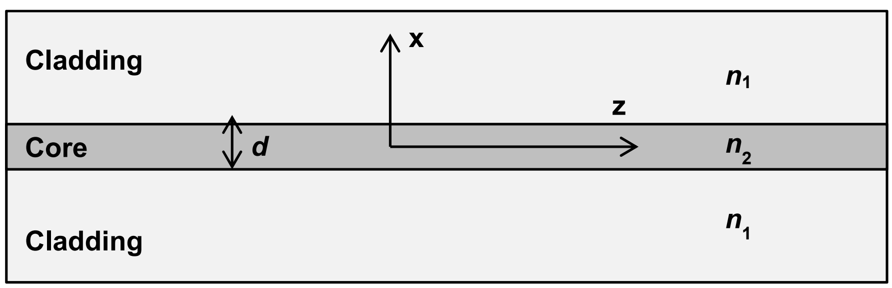
Chapter 8

Integrated Optical Waveguides

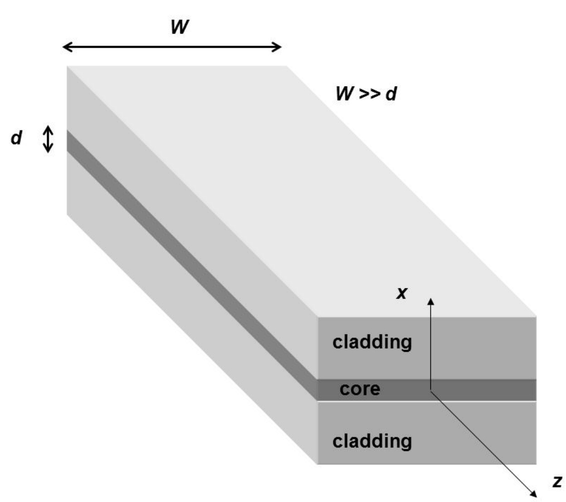
7.1 Dielectric Slab Waveguides

7.1.1 Introduction:

A variety of different integrated optical waveguides are used to confine and guide light on a chip. The most basic optical waveguide is a slab waveguides shown below. The structure is uniform in the y-direction. Light is guided inside the core region by total internal reflection at the core-cladding interfaces.



Most actual waveguides are not uniform and infinite in the y-direction but can be approximated as slab waveguides if their width W is much larger than the core thickness d, as shown below.



A better description of the guided light is in terms of the optical modes. The slab waveguide supports two different kinds of propagating modes:

i. TE (transverse electric) mode: In this mode, the electric field has no component in the direction of propagation.

ii. TM (transverse magnetic) modes: In this mode, the magnetic field has no component in the direction of propagation.

To study these modes we start from Maxwell's equations. The complex form of Maxwell's equations is,

$$\nabla \times \vec{E} = i\omega\mu_o \vec{H}$$

$$\nabla \times \vec{H} = -i\omega\varepsilon_o n^2(x)\vec{E}$$

$$\Rightarrow \nabla \times \nabla \times \vec{E} = \frac{\omega^2}{c^2} n^2(x)\vec{E}$$

Since $\nabla \times \nabla \times \vec{E} = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E}$. In general $\nabla \cdot \vec{E} \neq 0$. Rather $\nabla \cdot [n^2(x)\vec{E}] = 0$. But if index is piecewise uniform in different regions then inside each region one may assume $\nabla \cdot \vec{E} = 0$. So we have in each region,

$$-\nabla^2 \vec{E} = \frac{\omega^2}{c^2} n^2(x)\vec{E}$$

Similarly, with the assumption of piecewise uniform index we can write for the $\vec{H}$ field,

$$-\nabla^2 \vec{H} = \frac{\omega^2}{c^2} n^2(x)\vec{H}$$

7.1.2 TE Modes:

For TE modes, the electric field can be written as,

$$\vec{E}(x,z) = \hat{y} E_o \phi(x) e^{i\beta z}$$

In each region of piecewise uniform index (core and cladding), $\phi(x)$ satisfies,

$$\left[-\frac{\partial^2}{\partial x^2} - \frac{\omega^2}{c^2} n^2(x) \right] \phi(x) = -\beta^2 \phi(x)$$

Given a value for the frequency ω, we can find all solutions of the above equation which is an eigenvalue equation with eigenfunction $\phi(x)$ and eigenvalue $-\beta^2$. Once we have the electric field, the magnetic field $\vec{H}$ can be found as follows,

$$\vec{H}(x,z) = \frac{\nabla \times \vec{E}}{i\omega\mu_o} = \frac{\left(\hat{x}\frac{\partial}{\partial x} + i\beta\hat{z} \right) \times \vec{E}(x,z)}{i\omega\mu_0}$$

$$= \left\{ \frac{\hat{z}E_o}{i\omega\mu_o} \frac{\partial \phi(x)}{\partial x} - \hat{x}\frac{E_o \beta}{\omega\mu_o} \phi(x) \right\} e^{i\beta z}$$

The boundary conditions needed to solve the eigenvalue equation above are as follows:

i) y-component of the electric field is continuous at the core-cladding interfaces
ii) z-component of the magnetic field is continuous at the core-cladding interfaces
iii) x-component of the magnetic field is continuous at the core-cladding interfaces (this is automatically satisfied when (i) above is satisfied)

The solutions are labeled with the integer index m ($m = 0,1,2,3,\ldots$). So the field for the TE$_m$ mode is,

$$\vec{E}(x,z) = \hat{y} E_o \phi_m(x,\omega) e^{i\beta_m(\omega) z}$$

where the dependence of the eigenfunctions and the propagation vector β on the frequency ω is explicitly indicated. For the TE modes we assume the solution,

$$\phi(x) = \begin{cases} C_1 e^{-\gamma(x-d/2)} & x \geq d/2 \\ C_2 \begin{bmatrix} \cos(kx) \\ \sin(kx) \end{bmatrix} & |x| \leq d/2 \\ C_1 e^{\gamma(x+d/2)} & x \leq -d/2 \end{cases}$$

Plugging the above solutions in the equation,

$$\left[-\frac{\partial^2}{\partial x^2}-\frac{\omega^2}{c^2}n^2(x)\right]\phi(x)=-\beta^2\phi(x)$$

we get,

$$\left.\begin{aligned}\beta^2+k^2&=\frac{\omega^2}{c^2}n_2^2\\ \beta^2-\gamma^2&=\frac{\omega^2}{c^2}n_1^2\end{aligned}\right\}\Rightarrow\gamma=\sqrt{\frac{\omega^2}{c^2}(n_2^2-n_1^2)-k^2}$$

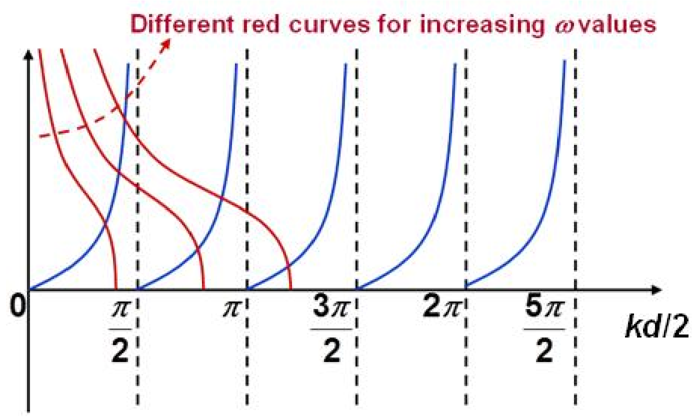
Using the boundary conditions give,

$$\tan\left(\frac{kd}{2}\right)=\frac{\gamma}{k}\quad\text{(cosine solutions – even TE modes)}$$

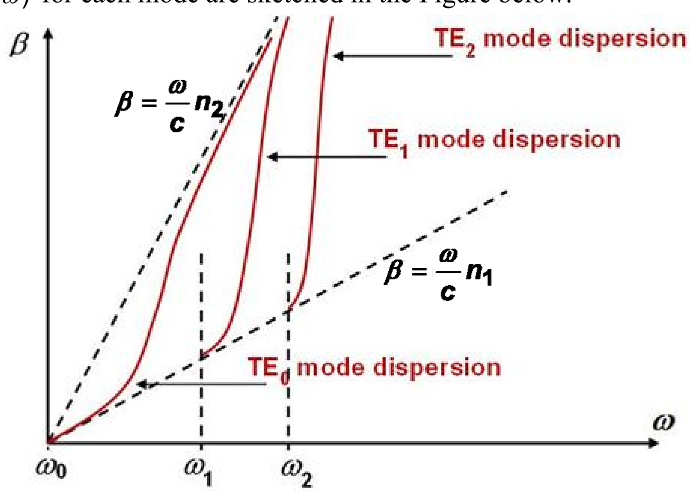
$$-\cot\left(\frac{kd}{2}\right)=\frac{\gamma}{k}\quad\text{(sine solutions – odd TE modes)}$$

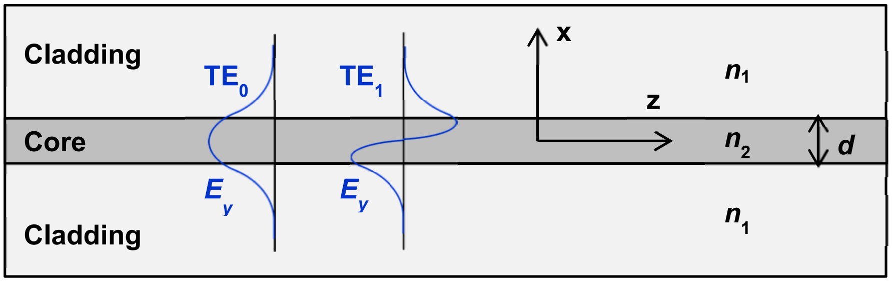
A solution can be obtained graphically by plotting the left and right sides of the above equations, as shown below where the left side is plotted using blue lines and right side using red lines. Below a certain frequency ω_m the m-the mode ceases to exist. This cut-off frequency is,

$$\omega_m=m\frac{\pi}{d}\frac{c}{\sqrt{n_2^2-n_1^2}}$$



The values of $\beta_m(\omega)$ for each mode are sketched in the Figure below.





The propagation vectors $\beta_m(\omega)$ behave as $\frac{\omega}{c}n_1$ near then cut-off frequencies, and asymptotically approach the $\frac{\omega}{c}n_2$ curve at high frequencies. In most applications, waveguide dimensions are chosen such that it supports only one mode. Unless necessary we will usually drop the mode index "m".

7.1.3 TM Modes:

Similar analysis can be done for the TM modes. Assume,

$$\vec{H}(x,z) = \hat{y} H_o \phi(x) e^{i\beta z}$$

$$\vec{E}(x,z) = \frac{\nabla \times \vec{H}}{-i\omega\varepsilon_o n^2(x)} = \left\{ \hat{z} \frac{H_o}{-i\omega\varepsilon_o n^2(x)} \frac{\partial \phi}{\partial x} + \hat{x} \frac{H_o \beta}{\omega\varepsilon_o n^2(x)} \phi(x) \right\} e^{i\beta z}.$$

For piecewise uniform index $n(x)$, $\phi(x)$, satisfies,

$$\left(-\frac{\partial_2}{\partial x^2} - \frac{\omega^2}{c^2} n^2(x) \right) \phi(x) = -\beta^2 \phi(x)$$

Assume solution of the form,

$$\phi(x) = \begin{cases} C_1 e^{-\gamma(x-d/2)} & x \ge d/2 \\ C_2 \begin{bmatrix} \cos(kx) \\ \sin(kx) \end{bmatrix} & |x| \le d/2 \\ C_1 e^{\gamma(x+d/2)} & x \le -d/2 \end{cases} \qquad \begin{cases} \beta^2 + k^2 = \frac{\omega^2}{c^2} n_2^2 \\ \beta^2 - \gamma^2 = \frac{\omega^2}{c^2} n_1^2 \end{cases}$$

The boundary conditions needed to solve the eigenvalue equation above are as follows:

i) y-component of the magnetic field is continuous at the core-cladding interfaces
ii) z-component of the electric field is continuous at the core-cladding interfaces
iii) x-component of the electric field weighted by the square of the index is continuous at the core-cladding interfaces (this is automatically satisfied when (i) above is satisfied)

Using the boundary conditions give,

$$\tan\left(\frac{kd}{2}\right) = \frac{n_2^2}{n_1^2} \frac{\gamma}{k} \quad \text{(cosine solutions – even TM modes)}$$

$$-\cot\left(\frac{kd}{2}\right) = \frac{n_2^2}{n_1^2} \frac{\gamma}{k} \quad \text{(sine solutions – odd TM modes)}$$

The above relations are similar to those obtained for the TE modes other than the factors containing the squares of the core and cladding refractive indices. The general behavior of the TM modes is similar to the TE modes.

7.1.4 Effective Index and Group Index:
The **effective index** $n_{eff}(\omega)$ of a mode is defined by the relation,

$$\beta(\omega) = \frac{\omega}{c} n_{eff}(\omega)$$

The effective index of each mode equals n_1 at their cut-off frequencies and approaches n_2 at high frequencies. The group velocity of a mode is,

$$\frac{1}{v_g(\omega)} = \frac{\partial \beta}{\partial \omega}$$

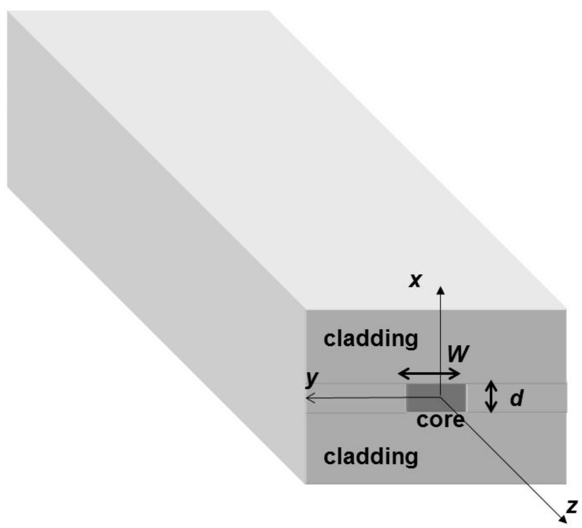
v_g is written as,

$$v_g(\omega) = \frac{c}{n_g(\omega)}$$

where $n_g(\omega)$ is called the **group index** of the mode.

7.2 Two Dimensional Waveguides

7.2.1 Introduction:
In slab waveguides, light is confined in only one dimension (the x-dimension in our notation) as it travels in the z-direction. In most actual waveguides light is confined in two dimensions (x and y-dimensions) and travels in the z-direction. For example, the cross-section of a rectangular waveguide is shown below.



Unfortunately, the exact modes of such 2D waveguides are not as easy to complete as those of slab waveguide. The modes are neither TE nor TM. All modes have a small component of $\vec{E}$ and $\vec{H}$ fields in the direction of propagation. The $\vec{E}$ and the $\vec{H}$ fields can be written as,

$$\vec{E}(x,y,z) = \left\{\hat{x}E_x(x,y) + \hat{y}E_y(x,y) + \hat{z}E_z(x,y)\right\} e^{i\beta z} = \left\{ \vec{E}_t(x,y) + \hat{z}E_z(x,y) \right\} e^{i\beta z}$$

$$\vec{H}(x,y,z) = \left\{\hat{x}H_x(x,y) + \hat{y}H_y(x,y) + \hat{z}H_z(x,y)\right\} e^{i\beta z} = \left\{ \vec{H}_t(x,y) + \hat{z}H_z(x,y) \right\} e^{i\beta z}$$

We have defined the transverse components of the fields as follows,

$$\vec{E}_t(x,y) = \hat{x}E_x(x,y) + \hat{y}E_y(x,y)$$

$$\vec{H}_t(x,y) = \hat{x}H_x(x,y) + \hat{y}H_y(x,y)$$

Since the exact solution is difficult and cumbersome, several levels of approximations are commonly used and are discussed below.

7.2.2 Mode Solutions for 2D Waveguides:
We need to solve the equation,

$$\nabla \times \nabla \times \vec{E}(x,y,z) = \frac{\omega^2}{c^2} n^2(x,y)\vec{E}(x,y) \qquad \left\{ \vec{H} = \frac{\nabla \times \vec{E}}{i\omega\mu_o} \right\}$$

subject to all the proper boundary conditions for the $\vec{E}$ and $\vec{H}$ fields at all the interfaces.
The operator ∇ is,

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

Define,

$$\nabla_t = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} \Rightarrow \nabla = \nabla_t + \hat{z}\frac{\partial}{\partial z}$$

From $\nabla \times \vec{E} = i\omega\mu_o\vec{H}$, it follows that,

$$(\nabla \times \vec{E}).\hat{z} = i\omega\mu_o\vec{H}.\hat{z} = i\omega\mu_o H_z e^{i\beta z}$$

$$\Rightarrow (\nabla_t \times \vec{E}_t).\hat{z} = i\omega\mu_o H_z$$

$$\Rightarrow H_z = \frac{(\nabla_t \times \vec{E}_t).\hat{z}}{i\omega\mu_o} \qquad (1)$$

Similarly it can be shown that,

$$E_z = \frac{(\nabla_t \times \vec{H}_t).\hat{z}}{-i\omega\varepsilon_o n^2} \qquad (2)$$

Equations (1) and (2) show that knowing the transverse components of the field is enough since the z-components can be determined from the transverse components. We need to solve,

$$\nabla \times \nabla \times \vec{E} = \frac{\omega^2}{c^2} n^2(x,y)\vec{E}$$

$$\Rightarrow \nabla\left(\nabla.\vec{E}\right) - \nabla^2\vec{E} = \frac{\omega^2}{c^2} n^2(x,y)\vec{E}$$

Taking the transverse component gives,

$$\nabla_t(\nabla.\vec{E}) - \nabla^2\vec{E}_t e^{i\beta z} - \frac{\omega^2}{c^2} n^2\vec{E}_t e^{i\beta z} = 0 \qquad (3)$$

We need to find convenient and useful expressions for the first two terms on the right hand side. Now,

$$\nabla.\vec{E} = \nabla_t.\vec{E}_t e^{i\beta z} + i\beta E_z e^{i\beta z}$$

but,

$$\nabla.(n^2\vec{E}) = 0$$

$$\Rightarrow \nabla_t.(n^2\vec{E}_t) + i\beta n^2 E_z = 0$$

$$\Rightarrow i\beta E_z = -\frac{\nabla_t.(n^2\vec{E}_t)}{n^2}$$

$$\Rightarrow \nabla.\vec{E} = \nabla_t.\vec{E}_t e^{i\beta z} + i\beta E_z e^{i\beta z} = \nabla_t.\vec{E}_t e^{i\beta z} - \frac{\nabla_t.(n^2\vec{E}_t)}{n^2} e^{i\beta z}$$

$$\Rightarrow \nabla_t(\nabla.\vec{E}) = \nabla_t\left(\nabla_t.\vec{E}_t\right)e^{i\beta z} - \nabla_t\left[\frac{\nabla_t.(n^2\vec{E}_t)}{n^2}\right]e^{i\beta z}$$

The other term in Equation (3) is,

$$\nabla^2 \vec{E}_t e^{i\beta z} = \nabla_t^2 \vec{E}_t e^{i\beta z} - \beta^2 \vec{E}_t e^{i\beta z}$$

Using the results above, Equation (3) becomes,

$$-\nabla_t^2 \vec{E}_t + \nabla_t \left[\nabla_t . \vec{E}_t - \frac{1}{n^2} \nabla_t . (n^2 \vec{E}_t) \right] - \frac{\omega^2}{c^2} n^2 \vec{E}_t = -\beta^2 \vec{E}_t$$

The above eigenvalue equation is what one needs to solve to get the exact solution. This equation can be put in the form,

$$\begin{bmatrix} \hat{P}_{xx} & \hat{P}_{xy} \\ \hat{P}_{yx} & \hat{P}_{yy} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} = -\beta^2 \begin{bmatrix} E_x \\ E_y \end{bmatrix} \qquad (4)$$

where the differential operators are.

$$\hat{P}_{xx} E_x = -\frac{\partial}{\partial x} \left[\frac{1}{n^2} \frac{\partial (n^2 E_x)}{\partial x} \right] - \frac{\partial^2 E_x}{\partial y^2} - \frac{\omega^2}{c^2} n^2 E_x$$

$$\hat{P}_{xy} E_y = -\frac{\partial}{\partial x} \left[\frac{1}{n^2} \frac{\partial (n^2 E_y)}{\partial y} \right] + \frac{\partial^2 E_y}{\partial x \partial y}$$

$$\hat{P}_{yy} E_y = -\frac{\partial^2 E_y}{\partial x^2} - \frac{\partial}{\partial y} \left[\frac{1}{n^2} \frac{\partial (n^2 E_y)}{\partial y} \right] - \frac{\omega^2}{c^2} n^2 E_y$$

$$\hat{P}_{yx} E_x = -\frac{\partial}{\partial y} \left[\frac{1}{n^2} \frac{\partial (n^2 E_x)}{\partial x} \right] + \frac{\partial^2 E_x}{\partial y \partial x}$$

The above equation is an eigenvalue equation and its solution gives the transverse components of the electric field $\vec{E}$ for the mode and the corresponding propagation constant $\beta(\omega)$. $E_z(x,y)$ can be obtained from $E_x(x,y)$ and $E_y(x,y)$ as already explained, and $\vec{H}$ field can be obtained from the relation,

$$\vec{H} = \frac{(\nabla \times \vec{E})}{i\omega\mu_0}$$

For piecewise uniform indices, all derivatives of the index $n(x,y)$ can be dropped provided appropriate boundary conditions are used at all the interfaces.

7.2.3 The Semi-Vectorial Approximation:

In many cases of practical interest one transverse component of the electric field (either E_x or E_y) dominates over the other component. In such cases, we may assume that the other transverse field component is zero. For example, if we know a priori that E_x dominates then we may assume that E_y is zero and solve the much simpler eigenvalue equation,

$$\hat{P}_{xx} E_x = -\beta^2 E_x$$

$$\Rightarrow -\frac{\partial}{\partial x} \left[\frac{1}{n^2} \frac{\partial (n^2 E_x)}{\partial x} \right] - \frac{\partial^2 E_x}{\partial y^2} - \frac{\omega^2}{c^2} n^2(x,y) E_x) = -\beta^2 E_x.$$

The above equation is called the semi-vectorial approximation. For piecewise uniform dielectrics we can also write it as,

$$-\frac{\partial^2}{\partial x^2} E_x - \frac{\partial^2}{\partial y^2} E_x - \frac{\omega^2}{c^2} n^2(x,y) E_x = -\beta^2 E_x$$

$$\Rightarrow -\nabla_t^2 E_x - \frac{\omega^2}{c^2} n^2(x,y) E_x = -\beta^2 E_x$$

provided we take care to impose the boundary conditions on $E_x(x,y)$ at all the dielectric interfaces as appropriate for the x-component of the electric field. Once the dominant $E_x(x,y)$ component has been found, the remaining field components can be found as follows,

$$\nabla .\left(n^2\vec{E}\right)=0$$

$$\Rightarrow E_z = \frac{i}{\beta}\frac{1}{n^2}\frac{\partial\left(n^2 E_x\right)}{\partial x}$$

$$\vec{H} = \frac{(\nabla\times\vec{E})}{i\omega\mu_o}$$

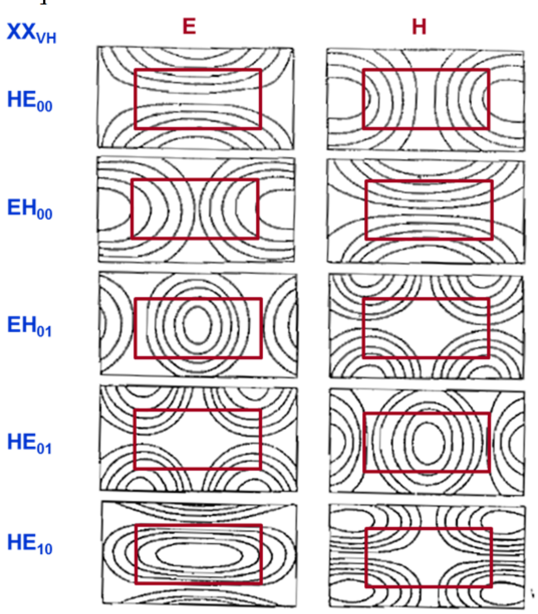
$$\Rightarrow H_y = \frac{\beta}{\omega\mu_o}E_x - \frac{1}{\beta\omega\mu_o}\frac{\partial}{\partial x}\left[\frac{1}{n^2}\frac{\partial\left(n^2 E_x\right)}{\partial x}\right]$$

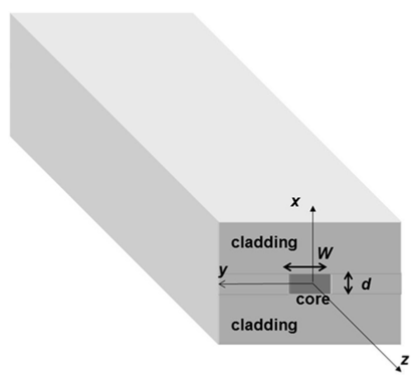
$$\Rightarrow H_x = \frac{1}{\beta\omega\mu_o}\frac{\partial}{\partial y}\left[\frac{1}{n^2}\frac{\partial\left(n^2 E_x\right)}{\partial x}\right]$$

$$\nabla .\vec{H}=0$$

$$\Rightarrow H_z = -\frac{1}{i\omega\mu_o}\frac{\partial E_x}{\partial y}$$

When the horizontal electric field component dominates the modes are sometimes called HE_{pq} modes or TE_{pq} modes (with a slight abuse of terminology). When the vertical electric field component dominates the modes are called EH_{pq} modes or TM_{pq} modes (again with a slight abuse of terminology). The two subscripts p and q indicate the number of nodes the dominant electric field component has in the horizontal and vertical directions, respectively.





7.2.4 The Scalar Field Approximation:

If one component of the transverse electric field dominates over the other transverse component and the index differences among different regions in the structure are also relatively small, then one may use the semi-vectorial approximation and also do away with the boundary conditions on the normal component of the electric field at all dielectric interfaces. For example, if we know a priori that $E_x(x,y)$ dominates then we may solve the equation,

$$-\frac{\partial^2}{\partial x^2}E_x(x,y)-\frac{\partial^2}{\partial y^2}E_x(x,y)-\frac{\omega^2}{c^2}n^2(x,y)E_x(x,y)=-\beta^2 E_x(x,y)$$

$$\Rightarrow -\nabla_t^2 E_x(x,y)-\frac{\omega^2}{c^2}n^2(x,y)E_x(x,y)=-\beta^2 E_x(x,y)$$

assuming that the field and its derivative are continuous across all dielectric interfaces. This is called the scalar field approximation. Once the dominant electric field component has been found, the remaining field components can be found as in the case of the semi-vectorial approximation. Scalar field approximation seems crude but it gives very accurate answers for the propagation vector $\beta(\omega)$ (or the effective index $n_{eff}(\omega)$) as long as one is for away from the mode cut-off frequency. It is also very accurate in calculating mode confinement factors – as we will see in later Chapters. For most part of this course we will use the scalar field approximation to keep the computational overhead low. One disadvantage of the scalar field approximation is that it does not tell accurately whether the single-mode condition holds since the scalar field approximation is not accurate near mode cut-off.

7.2.5 Slab Waveguide Approximation:

If the aspect ratio of the waveguide is such that one dimension is much larger than the other dimension, then the modes and the corresponding wavevectors and effective indices can be approximated by those of the corresponding slab waveguide as discussed earlier.

7.2.6 Energy and Power in Waveguides:

The energy flow for electromagnetic fields is given by the complex Poynting vector,

$$\vec{S}(\vec{r})=\frac{1}{2}\mathrm{Re}[\vec{E}(\vec{r})\times\vec{H}^*(\vec{r})]$$

$$\vec{S}(\vec{r})=\frac{1}{4}[\vec{E}(\vec{r})\times\vec{H}^*(\vec{r})+\vec{E}^*(\vec{r})\times\vec{H}(\vec{r})]$$

For waveguides electric and magnetic fields are of the form,

$$\vec{E}(x,y,z)=\left[\hat{x}E_x(x,y)+\hat{y}E_y(x,y)+\hat{z}E_z(x,y)\right]e^{i\beta z}$$

$$\vec{H}(x,y,z)=\left[\hat{x}H_x(x,y)+\hat{y}H_y(x,y)+\hat{z}H_z(x,y)\right]e^{i\beta z}$$

The total energy flow (or power) in a waveguide is obtained by integrating the Poynting vector over the cross-section of the waveguide,

$$P(z)=\iint\vec{S}\,.\,\hat{z}\;dxdy=\frac{1}{4}\iint[\vec{E}\times\vec{H}^*+\vec{E}^*\times\vec{H}]\,.\,\hat{z}\;\;dxdy=\frac{1}{4}\iint[\vec{E}_t\times\vec{H}_t{}^*+\vec{E}_t{}^*\times\vec{H}_t]\,.\,\hat{z}\;\;dxdy$$

Assuming the medium to be dispersive, the energy per unit length of the waveguide is,

$$W(z)=\iint\left[\frac{1}{4}\frac{\partial(\omega\varepsilon_o\varepsilon)}{\partial\omega}\vec{E}.\vec{E}^*+\frac{1}{4}\mu_o\vec{H}.\vec{H}^*\right]dxdy$$

$$=\iint\left[\frac{1}{4}\varepsilon_o n\left(2n_g^M-n\right)\vec{E}.\vec{E}^*+\frac{1}{4}\mu_o\vec{H}.\vec{H}^*\right]dxdy$$

Here, $n_g^M(\omega)$ is the material group index, defined earlier as,

$$n_g^M(\omega)=n(\omega)+\omega\frac{dn(\omega)}{d\omega}$$

The superscript "M" is intended so as not to cause confusion with the group index of the optical mode. The following relation can also be proven,

$$\frac{1}{4}\iint\varepsilon_o n^2\,\vec{E}.\vec{E}^*\,dxdy=\frac{1}{4}\iint\mu_o\vec{H}.\vec{H}^*\,dxdy$$

Therefore, the energy per unit length can be written as,

$$W(z) = \frac{1}{2} \iint \varepsilon_o n\, n_g^M \; \vec{E}.\vec{E}^* \, dxdy$$

The **effective index** $n_{eff}(\omega)$ of a mode is,

$$\beta(\omega) = \frac{\omega}{c} n_{eff}(\omega)$$

The group velocity of a mode is,

$$\frac{1}{v_g(\omega)} = \frac{\partial \beta}{\partial \omega}$$

v_g is frequently expressed in terms of the group index of the mode,

$$v_g(\omega) = \frac{c}{n_g(\omega)}$$

where $n_g(\omega)$ is called the **group index**. One can prove the following relation between the power $P(z)$ and the energy per unit length $W(z)$,

$$P(z) = v_g W(z)$$

which can also be written as,

$$\frac{n_g}{c} = \frac{W(z)}{P(z)} = \frac{\iint \varepsilon_o n\, n_g^M \; \vec{E}.\vec{E}^* \, dxdy}{\iint \mathrm{Re}\left[\vec{E}_t \times \vec{H}_t^*\right]. \hat{z} \, dxdy}$$

In the slab-waveguide approximation, assuming TE modes with the transverse component of the electric field given by $\phi(x,y)$, one obtains the following expressions for various quantities of interest,

$$W(z) = \frac{1}{2} \iint \varepsilon_o n\, n_g^M \; \vec{E}.\vec{E}^* \, dxdy = \frac{1}{2} \iint \varepsilon_o n\, n_g^M \; |\phi|^2 \, dxdy$$

$$P(z) = \frac{1}{4} \iint [\vec{E}_t \times \vec{H}_t^* + \vec{E}_t^* \times \vec{H}_t] . \hat{z} \;\; dxdy = \frac{1}{2} \frac{\beta}{\omega \mu_o} \iint |\phi|^2 \, dxdy$$

$$n_g \;\, n_{eff} = \frac{\iint n\, n_g^M \; |\phi|^2 \, dxdy}{\iint |\phi|^2 \, dxdy}$$

7.2.7 Properties of Waveguide Modes and Orthogonality of Modes:

Frequently, solutions in various cases involve expansions in terms of all the waveguide modes. In such cases, knowledge of the orthogonality of the modes is useful. The electric and the magnetic fields for the m-th mode can be written as,

$$\vec{E}^m = \left\{ \vec{E}_t^m(x,y) + \hat{z} E_z^m(x,y) \right\} e^{i\beta_m(\omega) z}$$

$$\vec{H}^m = \left\{ \vec{H}_t^m(x,y) + \hat{z} H_z^m(x,y) \right\} e^{i\beta_m(\omega) z}$$

Some useful properties of the waveguides modes are listed below:

i) When the indices are real, the propagation vectors are also real, and the transverse field components can be chosen to be real as well. Equations (1) and (2) show that in this case the z-components of the fields are purely imaginary.

ii) When the indices are real, the complex conjugate of the electric field mode gives the field for the mode propagating in the opposite (time-reversed) direction. For example, if,

$$\vec{E}^m = \left\{ \vec{E}_t^m(x,y) + \hat{z} E_z^m(x,y) \right\} e^{i\beta_m(\omega) z}$$

represents the field for the forward propagating mode then,

$$\left(\vec{E}^m\right)^* = \left\{ \vec{E}_t^m(x,y) - \hat{z} E_z^m(x,y) \right\} e^{-i\beta_m(\omega) z}$$

represents the field for the backward propagating mode.

iii) When the indices are real, the negative of the complex conjugate of the magnetic field mode gives the field for the mode propagating in the opposite (time-reversed) direction. For example, if,

$$\vec{H}^m = \left\{ \vec{H}_t^m(x,y) + \hat{z}H_z^m(x,y) \right\} e^{i\beta_m(\omega)z}$$

represents the field for the forward propagating mode then,

$$-\left(\vec{H}^m\right)^* = \left\{ -\vec{H}_t^m(x,y) + \hat{z}H_z^m(x,y) \right\} e^{-i\beta_m(\omega)z}$$

represents the field for the backward propagating mode.

iv) Consider two different modes, "m" and "p" with the same propagation vector but different frequencies then the orthogonality between the mode fields is expressed as,

$$\iint dxdy\, n^2(x,y)\, \vec{E}^m . \left(\vec{E}^p\right)^* = 0 \quad \text{for } m \neq p$$

$$\iint dxdy\;\; \vec{H}^m . \left(\vec{H}^p\right)^* = 0 \quad \text{for } m \neq p$$

v) Consider two different modes, "m" and "p" with the same frequency but different propagation vectors then the orthogonality between the mode fields is expressed as,

$$\iint dxdy \left[\vec{E}_t^m \times \left(\vec{H}_t^p\right)^* \right] . \hat{z} = 0 \quad \text{for } m \neq p$$

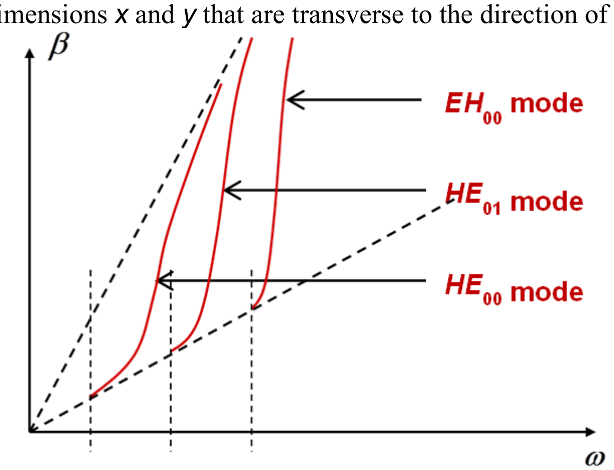
vi) The most general way of expanding a time harmonic field of a particular frequency inside a waveguide is in terms of the waveguide modes,

$$\vec{E}(x,y,z) = \sum_m a_m \left\{ \vec{E}_t^m(x,y) + \hat{z}E_z^m(x,y) \right\} e^{i\beta_m(\omega)z} + \sum_m b_m \left\{ \vec{E}_t^m(x,y) - \hat{z}E_z^m(x,y) \right\} e^{-i\beta_m(\omega)z}$$

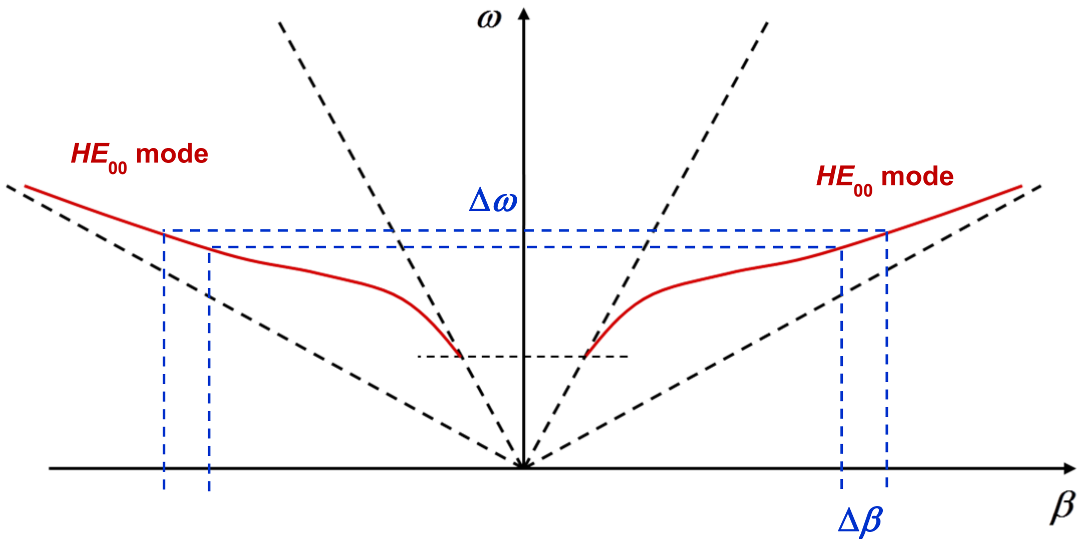
$$\vec{H}(x,y,z) = \sum_m a_m \left\{ \vec{H}_t^m(x,y) + \hat{z}H_z^m(x,y) \right\} e^{i\beta_m(\omega)z} + \sum_m b_m \left\{ -\vec{H}_t^m(x,y) + \hat{z}H_z^m(x,y) \right\} e^{-i\beta_m(\omega)z}$$

7.2.8 Longitudinal vs Transverse Modes of a Waveguide:

Consider the model dispersion relations shown below for the first three modes of a waveguide. The HE and EH modes are called the transverse modes of the waveguide since these modes describe the field profile in dimensions x and y that are transverse to the direction of propagation.



The field profile in the direction of propagation is described by the propagation vector β. The dispersion for the lowest HE_{00} mode is plotted below in a slightly different way that also shows the dispersion for negative values of the propagation vector (propagation in the –z-direction). Different values of β correspond to different longitudinal modes of the waveguide.



Periodic boundary conditions can be used to find the number of different longitudinal modes corresponding to a transverse mode in an interval $\Delta\omega$ of frequency. The problem is identical to finding the density of states for photons in one dimension. If the length of the waveguide is L then there are $L\,\Delta\beta/2\pi$ different longitudinal modes in an interval $\Delta\beta$. Since $v_g(\omega)=\Delta\omega/\Delta\beta$, there are $2\times(L/v_g)\Delta\omega/2\pi$ different longitudinal modes in an interval $\Delta\omega$. The factor of 2 accounts for the forward and backward propagating modes.

7.2 Perturbation Theory for Waveguides

7.2.7 Perturbation Theory for Waveguides:

Often one is interested in the change in the propagation vector of a mode that comes from a small change in some parameter, such as the frequency or the refractive index of the medium. The change in the propagation vector can be obtained from the complex electromagnetic variational theorem and the result is,

$$\Delta\beta = \frac{\iint \left[\Delta\left(\omega\varepsilon_o n^2\right)\vec{E}.\vec{E}^* + \Delta(\omega\mu_0)\vec{H}.\vec{H}^*\right] dxdy}{2\iint \mathrm{Re}\left[\vec{E}_t \times \vec{H}_t^*\right].\,\hat{z}\; dxdy}$$

For example, if the frequency is changed then using the above Equation one obtains the familiar result,

$$\frac{\Delta\beta}{\Delta\omega} = v_g = \frac{n_g}{c} = \frac{W(z)}{P(z)} = \frac{\iint \varepsilon_o n\, n_g^M\; \vec{E}.\vec{E}^*\, dxdy}{\iint \mathrm{Re}\left[\vec{E}_t \times \vec{H}_t^*\right].\,\hat{z}\; dxdy}$$

If the refractive index is changed then one obtains,

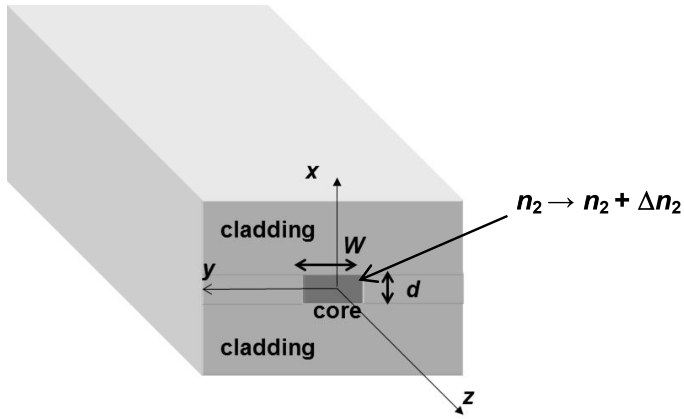
$$\Delta\beta = \omega\frac{\iint \left[\varepsilon_o n\; \Delta n\; \vec{E}.\vec{E}^*\right] dxdy}{\iint \mathrm{Re}\left[\vec{E}_t \times \vec{H}_t^*\right].\,\hat{z}\; dxdy}$$

In the slab waveguide approximation, assuming a TE mode and the transverse component of the electric field given by $\phi(x,y)$, one obtains the following expression for the change in the propagation vector with the refractive index,

$$\Delta\beta = \frac{\omega}{c}\frac{1}{n_{eff}}\frac{\iint n\;\Delta n\;|\phi|^2\; dxdy}{\iint |\phi|^2\; dxdy}$$

7.2.8 Mode Confinement Factors:

Consider the waveguide shown below. The refractive index of the core is n_2 and that of the cladding is n_1.



Suppose the refractive index of the core changes by Δn_2. The change in the propagation vector can be found as follows,

$$
\begin{aligned}
\Delta\beta &= \omega \frac{\iint \left[\varepsilon_o n\, \Delta n\, \vec{E}.\vec{E}^*\right] dxdy}{\iint \mathrm{Re}\left[\vec{E}_t \times \vec{H}_t^*\right].\hat{z}\, dxdy} \\
&= \omega \frac{\iint_{\text{core}} \left[\varepsilon_o n\, \Delta n\, \vec{E}.\vec{E}^*\right] dxdy}{\iint \mathrm{Re}\left[\vec{E}_t \times \vec{H}_t^*\right].\hat{z}\, dxdy} = \omega (n_2 \Delta n_2) \frac{\iint_{\text{core}} \left[\varepsilon_o\, \vec{E}.\vec{E}^*\right] dxdy}{\iint \mathrm{Re}\left[\vec{E}_t \times \vec{H}_t^*\right].\hat{z}\, dxdy} \\
&= \omega \left(\frac{n_2 \Delta n_2}{n_2\, n_{2g}^M}\right) \frac{\iint_{\text{core}} \left[\varepsilon_o\, n_2\, n_{2g}^M\, \vec{E}.\vec{E}^*\right] dxdy}{\iint \mathrm{Re}\left[\vec{E}_t \times \vec{H}_t^*\right].\hat{z}\, dxdy} \\
&= \omega \left(\frac{n_2 \Delta n_2}{n_2\, n_{2g}^M}\right) \frac{\iint_{\text{core}} \left[\varepsilon_o\, n_2\, n_{2g}^M\, \vec{E}.\vec{E}^*\right] dxdy}{\iint \left[\varepsilon_o\, n\, n_g^M\, \vec{E}.\vec{E}^*\right] dxdy} \frac{\iint \left[\varepsilon_o\, n\, n_g^M\, \vec{E}.\vec{E}^*\right] dxdy}{\iint \mathrm{Re}\left[\vec{E}_t \times \vec{H}_t^*\right].\hat{z}\, dxdy} \\
&= \omega \left(\frac{\Delta n_2}{n_{2g}^M}\right) \frac{\iint_{\text{core}} \left[\varepsilon_o\, n_2\, n_{2g}^M\, \vec{E}.\vec{E}^*\right] dxdy}{\iint \left[\varepsilon_o\, n\, n_g^M\, \vec{E}.\vec{E}^*\right] dxdy} \left(\frac{n_g}{c}\right) \\
&= \frac{\omega}{c} \Gamma_2 \left(\frac{n_g}{n_{2g}^M}\right) \Delta n_2
\end{aligned}
$$

Note that n_{2g}^M is the material group index of the waveguide core, and n_g is the group index of the waveguide mode. The overlap integral Γ_2, defined as,

$$
\Gamma_2 = \frac{\iint_{\text{core}} \left[\varepsilon_o\, n_2\, n_{2g}^M\, \vec{E}.\vec{E}^*\right] dxdy}{\iint \left[\varepsilon_o\, n\, n_g^M\, \vec{E}.\vec{E}^*\right] dxdy}
$$

represents the fraction of the mode energy density confined in the waveguide core. Γ_2 is called the transverse mode confinement factor for the waveguide core. The change in the propagation vector is, as expected, proportional to Δn_2 and also to the fraction of the modal energy inside the core.

Suppose now the refractive index of the cladding also changes by Δn_1 then the total change in the propagation vector can be written as a simple sum,

$$\Delta\beta = \frac{\omega}{c}\,\Gamma_1\left(\frac{n_g}{n_{1g}^M}\right)\Delta n_1 + \frac{\omega}{c}\,\Gamma_2\left(\frac{n_g}{n_{2g}^M}\right)\Delta n_2$$

In the slab waveguide approximation, assuming a TE mode and the transverse component of the electric field given by $\phi(x,y)$, one obtains the following expression for the transverse mode confinement factor for the core,

$$\Gamma_2 = \frac{\iint_{\text{core}} n_2\, n_{2g}^M |\phi|^2\, dxdy}{\iint n\, n_g^M\, |\phi|^2\, dxdy}$$

The waveguide perturbation theory can be used to calculate the change in the propagation vector in the presence of material loss (or gain). Suppose the core of the waveguide becomes lossy and the imaginary part of the core refractive index acquires a non-zero value given by,

$$n_2 \rightarrow n_2 + i\,\frac{c}{\omega}\frac{\alpha_2}{2}$$

In this case, we can take the index perturbation Δn_2 to be,

$$\Delta n_2 = i\,\frac{c}{\omega}\frac{\alpha_2}{2}$$

The change in the waveguide propagation constant becomes,

$$\Delta\beta = \frac{\omega}{c}\,\Gamma_2\left(\frac{n_g}{n_{2g}^M}\right)\Delta n_2 = \frac{\omega}{c}\,\Gamma_2\left(\frac{n_g}{n_{2g}^M}\right)\left(i\,\frac{c}{\omega}\frac{\alpha_2}{2}\right) = i\,\Gamma_2\left(\frac{n_g}{n_{2g}^M}\right)\frac{\alpha_2}{2} = i\,\Gamma_2\,\frac{\tilde{\alpha}_2}{2} = i\,\frac{\tilde{\alpha}}{2}$$

where,

$$\tilde{\alpha}_2 = \left(\frac{n_g}{n_{2g}^M}\right)\alpha_2 \qquad \tilde{\alpha} = \Gamma_2\,\tilde{\alpha}_2$$

The propagation vector β acquires a small imaginary part because of optical loss in the waveguide core. The imaginary part of the propagation vector will cause the wave energy to decay with distance as it propagates in the waveguide.