ECE/ENGRD 2100

Introduction to Circuits for ECE

Lecture 17

First Order RC and RL Circuits – Solution using Differential Equation Approach and Intuitive Approach

Announcements

- Recommended Reading:
  - Textbook Chapter 7

- Upcoming due dates:
  - Prelab 3 due by 12:20 pm on Tuesday March 5, 2019
  - Homework 3 due by 11:59 pm on Monday March 11, 2019
  - Lab report 3 due by 11:59 pm on Friday March 15, 2019

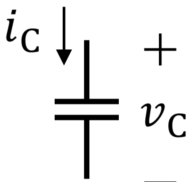
- Lab 3 is this week (starting Tuesday March 5, 2019)

Capacitors and Inductors Summary

We will only use linear C & L

Capacitor

$$\boxed{C \equiv \frac{dq_C}{dv_C}}$$



$$\underline{i_C = C\frac{dv_C}{dt}}$$

$$\underline{v_C(t_1) = v_C(t_0) + \frac{1}{C}\int_{t_0}^{t_1} i_C(t)\,dt}$$

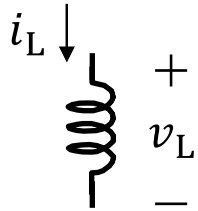
$$v_C(t_1^+) = v_C(t_1^-) \quad \text{if} \;\; i_C(t_1) < \infty$$

Energy Stored

$$\boxed{w_C = \frac{1}{2}Cv_C^2}$$

Inductor

$$\boxed{L \equiv \frac{d\lambda_L}{di_L}}$$



$$\underline{v_L = L\frac{di_L}{dt}}$$

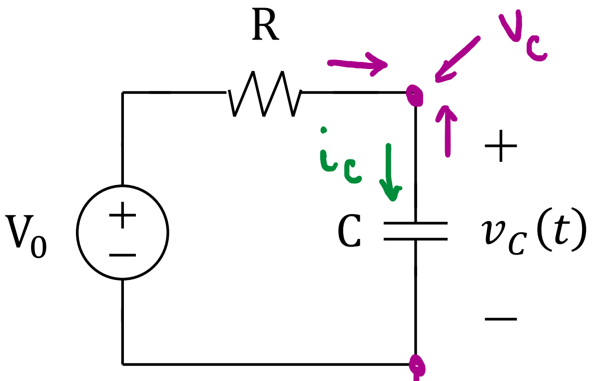
$$\underline{i_L(t_1) = i_L(t_0) + \frac{1}{L}\int_{t_0}^{t_1} v_L(t)\,dt}$$

$$i_L(t_1^+) = i_L(t_1^-) \quad \text{if} \;\; v_L(t_1) < \infty$$

Energy Stored

$$\boxed{w_L = \frac{1}{2}Li_L^2}$$

First Order Circuit using a Capacitor



$$\boxed{v_C(0) = 0}$$

Initial Condition

$$i_c = C\frac{dv_c}{dt}$$

$$\frac{V_0 - v_c}{R} - i_c = 0$$

$$i_c + \frac{v_c}{R} = \frac{V_0}{R}$$

$$\Rightarrow \boxed{C\frac{dv_c}{dt} + \frac{v_c}{R} = \frac{V_0}{R}}$$

Differential Equation

First Order Linear Constant Coefficient Differential Eqn.

Solving Linear Constant Coefficient Differential Eqn

Differential Equation (Non-Homogeneous):

v_c or i_L

$$A_n \frac{d^n y}{dt^n} + A_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + A_1 \frac{dy}{dt} + A_0 y = f(t)$$

Solution:

Homogeneous Soln

Particular

$$y(t) = \underline{\underline{y_h}}(t) + y_p(t)$$

Solution to D.E.
with $f(t)=0$

Homogeneous Solution

Homogeneous Differential Equation: $\underline{\underline{f(t) = 0}}$

$$A_n \frac{d^n y}{dt^n} + A_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + A_1 \frac{dy}{dt} + A_0 y = 0$$

Solution to Homogeneous Differential Equation is of the Form:

unknown constant

$$y_h(t) = Ke^{st}$$

unknown constant

Particular Solution

Non-Homogeneous Differential Equation:

$$A_n \frac{d^n y}{dt^n} + A_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + A_1 \frac{dy}{dt} + A_0 y = f(t)$$

Particular Solution to Non-Homogeneous Differential Equation:

Any solution to this
is a particular solution

How to find it: Guess

Particular Solution

Non-Homogeneous Differential Equation:

$$A_n \frac{d^n y}{dt^n} + A_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + A_1 \frac{dy}{dt} + A_0 y = f(t)$$

Particular Solution to Non-Homogeneous Differential Equation:

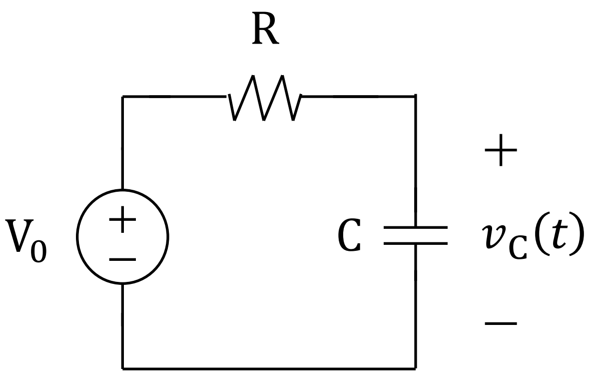
Has the same form as $f(t)$

Examples

If $f(t) = \mathrm{V}_0 =$ Constant ⇨ Guess $y_p(t) = \mathrm{B} =$ Constant

If $f(t) = \mathrm{V}_1 t =$ Ramp ⇨ Guess $y_p(t) = \mathrm{B}_0 + \mathrm{B}_1 t$

RC Circuit Example – Homogeneous Solution



$v_C(0) = 0$

$$C\frac{dv_C}{dt} + \frac{v_C}{R} = \frac{V_0}{R}$$

Homogeneous D.E. : $C\frac{dv_{c,h}}{dt} + \frac{v_{c,h}}{R} = 0$ — ①

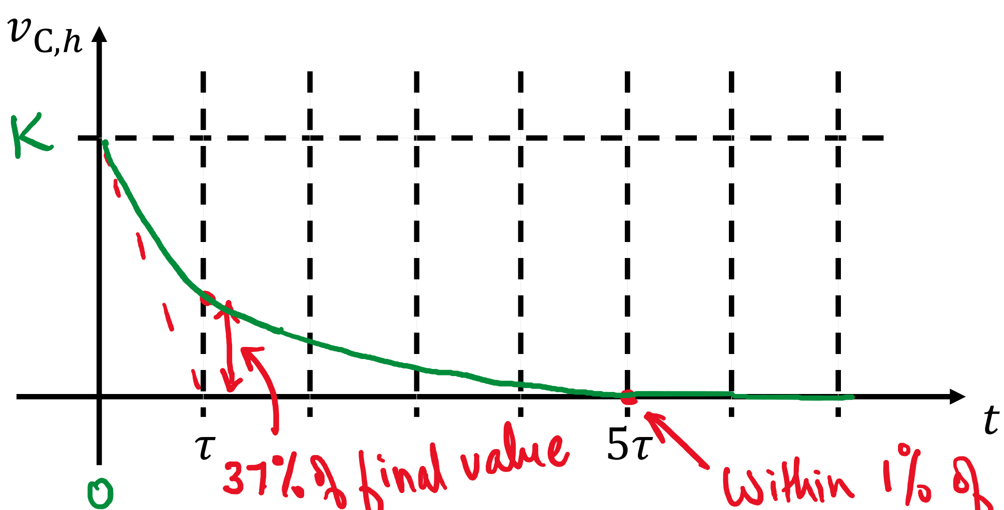
$v_{c,h} = Ke^{st}$

Substitute into ① $\quad C\cancel{K}s\cancel{e^{st}} + \frac{\cancel{Ke^{st}}}{R} = 0 \Rightarrow \boxed{s = -\frac{1}{RC}}$

$v_{c,h}(t) = Ke^{-t/RC}$

Time Constant, $\tau = RC$

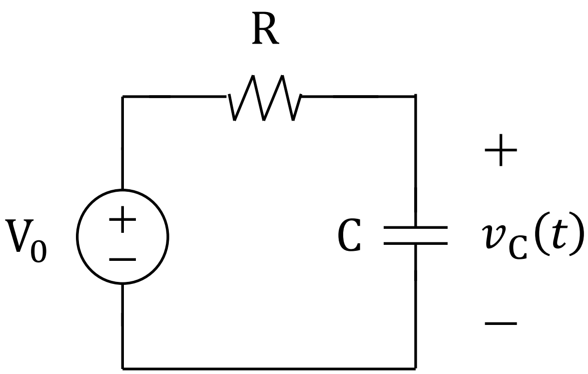
$v_{c,h}(t) = \underline{Ke^{-t/\tau}}$ ← Decaying Exponential



37% of final value

Within 1% of final value by 5τ

RC Circuit Example – Particular Solution



$v_C(0) = 0$

$$C\frac{dv_C}{dt} + \frac{v_C}{R} = \overset{f(t)}{\frac{V_0}{R}} \quad — \; (2)$$

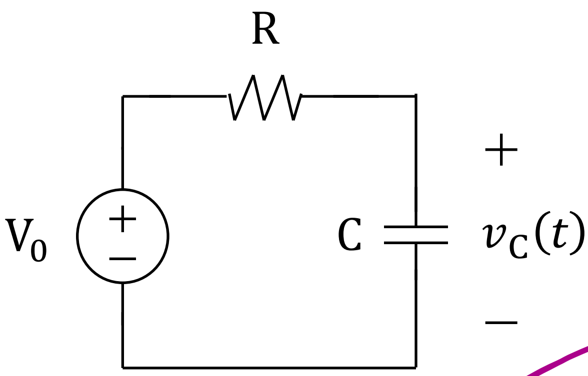
Guess, $v_{c,p} = B$

Substitute into (2) : $0 + \dfrac{B}{R} = \dfrac{V_0}{R}$

$\Rightarrow \boxed{B = V_0}$

$$\boxed{v_{c,p} = V_0}$$

RC Circuit Example – Total Solution



$$v_C(t) = \underbrace{Ke^{-t/RC}}_{v_{c,h}} + \underbrace{V_0}_{v_{c,p}}$$

$$v_C(0) = 0$$

Initial Condition

$$0 = K\underset{1}{\underset{\shortparallel}{e^{0}}} + V_0 = \boxed{K = -V_0}$$

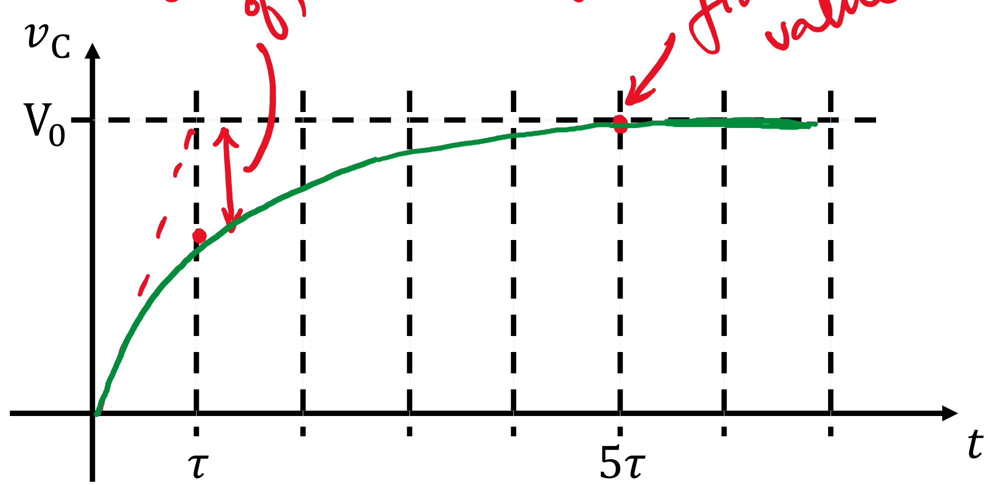
$$\tau = RC$$

$$v_c(t) = -V_0 e^{-t/\tau} + V_0$$

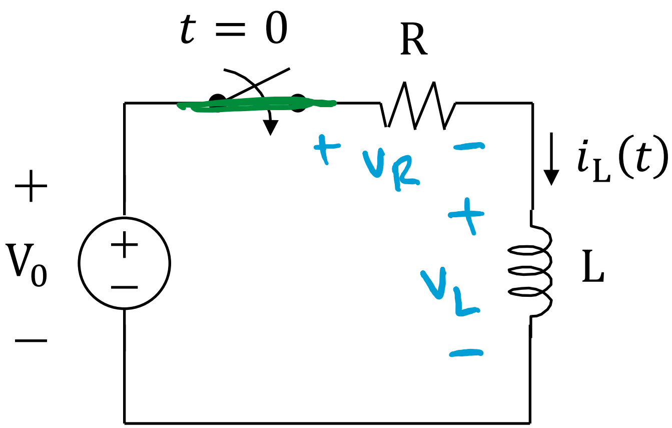
$$v_c(t) = V_0\underbrace{\left(1 - e^{-t/\tau}\right)}_{\text{rising exponential}}$$

within 37% of final value

within 1% of final value



RL Circuit Example – Differential Equation



Find $i_L(t)$ for $t \geq 0$

$v_L = L\frac{di_L}{dt}$

$v_R = Ri_L$

D.E. for $t \geq 0$

$v_L + v_R = V_0$

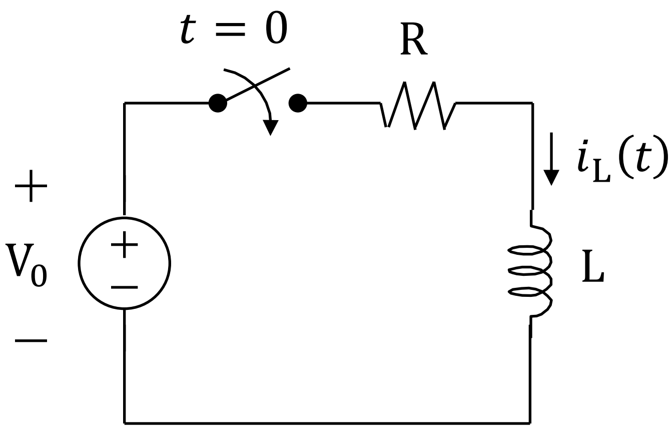
$\Downarrow$

$$\boxed{L\frac{di_L}{dt} + Ri_L = V_0}$$

at $t = 0^- \Rightarrow i_L(0^-) = 0$

$\underline{\underline{i_L(0^+)}} = i_L(0^-) = \underline{\underline{0}}$

RL Circuit Example – Homogenous Solution



$$L\frac{di_L}{dt} + Ri_L = V_0$$

Diff. Homogeneous Eqn

$$L\frac{di_L}{dt} + Ri_L = 0$$

$$i_{L,h} = Ke^{st}$$

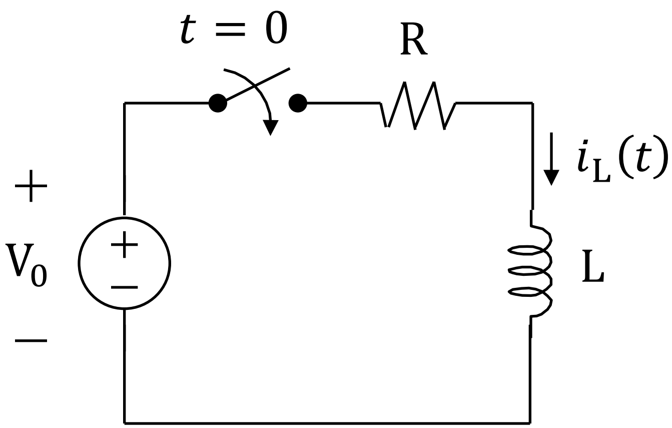
$$L\cancel{K}s\cancel{e^{st}} + R\cancel{K}\cancel{e^{st}} = 0$$

$$s = -\frac{R}{L}$$

$$i_{L,h}(t) = Ke^{-Rt/L} = \underline{\underline{Ke^{-t/\tau}}}$$

$$\boxed{\tau = \frac{L}{R}}$$

RL Circuit Example – Particular Solution



$$L\frac{di_L}{dt} + Ri_L = V_0$$

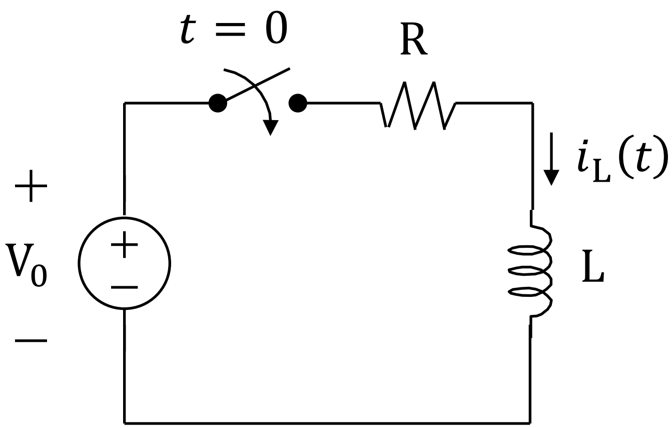
$i_{L,p} = B$ ← Guess

$0 + RB = V_0$

$\Rightarrow \quad B = \frac{V_0}{R}$

$$\boxed{i_{L,p} = \frac{V_0}{R}}$$

RL Circuit Example – Total Solution



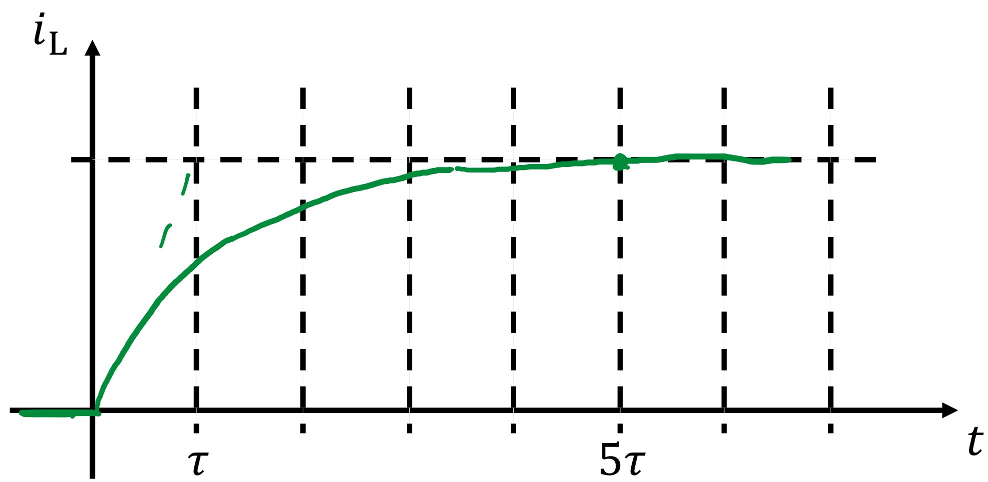
$i_L(0^+) = 0$

$0 = Ke^0 + \frac{V_0}{R}$

$\Rightarrow K = -\frac{V_0}{R}$

$$i_L(t) = \underbrace{Ke^{-t/\tau}}_{i_{L,h}} + \underbrace{\frac{V_0}{R}}_{i_{L,p}} \qquad \text{where} \qquad \tau = \frac{L}{R}$$

$$i_L(t) = \frac{V_0}{R}\left(1 - e^{-t/\tau}\right)$$



Intuitive Approach

Total Solution: $i(t) = i_h(t) + i_p(t)$

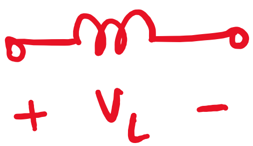
Mathematical Interpretation		Physical Interpretation
Homogeneous Solution	$\xleftrightarrow{i_h(t)}$	Natural Response
Particular Solution	$\xleftrightarrow{i_p(t)}$	Driven Response

Intuitive Solution for dc Steady State

Inductor $v_L = \mathrm{L}\dfrac{di_L}{dt}$

$i_L = \text{constant} \Rightarrow v_L = 0$

dc (constant)



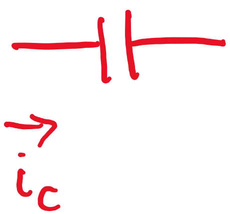
$\Longrightarrow$

short circuit

Capacitor $i_C = \mathrm{C}\dfrac{dv_C}{dt}$

$v_C = \text{constant} \Rightarrow i_C = 0$

dc (constant)



$\Longrightarrow$

open ckt