Chapter 14

Carrier Density Dependent Index, Frequency Chirp and FM Modulation, and Frequency Linewidth in Semiconductor Lasers

14.1 Introduction

14.1.1 Carrier Density Dependent Refractive Index in Semiconductor Lasers:
In Chapter 6, the following expression was obtained for the dielectric constant of semiconductors,

$$\varepsilon_{\text{total}}(\omega) = \varepsilon_{\text{inter}}(\omega) + \varepsilon_{\text{intra}}(\omega)$$

$$\varepsilon_{\text{inter}}(\omega) = \varepsilon_o - 2\left(\frac{q}{m}\right)^2 \hbar^2 \sum_{r,s} 2 \times \int_{\text{FBZ}} \frac{d^3\vec{k}}{(2\pi)^3} \left|\vec{P}_{rs}\,.\,\hat{n}\right|^2 \left[f_s(\vec{k}) - f_r(\vec{k})\right] \frac{\left(E_r(\vec{k}) - E_s(\vec{k})\right)^{-1}}{\omega^2 - \left(E_r(\vec{k}) - E_s(\vec{k})\right)^2}$$

$$\varepsilon_{\text{intra}}(\omega) = i\,\frac{nq^2\tau/m_e}{\omega(1 - i\omega\tau_e)} + i\,\frac{pq^2\tau/m_h}{\omega(1 - i\omega\tau_h)}$$

The dielectric constant and the refractive index depend on the electron and hole densities. This dependence is explicit in the intraband contribution to the dielectric constant and implicit through the carrier distribution functions in the interband contribution. The above expressions are rather complicated. In semiconductor laser, changes in the carrier density affect both the gain as well as the refractive index. In other words, changes in the carrier density affect both the real and the imaginary parts of the refractive index. The complex refractive index is,

$$n(\omega) = n'(\omega) + in''(\omega) = n'(\omega) - i\,\frac{g(\omega)}{2}\frac{\omega}{c}$$

We define a quantity $\alpha(\omega)$ as follows,

$$\alpha(\omega) = -\frac{dn'(\omega)/dn}{dn''(\omega)/dn} = -\frac{4\pi}{\lambda}\frac{dn'(\omega)/dn}{dg(\omega)/dn}$$

The above expression is useful since it can be used to express changes in the real part of the refractive index in terms of the material differential gain,

$$\frac{dn'(\omega)}{dn} = -\alpha(\omega)\frac{\lambda}{4\pi}\frac{dg(\omega)}{dn}$$

The values of $\alpha(\omega)$ near the peak gain frequency can range anywhere from 2 to 10 for most commonly used III-V gain materials with values in the 4 to 7 range being typical. Also, note that the refractive index decreases with an increase in the carrier density.

14.1.2 Cavity Mode Frequency Shift with a Carrier Density Change:
In Chapter 12, the following expression was derived for the change in the cavity mode frequency due to a change in the cavity refractive index,

$$\Delta\omega = -\omega \frac{\int n'(\omega,\vec{r})\,\Delta n'(\omega,\vec{r})\,\vec{E}^*(\vec{r}).\vec{E}(\vec{r})\,d^3\vec{r}}{\int n'(\omega,\vec{r})\,n_g^M(\omega,\vec{r})\,\vec{E}^*(\vec{r}).\vec{E}(\vec{r})\,d^3\vec{r}}$$

If the carrier density in the active region changes by Δn, then the change in the active region refractive index is,

$$\Delta n'(\omega) = -\alpha(\omega)\frac{\lambda}{4\pi}\frac{dg(\omega)}{dn}\Delta n$$

The corresponding change in the cavity mode frequency is,

$$\Delta\omega = -\omega \frac{\int n'(\omega,\vec{r})\,\Delta n'(\omega,\vec{r})\,\vec{E}^*(\vec{r}).\vec{E}(\vec{r})\,d^3\vec{r}}{\int n'(\omega,\vec{r})\,n_g^M(\omega,\vec{r})\,\vec{E}^*(\vec{r}).\vec{E}(\vec{r})\,d^3\vec{r}} = \alpha(\omega)\frac{\Gamma_a v_g^M (dg/dn)}{2}\Delta n$$

If the cavity is a Fabry-Perot then the above expression can also be written in terms of the mode group velocity v_g,

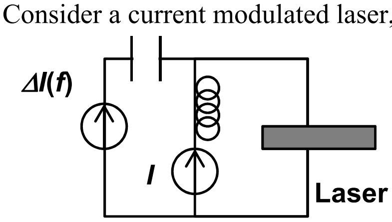
$$\Delta\omega = \alpha\frac{\Gamma_a v_g (d\tilde{g}/dn)}{2}\Delta n$$

For example, in a 1550 nm laser, if $\Delta n = 5\times 10^{17}$ cm^{-3}, $\Gamma_a = .075$, $n_g = 3.6$, $d\tilde{g}/dn = 10^{-15}$ cm^2, and $\alpha = 5$, the change in the cavity frequency is 124 GHz which is around 1 nm. Applications that require a stable frequency optical source demand that the carrier density in the laser active region be free of fluctuations.

14.2 Direct Current Frequency Modulation in Semiconductor Lasers

14.2.1 Introduction:

Consider a current modulated laser, as shown below.



A direct current modulated semiconductor laser

In Chapter 11, the following expressions were derived for the small signal change in the carrier density and the photon density in a response to a current modulation,

$$I(t) = I + \mathrm{Re}\{\Delta I(f)e^{-i2\pi ft}\}$$

$$n(t) = n + \mathrm{Re}\{\Delta n(f)e^{-i2\pi ft}\}$$

$$n_p(t) = n_p + \mathrm{Re}\{\Delta n_p(f)e^{-i2\pi ft}\}$$

$$\begin{bmatrix}\Delta n(f)\\ \Delta n_p(f)\end{bmatrix} = \eta_i \frac{\Delta I(f)}{qV_a}\begin{bmatrix}-i2\pi f\\ \dfrac{\omega_R^2}{\Gamma_a\tau_p}\end{bmatrix}H(f)$$

Suppose the cavity mode frequency is also time dependent,

$$\omega(t) = \omega + \mathrm{Re}\{\Delta\omega(f)e^{-i2\pi ft}\}$$

Then,

$$\Delta\omega(f) = \alpha \frac{\Gamma_a v_g \left(d\tilde{g}/dn\right)}{2} \Delta n(f) = \alpha \frac{\Gamma_a v_g \left(d\tilde{g}/dn\right)}{2} \frac{-i2\pi f}{\omega_R^2} H(f)\, \eta_i \frac{\Delta I(f)}{qV_a}$$

The above equation describes the frequency modulation (FM) response of the laser. Because of the carrier density dependent refractive index, the frequency of the lasing mode is modulated in response to a current modulation. We can also write,

$$\begin{aligned} \Delta\omega(f) &= \alpha \frac{\Gamma_a v_g \left(d\tilde{g}/dn\right)}{2} \Delta n(f) \\ &= \alpha \frac{\Gamma_a v_g \left(d\tilde{g}/dn\right)}{2} \frac{-i2\pi f}{\omega_R^2\, \Gamma_a \tau_p} \Delta n_p(f) \\ &= \alpha(-i2\pi f) \frac{\Delta n_p(f)}{2n_p} \end{aligned}$$

In time domain, one can write the above expression as,

$$\Delta\omega(t) = \frac{\alpha}{2n_p} \frac{d\Delta n_p(t)}{dt} = \frac{\alpha}{2N_p} \frac{d\Delta N_p(t)}{dt} = \frac{\alpha}{2P} \frac{d\Delta P(t)}{dt}$$

The instantaneous frequency shift is proportional to the rate of change of the photon density or the rate of change of the laser output power. Frequency modulation (FM modulation) is also used to transfer information in fiber optical networks. In such schemes, detection and information retrieval at the receiving end is more complicated compared to the schemes that use intensity modulation.

14.3 Laser Phase Noise and Laser Frequency Linewidth

14.3.1 Introduction to Phase Noise:

Consider a sinusoidal signal whose phase is changing with time,

$$A(t) = A_o \cos(\omega_o t + \phi(t))$$

The instantaneous frequency is given as,

$$\omega(t) = \frac{d(\omega_o t + \phi(t))}{dt} = \omega_o + \frac{d\phi(t)}{dt}$$

A time varying phase is therefore equivalent to a frequency variation. The term "phase noise" refers to a time varying phase with a zero average value for the phase difference,

$$\langle \phi(t_1) - \phi(t_2) \rangle = 0$$

However, the higher order correlation functions of the phase difference are not all zero. For example,

$$\left\langle [\phi(t_1) - \phi(t_2)]^2 \right\rangle \neq 0$$

The above phase correlation function is closely related to the frequency linewidth of the sinusoidal signal, as discussed below. If the phase noise has Gaussian statistics then all higher order correlation functions of the phase difference can be written in terms of the second order correlation function given above. For example,

$$\left\langle [\phi(t_1) - \phi(t_2)]^n \right\rangle = \begin{cases} \left\langle [\phi(t_1) - \phi(t_2)]^2 \right\rangle (n-1)!! & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

The double factorial sign represents the product of all odd integers from 1 to $n-1$.

14.3.2 Frequency Spectrum of Signals with Phase Noise:

The frequency spectrum $S(\omega)$ of a sinusoidal signal $A(t)$ is defined as follows,

$$S(\omega)=\int_{-\infty}^{\infty}\frac{\langle A(t+\tau)A(t)\rangle}{\sqrt{\langle A(t+\tau)A(t+\tau)\rangle\langle A(t)A(t)\rangle}}e^{i\omega\tau}d\tau$$

The angled brackets imply ensemble averaging. Ensemble averaging can be replaced by time averaging if the signal is ergodic. Suppose,

$$A(t)=A_o\cos(\omega_o t+\phi(t))$$

then,

$$\langle A(t)A(t)\rangle=\frac{A_o^2}{2}=\langle A(t+\tau)A(t+\tau)\rangle$$

and,

$$\langle A(t+\tau)A(t)\rangle=\frac{A_o^2}{2}\left[\langle\cos(2\omega_o t+\omega_o\tau+\phi(t+\tau)+\phi(t))\rangle+\langle\cos(\omega_o\tau+\phi(t+\tau)-\phi(t))\rangle\right]$$

The first term in the square brackets goes to zero upon averaging. The second term is,

$$\langle\cos(\omega_o\tau+\phi(t+\tau)-\phi(t))\rangle=\frac{e^{i\omega_o\tau}}{2}\left\langle e^{i[\phi(t+\tau)-\phi(t)]}\right\rangle+\frac{e^{-i\omega_o\tau}}{2}\left\langle e^{-i[\phi(t+\tau)-\phi(t)]}\right\rangle$$

If there is no phase noise then,

$$S(\omega)=\int_{-\infty}^{\infty}\frac{\langle A(t+\tau)A(t)\rangle}{\sqrt{\langle A(t+\tau)A(t+\tau)\rangle\langle A(t)A(t)\rangle}}e^{i\omega\tau}d\tau=\frac{1}{2}\left[2\pi\delta(\omega+\omega_o)+2\pi\delta(\omega-\omega_o)\right]$$

The frequency spectrum consists of two delta functions at the frequency of the sinusoidal signal. If the phase noise is assumed to obey Gaussian statistics, then,

$$\left\langle e^{i[\phi(t+\tau)-\phi(t)]}\right\rangle=e^{-\left\langle[\phi(t+\tau)-\phi(t)]^2\right\rangle/2}$$

$$\left\langle e^{-i[\phi(t+\tau)-\phi(t)]}\right\rangle=e^{-\left\langle[\phi(t+\tau)-\phi(t)]^2\right\rangle/2}$$

A commonly encountered case is when the phase noise correlation represents phase diffusion with the following correlation function,

$$\left\langle[\phi(t+\tau)-\phi(t)]^2\right\rangle=\gamma\,|\tau|$$

The frequency spectrum in this case becomes,

$$S(\omega)=\int_{-\infty}^{\infty}\frac{\langle A(t+\tau)A(t)\rangle}{\sqrt{\langle A(t+\tau)A(t+\tau)\rangle\langle A(t)A(t)\rangle}}e^{i\omega\tau}d\tau=\frac{1}{2}\left[\frac{\gamma}{(\omega+\omega_o)^2+(\gamma/2)^2}+\frac{\gamma}{(\omega-\omega_o)^2+(\gamma/2)^2}\right]$$

In the presence of phase noise, the delta functions have broadened into Lorentzians with a full width at half maximum (FWHM) equal to γ which is also called the linewidth of the spectrum.

14.3.3 Laser Phase Coherence:

So far our discussion on lasers has been focused on the photon density or the total number of photons inside the laser cavity. Lasing was identified with the buildup of a large photon population inside the laser cavity above threshold. If this were all there was to lasers then lasers would not have been any more interesting than high power incandescent light sources. Laser light has phase coherence. We discuss this phase coherence in below.

We assume that the field $\vec{E}_o(\vec{r})$ of the cavity eigenmode is normalized such that,

$$\frac{1}{2}\int\varepsilon_o n n_g^M\vec{E}_o^*(\vec{r}).\vec{E}_o(\vec{r})\,d^3\vec{r}=\hbar\omega_o$$

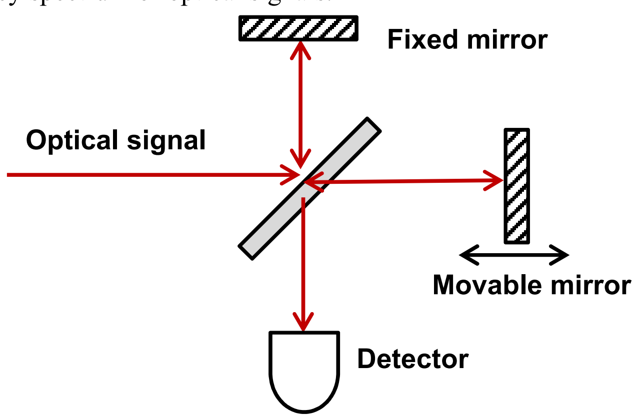
where ω_o is the cavity mode frequency. The field inside a laser cavity above threshold can be written as,

$$\vec{E}(\vec{r},t) = \mathrm{Re}\left\{\sqrt{N_p(t)}\,\vec{E}_o(\vec{r})\,e^{-i\omega_o t - i\phi(t)}\right\}$$

Here, $N_p(t)$ is the number of photons inside the cavity and $\phi(t)$ is the phase of the field. It is not difficult to see, given our normalization of the cavity eigenmode, that the total energy inside the cavity will equal $\hbar\omega_o\, N_p(t)$. Note that the field amplitude is proportional to $\sqrt{N_p(t)}$. In a stimulated emission process, the field of the emitted photon has the same phase as the phase of the cavity field that stimulated the transition. Therefore, when photons multiply via stimulated emission the phase is preserved. However, the phase of the field of a spontaneously emitted photon has no relationship with the phase of the field inside the cavity. Every spontaneously emitted photon has a random phase with respect to the phase of the field inside the cavity. Below threshold, the photons inside the laser cavity are mostly a result of spontaneous emission and therefore cavity field has no well-defined phase. Above threshold, the photons inside the laser cavity are mostly a result of stimulated emission and the cavity field, therefore, has a well-defined phase. In addition, the phase $\phi(t)$ of the cavity field remains constant for long durations and we call this property "phase coherence". Phase coherence is not unique to laser fields. Phase coherent electromagnetic signals in the KHz, MHz and GHz frequency ranges have been around for over a century (e.g. radio waves). But phase coherent sources at the optical frequencies were missing until the laser was invented.

14.3.4 Phase Noise, Frequency Spectrum of Optical Fields, and Coherence Time:

In Section 13.3.2, the frequency spectrum of sinusoidal signals was related to their phase noise. The same holds true for optical signals. The interferometric scheme shown below can be used to measure the frequency spectrum of optical signals.



The input light is split into two using a 50/50 beam splitter, reflected off mirrors, and then combined again with the same beam splitter. The resulting intensity is detected with a photodetector. One of the mirrors is movable and is used to introduce a delay in one of the split signals with respect to the other one. The average photodetector current $\langle I(t)\rangle$ is proportional to the square magnitude of the field composed of the optical signal and its time-delayed version,

$$\langle I(t,\tau)\rangle \propto \left\langle \left|\vec{E}(t)+\vec{E}(t+\tau)\right|^2 \right\rangle = \left\langle \vec{E}(t)\,.\,\vec{E}(t)\right\rangle + \left\langle \vec{E}(t+\tau)\,.\,\vec{E}(t+\tau)\right\rangle + 2\left\langle \vec{E}(t)\,.\,\vec{E}(t+\tau)\right\rangle$$

If the optical signal has no amplitude noise, i.e. if,

$$\left\langle \vec{E}(t)\,.\,\vec{E}(t)\right\rangle = \left\langle \vec{E}(t+\tau)\,.\,\vec{E}(t+\tau)\right\rangle$$

then the frequency spectrum of the optical signal can be written in terms of the detector current,

$$S(\omega) = \int_{-\infty}^{\infty} \frac{\left\langle \vec{E}(t+\tau)\,.\,\vec{E}(t)\right\rangle}{\sqrt{\left\langle \vec{E}(t+\tau)\,.\,\vec{E}(t+\tau)\right\rangle\left\langle \vec{E}(t)\,.\,\vec{E}(t)\right\rangle}}\, e^{i\omega\tau} d\tau = \int_{-\infty}^{\infty} \left[2\frac{\langle I(t,\tau)\rangle}{I(t,\tau=0)} - 1 \right] e^{i\omega\tau} d\tau$$

The quantity,

$$2\frac{\langle I(t,\tau)\rangle}{I(t,\tau=0)}-1$$

is called the interferogram. The Fourier transform of the interferogram with respect to the delay τ gives the frequency spectrum of the optical signal. Suppose that,

$$\vec{E}(t)=\hat{n}E_o\cos(\omega_o t+\phi(t))$$

then,

$$\langle I(t,\tau)\rangle \propto \left\langle \vec{E}(t).\vec{E}(t)\right\rangle+\left\langle \vec{E}(t+\tau).\vec{E}(t+\tau)\right\rangle+2\left\langle \vec{E}(t).\vec{E}(t+\tau)\right\rangle$$
$$=E_o^2+E_o^2\langle\cos(\omega_o\tau+\phi(t+\tau)-\phi(t))\rangle$$

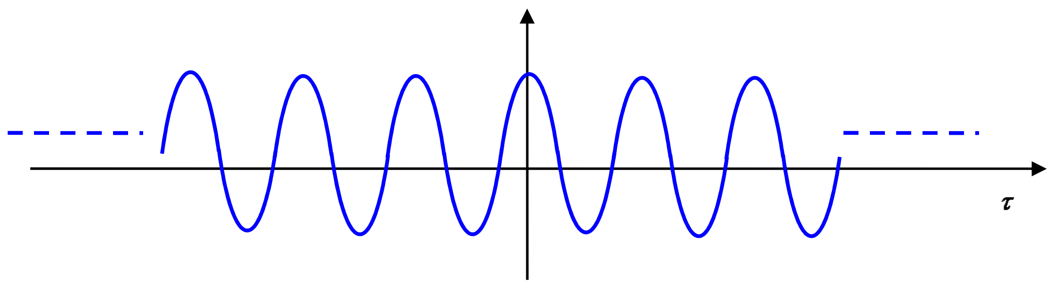
First suppose that there is no phase noise. In this case, the interferogram equals,

$$2\frac{\langle I(t,\tau)\rangle}{I(t,\tau=0)}-1=\cos(\omega_o\tau)$$

and the frequency spectrum equals,

$$S(\omega)=\frac{1}{2}\left[2\pi\delta(\omega+\omega_o)+2\pi\delta(\omega-\omega_o)\right]$$

The interferogram is sketched below and shows the interference fringes between the signal and its time delayed version that persist forever indicating perfect phase coherence.



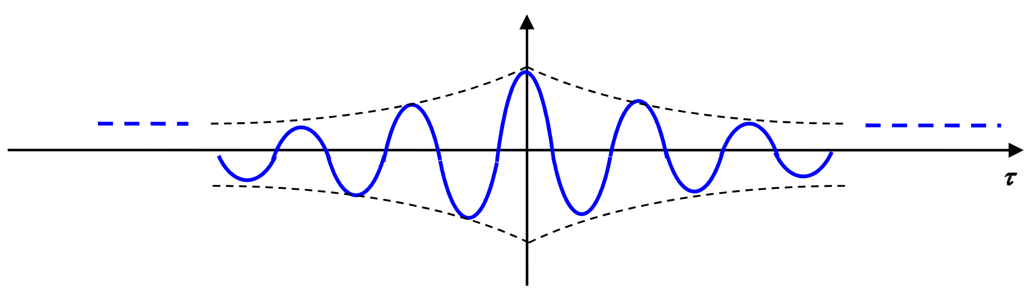
Now suppose the signals has phase noise and the phase noise has Gaussian statistics and the phase correlation function is given by,

$$\left\langle[\phi(t+\tau)-\phi(t)]^2\right\rangle=\gamma\,|\tau|$$

The interferogram equals,

$$2\frac{\langle I(t,\tau)\rangle}{I(t,\tau=0)}-1=e^{-\gamma|\tau|/2}\cos(\omega_o\tau)$$

The interferogram is sketched below. The envelope of the interferogram decays exponentially with a decay constant equal to $\gamma/2$. The quantity $1/\gamma$ is called the coherence time of the signal. It is the time scale over which the phase of the signal would likely not change.



As the delay τ is increased, and the phase of the signal diffuses and the heights of the peaks in the interferogram become smaller. The frequency spectrum equals,

$$S(\omega)=\frac{1}{2}\left[\frac{\gamma}{(\omega+\omega_o)^2+(\gamma/2)^2}+\frac{\gamma}{(\omega-\omega_o)^2+(\gamma/2)^2}\right]$$

Note that the FWHM of the spectral density is equal to the inverse of the coherence time. The FWHM of the spectral density is used as a measure of the coherence time of an optical signal.

14.3.5 Laser Phase Dynamics and Laser Linewidth:

Consider a laser operating above threshold with a steady state photon number in the cavity equal to N_p and we assume that $N_p >> 1$. The electric field can be written as,

$$\vec{E}(\vec{r},t)=\mathrm{Re}\left\{\sqrt{N_p}\,\vec{E}_o(\vec{r})\,e^{-i\omega_o t-i\phi(t)}\right\}$$

In order to study the dynamics of the phase $\phi(t)$ we need to figure out how stimulated and spontaneous emission events affect the laser phase. Consider a stimulated emission event at time t. Right after the stimulated emission event, the field becomes,

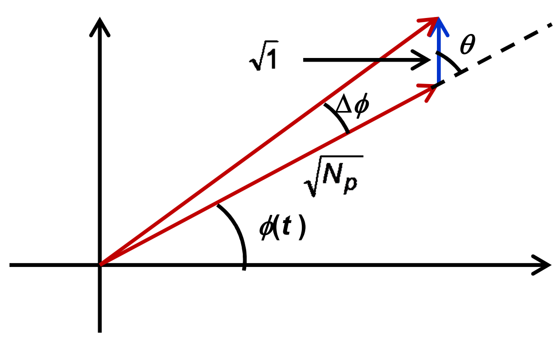
$$\vec{E}(\vec{r},t)=\mathrm{Re}\left\{\sqrt{N_p+1}\,\vec{E}_o(\vec{r})\,e^{-i\omega_o t-i\phi(t)}\right\}$$

Note that the phase of the field has not changed before and after the stimulated emission event. The cavity photon number is now one more than the steady state photon number N_p. The photon number will decay via relaxation oscillations until the photon number again reaches the steady state photon number N_p.

Now consider a spontaneous emission event at time t. The phase of the field of the added photon is not the same as that of the cavity field. Suppose the phase of the field of the added photon is θ with respect to the phase of the cavity field. Right after the spontaneous emission event, the field is,

$$\begin{aligned}\vec{E}(\vec{r},t)&\approx\mathrm{Re}\left\{\sqrt{N_p}\,\vec{E}_o(\vec{r})\,e^{-i\omega_o t-i\phi(t)}\right\}+\mathrm{Re}\left\{\sqrt{1}\,\vec{E}_o(\vec{r})\,e^{-i\omega_o t-i\phi(t)-i\theta}\right\}\\&=\mathrm{Re}\left\{\left[\sqrt{N_p}+\cos\theta-i\sin\theta\right]\vec{E}_o(\vec{r})\,e^{-i\omega_o t-i\phi(t)}\right\}\\&\approx\mathrm{Re}\left\{\sqrt{\left(\sqrt{N_p}+\cos\theta\right)^2+\sin^2\theta}\;\vec{E}_o(\vec{r})\,e^{-i\omega_o t-i\phi(t)-i\Delta\phi}\right\}\end{aligned}$$

$$\Delta\phi\approx\frac{\sin\theta}{\sqrt{N_p}}$$



In graphical representation, the electric field phasor in the complex plane before and after the spontaneous emission event are shown in the Figure above. The angle θ is to be considered a random variable with a uniform probability distribution in the interval $0\le\theta<2\pi$. Therefore,

$$\langle\cos\theta\rangle=\langle\sin\theta\rangle=0$$

$$\langle\cos^2\theta\rangle=\langle\sin^2\theta\rangle=\frac{1}{2}$$

Note that the average photon number after the spontaneous emission event is,

$$\left\langle \left(\sqrt{N_p} + \cos\theta\right)^2 + \sin^2\theta \right\rangle = N_p + 1$$

as expected. The increase in the amplitude of the field after spontaneous emission will decay via relaxation oscillations until the cavity field is,

$$\vec{E}(\vec{r},t) = \text{Re}\left\{\sqrt{N_p}\ \vec{E}_o(\vec{r})\, e^{-i\omega_o t - i\phi(t) - i\Delta\phi}\right\}$$

The photon number has returned to the steady state value. However, the phase has been shifted by,

$$\Delta\phi \approx \frac{\sin\theta}{\sqrt{N_p}}$$

The above analysis shows that each spontaneous emission event "kicks" the phase of the cavity field by an amount approximately equal to $\sin\theta/\sqrt{N_p}$. In any time interval τ, the total change in the phase of the cavity field can be obtained by summing over the contributions from all the spontaneous emission events that happen during this interval,

$$\phi(t+\tau) - \phi(t) \approx \frac{1}{\sqrt{N_p}} \sum_k \sin\theta_k$$

Since the phases θ_k are completely random and uncorrelated,

$$\langle \phi(t+\tau) - \phi(t) \rangle \approx \frac{1}{\sqrt{N_p}} \sum_k \langle \sin\theta_k \rangle = 0$$

$$\left\langle [\phi(t+\tau) - \phi(t)]^2 \right\rangle \approx \frac{1}{N_p} \sum_k \left\langle \sin^2\theta_k \right\rangle = \frac{1}{2N_p}\left(\Gamma_a v_g^M g n_{sp}\right)\tau$$

And if τ is allowed to be negative then,

$$\left\langle [\phi(t+\tau) - \phi(t)]^2 \right\rangle \approx \frac{1}{N_p} \sum_k \left\langle \sin^2\theta_k \right\rangle = \frac{1}{2N_p}\left(\Gamma_a v_g^M g n_{sp}\right)|\tau|$$

From the phase correlation function it follows that the frequency spectrum of the laser field is,

$$S(\omega) = \frac{1}{2}\left[\frac{\Delta\omega}{(\omega+\omega_o)^2 + (\Delta\omega/2)^2} + \frac{\Delta\omega}{(\omega-\omega_o)^2 + (\Delta\omega/2)^2}\right]$$

where the FWHM laser linewidth $\Delta\omega$ is given by,

$$\Delta\omega = \frac{1}{2N_p}\left(\Gamma_a v_g^M g n_{sp}\right)$$

The above expression is the Schawlow-Townes result for the laser linewidth. It expresses the fact that the laser linewidth is determined by the spontaneous emission rate since spontaneous emission events cause the phase of the cavity field to "diffuse" in time. Unfortunately, the Schawlow-Townes expression, although accurate for gas lasers, underestimates the experimentally measured linewidths of semiconductor lasers by more than an order of magnitude.

14.3.6 Phase Dynamics and Linewidth of Semiconductor Lasers:

In semiconductor lasers, photon number changes result in carrier density changes which lead to refractive index changes and which in turn lead to shifts in the cavity mode frequency. All these complicated dynamics are captured by the equation derived earlier in this Chapter,

$$\Delta\omega(t) = \frac{\alpha}{2N_p}\frac{d\Delta N_p(t)}{dt}$$

We can think of the time dependent frequency in terms of a time dependent phase,

$$\frac{d\Delta\psi(t)}{dt} = \Delta\omega(t) = \frac{\alpha}{2N_p}\frac{d\Delta N_p(t)}{dt}$$

Consider a spontaneous emission event at time t. Right after the spontaneous emission event the cavity field is,

$$\vec{E}(\vec{r},t) \approx \mathrm{Re}\left\{ \sqrt{\left(\sqrt{N_p} + \cos\theta\right)^2 + \sin^2\theta}\; \vec{E}_o(\vec{r})\, e^{-i\omega_o t - i\phi(t) - i\Delta\phi} \right\}$$

$$\Delta\phi \approx \frac{\sin\theta}{\sqrt{N_p}}$$

The change in the cavity photon number from the steady state value immediately after the spontaneous emission event is,

$$\Delta N_p = 2\sqrt{N_p}\cos\theta + 1 \approx 2\sqrt{N_p}\cos\theta$$

The cavity photon number will now undergo relaxation oscillations, in which the carrier density and the photon number oscillate 90-degrees out of phase, until the photon number reaches the steady state value again. During these relaxation oscillations the cavity mode frequency and the phase changes are governed by the equation,

$$\frac{d\Delta\psi(t)}{dt} = \Delta\omega(t) = \frac{\alpha}{2N_p}\frac{d\Delta N_p(t)}{dt}$$

Once the relaxation oscillations are over, and the photon number has returned to its steady state value, the net change in the phase is,

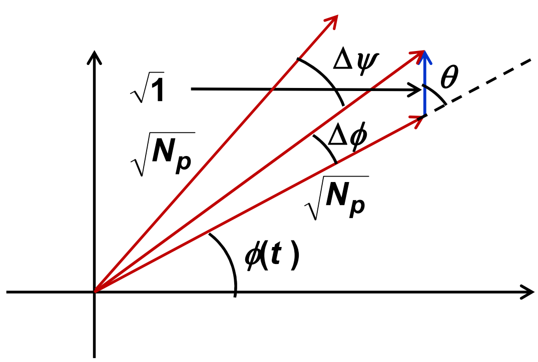
$$\Delta\psi = \frac{\alpha}{2N_p}\Delta N_p = -\frac{\alpha}{2N_p}2\sqrt{N_p}\cos\theta$$

When the relaxation oscillations are over, the cavity field is,

$$\vec{E}(\vec{r},t) = \mathrm{Re}\left\{\sqrt{N_p}\; \vec{E}_o(\vec{r})\, e^{-i\omega_o t - i\phi(t) - i\Delta\phi - i\Delta\psi}\right\}$$

The total phase change caused by the spontaneous emission event is,

$$\Delta\vartheta = \Delta\phi + \Delta\psi \approx \frac{\sin\theta}{\sqrt{N_p}} - \alpha\frac{\cos\theta}{\sqrt{N_p}}$$



In any time interval τ, the total change in the phase of the cavity field can be obtained by summing over the contributions from all the spontaneous emission events that happen during this interval,

$$\vartheta(t+\tau) - \vartheta(t) \approx \frac{1}{\sqrt{N_p}}\left(\sum_k \sin\theta_k - \alpha\cos\theta_k\right)$$

Since the phases θ_k are completely random and uncorrelated,

$$\left\langle \vartheta(t+\tau) - \vartheta(t)\right\rangle \approx 0$$

$$\left\langle \left[\vartheta(t+\tau) - \vartheta(t)\right]^2 \right\rangle \approx \frac{1}{N_p}\sum_k \left\langle \sin^2\theta_k \right\rangle + \alpha^2\frac{1}{N_p}\sum_k \left\langle \cos^2\theta_k \right\rangle = \frac{\left(1+\alpha^2\right)}{2N_p}\left(\Gamma_a v_g^M g n_{sp}\right)\tau$$

And if τ is allowed to be negative then,

$$\left\langle [\vartheta(t+\tau)-\vartheta(t)]^2 \right\rangle \approx \frac{1}{N_p}\sum_k \left\langle \sin^2\theta_k \right\rangle = \frac{\left(1+\alpha^2\right)}{2N_p}\left(\Gamma_a v_g^M g n_{sp}\right)|\tau|$$

It follows that the FWHM of the laser linewidth is,

$$\Delta\omega = \frac{\left(1+\alpha^2\right)}{2N_p}\left(\Gamma_a v_g^M g n_{sp}\right)$$

Since values of α are in the 4 to 7 range, the semiconductor laser frequency linewidth is 17 to 50 times larger than the Schawlow-Townes limit. The additional linewidth broadening is due to phase changes that accompany relaxation oscillations because of the carrier density dependent refractive index in semiconductors.