ECE/ENGRD 2100

Introduction to Circuits for ECE

Lecture 20

Undamped Second Order LC Circuits

Natural Response

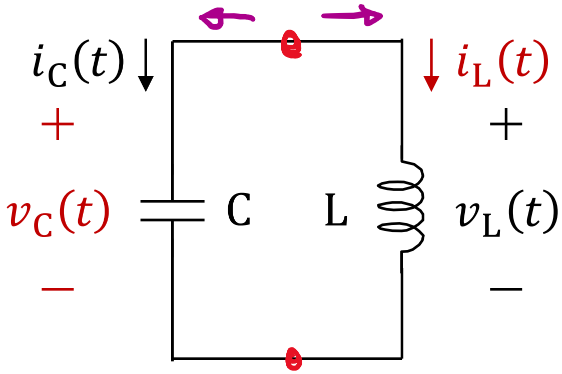
Announcements

- Recommended Reading:
  - Textbook Chapter 8

- Upcoming due dates:
  - Homework 3 due by 11:59 pm on Monday March 11, 2019
  - Lab report 3 due by 11:59 pm on Friday March 15, 2019

- Prelim 2 on Thursday March 28, 2019 from 7:30 – 9 pm in 203 Phillips
  - Email afridi@cornell.edu if have conflict
  - Will cover material through Lecture 24
  - Prelim is closed-book and closed-notes
  - Two double-sided page formula sheet is allowed
  - Bring a calculator

Undriven LC Circuit – Differential Equation Approach



$i_c = -i_L$

KVL $\Rightarrow v_c = v_L$

$v_L = L\dfrac{di_L}{dt}$

$\Rightarrow \quad v_c = L\dfrac{di_L}{dt}$

$i_c = C\dfrac{dv_c}{dt}$

$\Downarrow$

$\dfrac{dv_c}{dt} = \dfrac{i_c}{C} = \dfrac{-i_L}{C}$

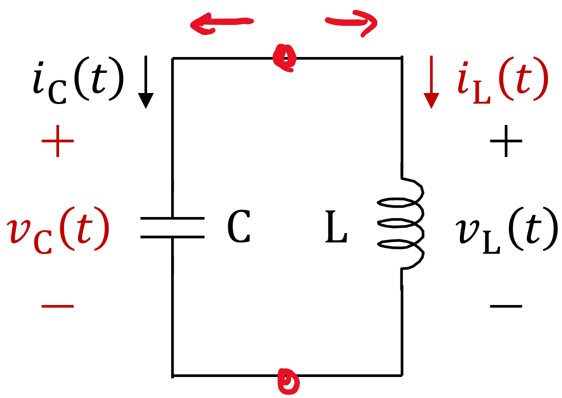
$\Rightarrow \quad \dfrac{dv_c}{dt} = L\dfrac{d^2 i_L}{dt^2}$

$-\dfrac{i_L}{C} = L\dfrac{d^2 i_L}{dt^2}$

$\Rightarrow \boxed{LC\dfrac{d^2 i_L}{dt^2} + i_L = 0}$

Diff. Eq. in terms of i_L

Undriven LC Circuit – Alternate Differential Equation



$$i_C = -i_L$$

$$i_C = C\frac{dv_C}{dt} \Rightarrow C\frac{dv_C}{dt} = -i_L$$

$$C\frac{d^2v_C}{dt^2} = -\frac{di_L}{dt}$$

$$\left.\begin{aligned} v_L &= L\frac{di_L}{dt} \\ v_L &= v_C \end{aligned}\right\} \Rightarrow \frac{di_L}{dt} = \frac{v_C}{L} \Rightarrow C\frac{d^2v_C}{dt^2} = -\frac{v_C}{L}$$

$$\boxed{LC\frac{d^2v_C}{dt^2} + v_C = 0}$$

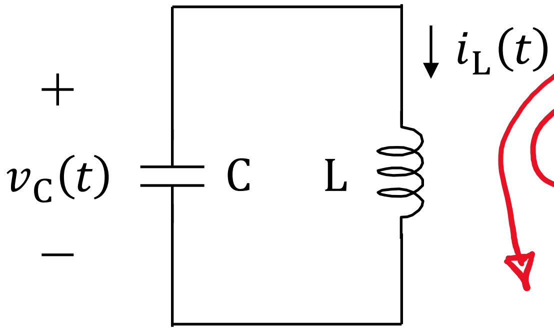
Differential equation in terms of v_C

$$\boxed{LC\frac{d^2i_L}{dt^2} + i_L = 0}$$

Differential equation in terms of i_L

Can Use either

LC Circuit – Natural Response



$$LC\frac{d^2v_C}{dt^2} + v_C = 0$$ ← Homogeneous Eq

Particular Soln = 0

↑

Driven Response (No drive)

$v_{c,h} = Ae^{st}$ ← Homogeneous Solution

$$LC\cancel{A}s^2\cancel{e^{st}} + \cancel{A}\cancel{e^{st}} = 0$$

$$\Rightarrow \quad s^2 = -\frac{1}{LC} \quad \Rightarrow \quad s = \frac{\pm\sqrt{-1}}{\sqrt{LC}}$$

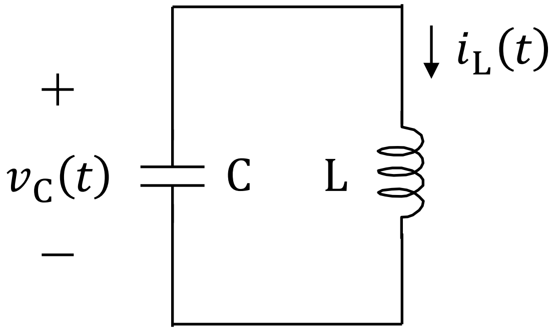
$$\boxed{j \equiv \sqrt{-1}} \quad \Rightarrow \quad s = \frac{\pm j}{\sqrt{LC}} \quad \Rightarrow \quad s = +j\omega_0 \text{ or } s = -j\omega_0$$

Define $\boxed{\frac{1}{\sqrt{LC}} \equiv \omega_0}$

$$v_{c,h}(t) = A_1 e^{+j\omega_0 t} + A_2 e^{-j\omega_0 t}$$

Undamped Natural Radial Frequency, ω_0

LC Circuit – Natural Response – Capacitor Voltage



$$v_C(t) = v_{C,h}(t) + \underbrace{v_{C,p}(t)}_{=0} = A_1 e^{j\omega_0 t} + A_2 e^{-j\omega_0 t}$$

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$v_c(t) = A_1\cos(\omega_0 t) + jA_1\sin(\omega_0 t) + A_2\cos(-\omega_0 t) + jA_2\sin(-\omega_0 t)$$

$$A_2\cos(\omega_0 t)$$

$$-jA_2\sin(\omega_0 t)$$

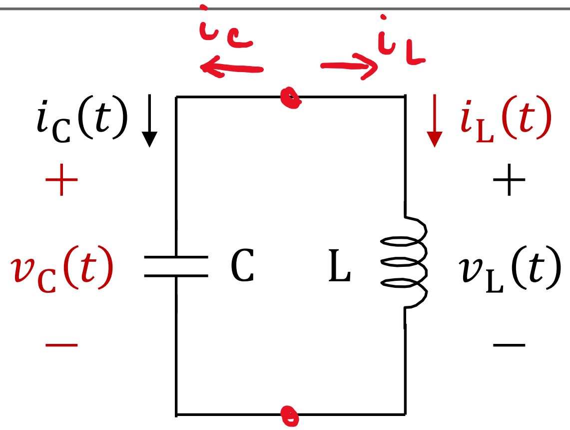
$$v_c(t) = \underbrace{(A_1 + A_2)}_{\equiv B_1}\cos(\omega_0 t) + \underbrace{j(A_1 - A_2)}_{\equiv B_2}\sin(\omega_0 t)$$

B_1 & B_2 will be real

$$v_c(t) = B_1\cos(\omega_0 t) + B_2\sin(\omega_0 t) = B\cos(\omega_0 t + \phi)$$

Undamped Natural Radial Freq.

LC Circuit – Natural Response – Inductor Current



$$v_C(t) = B_1 \cos(\omega_0 t) + B_2 \sin(\omega_0 t) = B \cos(\omega_0 t + \phi)$$

$$i_L(t) = -i_c = -C \frac{dv_c}{dt}$$

$$i_L(t) = +C B \omega_0 \sin(\omega_0 t + \phi)$$

$$\Downarrow$$

$$\boxed{i_L(t) = \frac{B}{Z_0} \sin(\omega_0 t + \phi)}$$

Initial Conditions

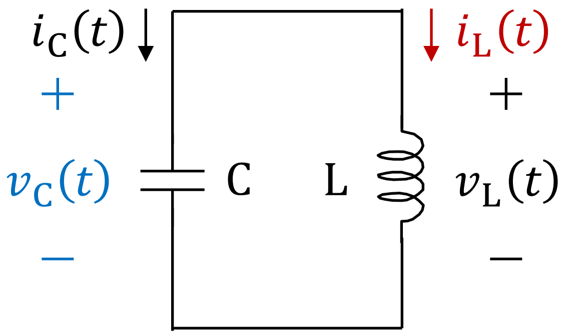
$v_c(0) = V_0$ & $i_L(0) = I_0$

$$C\omega_0 = C \frac{1}{\sqrt{LC}} = \sqrt{\frac{C}{L}} = \frac{1}{Z_0}$$

Define $\boxed{Z_0 \equiv \sqrt{\frac{L}{C}}}$

Characteristic Impedance, Z_0

LC Circuit – Natural Response Summary



cos sin

$$v_L = L\frac{di_L}{dt}$$

cos sin

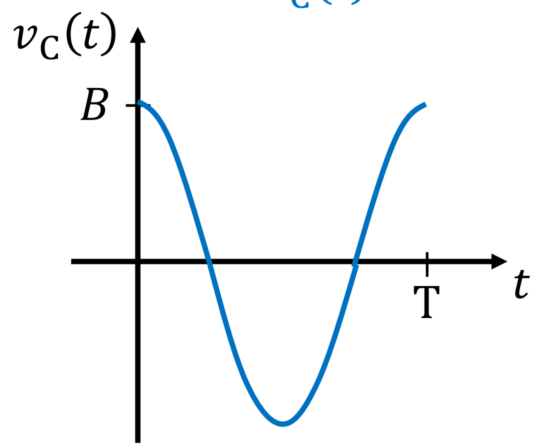
$$i_C = C\frac{dv_C}{dt}$$

voltage current current voltage

ELI the ICE man

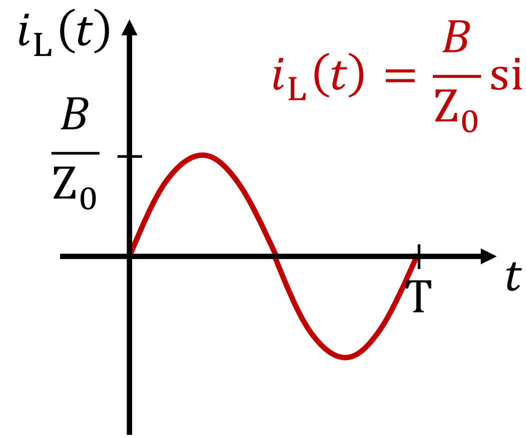
Inductor Capacitor

$$v_C(t) = B\cos(\omega_0 t + \phi)$$

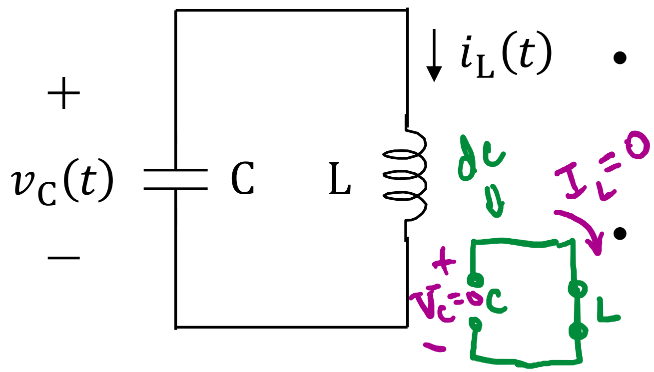


$$T = \frac{1}{f_0} = \frac{2\pi}{\omega_0}$$

$$i_L(t) = \frac{B}{Z_0}\sin(\omega_0 t + \phi)$$



Undriven LC Circuit – Intutive Approach



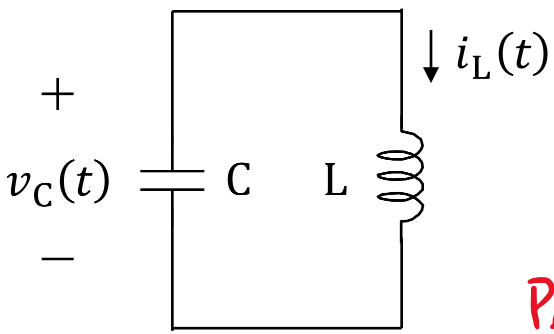
- Since there is no energy dissipating element in the circuit, $v_C(t)$ and $i_L(t)$ will oscillate without damping
- For total energy to remain constant, $v_C(t)$ and $i_L(t)$ oscillations must remain $90°$ out of phase

- Voltage of an inductor always leads its current, v_L $(= v_C)$ must lead i_L, i.e., if v_C oscillation is $\cos$ then i_L oscillation is $\sin$ (as $\cos$ leads $\sin$ by $90°$)
- To determine dc value $v_C(t)$ and $i_L(t)$ will oscillate around: V_C, I_L, assume circuit has a little damping, then ask what values of $v_C(t)$ and $i_L(t)$ would the circuit settle down to – $v_C(t)$ and $i_L(t)$ will oscillate around these values.
- Frequency of oscillation, $\omega_0 = 1/\sqrt{LC}$
- Relationship between amplitudes of oscillations of v_C and i_L is given by $\Delta V_C = Z_0 \Delta I_L$ where $Z_0 = \sqrt{L/C}$

$$v_c(t) = B\cos(\omega_0 t + \phi) + \overset{V_c}{\overset{\shortparallel}{0}}$$

$$i_L(t) = \frac{B}{Z_0}\sin(\omega_0 t + \phi) + \underset{I_L}{\underset{\shortparallel}{0}}$$

Relationship Between Amplitudes of Oscillations



Capacitor voltage amplitude of oscillation $\equiv \Delta V_C$

Inductor current amplitude of oscillation $\equiv \Delta I_L$

Peak Energy in Capacitor $= \frac{1}{2} C \Delta V_c^2$

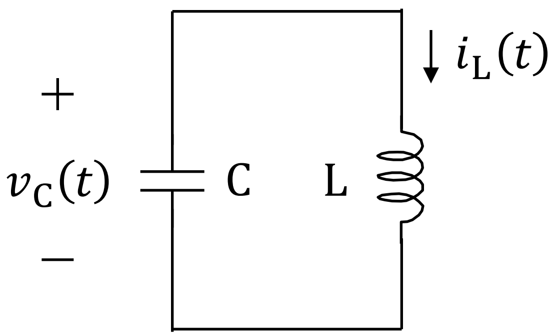
Peak Energy in Inductor $= \frac{1}{2} L \Delta I_L^2$

Energy Conservation: $\cancel{\frac{1}{2}} C \Delta V_c^2 = \cancel{\frac{1}{2}} L \Delta I_L^2$

$$\Rightarrow \quad \frac{\Delta V_c}{\Delta I_L} = \sqrt{\frac{L}{C}} = Z_0$$

$$\Delta V_C = Z_0 \Delta I_L$$

Undriven LC Circuit – Applying Initial Conditions



$$v_C(t) = B\cos(\omega_0 t + \phi) \Rightarrow v_C(0^+) = B\cos\phi$$

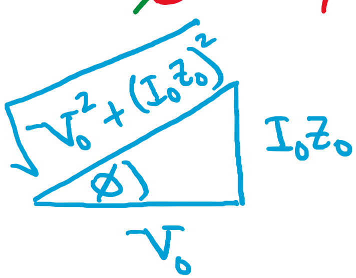
$$i_L(t) = \frac{B}{Z_0}\sin(\omega_0 t + \phi) \Rightarrow i_L(0^+) = \frac{B}{Z_0}\sin\phi$$

$v_C(0^+) = V_0 \qquad i_L(0^+) = I_0$

$$\Rightarrow \quad B\cos\phi = V_0$$

$$\frac{B}{Z_0}\sin\phi = I_0$$

$$\Rightarrow \frac{\frac{\cancel{B}}{Z_0}\sin\phi}{\cancel{B}\cos\phi} = \frac{I_0}{V_0} \Rightarrow \tan\phi = \frac{I_0 Z_0}{V_0} \Rightarrow \boxed{\phi = \tan^{-1}\left(\frac{I_0 Z_0}{V_0}\right)}$$



$$\Rightarrow \cos\phi = \frac{V_0}{\sqrt{V_0^2 + (I_0 Z_0)^2}}$$

$$\therefore B = \frac{V_0}{\cos\phi} = \cancel{V_0} \cdot \frac{\sqrt{V_0^2 + (I_0 Z_0)^2}}{\cancel{V_0}} \Rightarrow \boxed{B = \sqrt{V_0^2 + (I_0 Z_0)^2}}$$