = -\alpha \qquad s_2 = -\alpha$

$v_c(t) = A_1 e^{-\alpha t} + \underline{A_2 t e^{-\alpha t}}$

Only when $\alpha = \omega_0$



Characteristic Equation – Possible Solution – Case 3

$s = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$

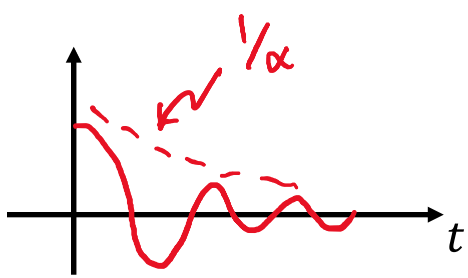
Case 3: $\alpha < \omega_0$

$$s_1 = -\alpha - j\underbrace{\sqrt{\omega_0^2 - \alpha^2}}_{\equiv\, \omega_d} \qquad s_2 = -\alpha + j\sqrt{\omega_0^2 - \alpha^2}$$

Damped Natural Radial Frequency, ω_d

$$v_c(t) = A_1 e^{(-\alpha - j\omega_d)t} + A_2 e^{(-\alpha + j\omega_d)t} = e^{-\alpha t}\left[A_1 e^{-j\omega_d t} + A_2 e^{+j\omega_d t}\right]$$

$$v_c(t) = e^{-\alpha t}\left[B_1 \cos \omega_d t + B_2 \sin \omega_d t\right]$$



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