

Chapter 12

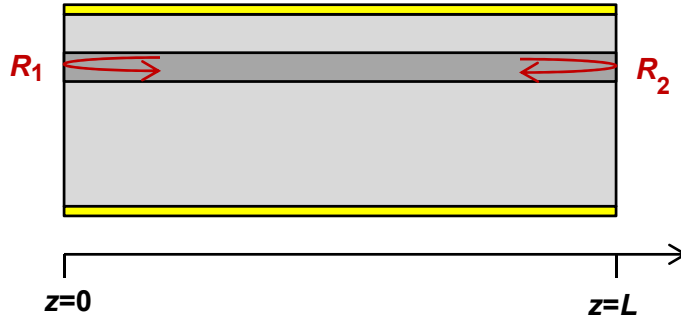
Laser Cavities and Microcavities: Vertical Cavity Surface Emitting Laser (VCSEL)

12.1 Introduction

12.1.1 Cavity Modes:

Consider a Fabry-Perot laser cavity with facet reflectivities R_1 and R_2 . The model gain per unit length is $\Gamma_a \tilde{g}$. The roundtrip condition for optical power for lasing is,

$$R_1 R_2 e^{(\Gamma_a \tilde{g} - \tilde{\alpha})2L} = 1$$



Suppose the complex field amplitude reflection coefficients at the two facets are r_1 and r_2 , and $R_1 = |r_1|^2$ and $R_2 = |r_2|^2$. We can write the reflection coefficients in terms of a magnitude and phase,

$$r_1 = \sqrt{R_1} e^{i\phi_1} \quad r_2 = \sqrt{R_2} e^{i\phi_2}$$

The roundtrip condition for the field amplitude for lasing is,

$$r_1 r_2 e^{\left(\Gamma_a \frac{\tilde{g}}{2} - \frac{\tilde{\alpha}}{2}\right)2L} e^{i\beta 2L} = 1$$

$$\Rightarrow |r_1| |r_2| e^{\left(\Gamma_a \frac{\tilde{g}}{2} - \frac{\tilde{\alpha}}{2}\right)2L} e^{i\beta 2L + \phi_1 + \phi_2} = 1$$

The above complex relation implies,

$$\sqrt{R_1} \sqrt{R_2} e^{\left(\Gamma_a \frac{\tilde{g}}{2} - \frac{\tilde{\alpha}}{2}\right)2L} = 1$$

and,

$$2\beta L + \phi_1 + \phi_2 = p2\pi \quad \{ p = \text{integer} \}$$

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The phase condition states that the roundtrip phase of a cavity mode must be an integral multiple of 2π . This implies that only those values of β are possible that satisfy,

$$\beta = \frac{p2\pi - \phi_1 - \phi_2}{2L}$$

The spacing $\Delta\beta$ between two adjacent allowed values of β (and therefore between two adjacent cavity modes) is,

$$\Delta\beta = \frac{\pi}{L} - \frac{\Delta\phi_1}{2L} - \frac{\Delta\phi_2}{2L} \approx \frac{\pi}{L}$$

The allowed values of β correspond to the longitudinal modes of the cavity. The spacing in frequency $\Delta\omega$ between the longitudinal modes can be found as follows,

$$\begin{aligned} \left(2 \frac{\partial\beta}{\partial\omega} L + \frac{\partial\phi_1}{\partial\omega} + \frac{\partial\phi_2}{\partial\omega} \right) \Delta\omega &= 2\pi \\ \Rightarrow \Delta\omega &= \frac{\pi}{\frac{L}{v_g} + \frac{1}{2} \left(\frac{\partial\phi_1}{\partial\omega} + \frac{\partial\phi_2}{\partial\omega} \right)} \approx \frac{2\pi}{2L/v_g} \end{aligned}$$

We can write,

$$\Delta\omega \approx \frac{2\pi}{T}$$

Here, T is the roundtrip time of the cavity. The frequency spacing between adjacent cavity modes is called the **free spectral range** of the cavity. It is more commonly expressed as wavelength spacing,

$$\Delta\lambda = \frac{\lambda^2}{2n_g L}$$

The above example shows that by following the wave through one roundtrip, one can obtain the lasing condition (and therefore the threshold gain) and also the allowed modes of the cavity. The cavity we considered was a simple Fabry-Perot cavity. For more complex cavities, such as the VCSEL cavity, one needs better and more sophisticated methods. In the next Section, we will introduce the S-matrix and the T-matrix approach and then we will use it later to analyze the VCSEL cavity.

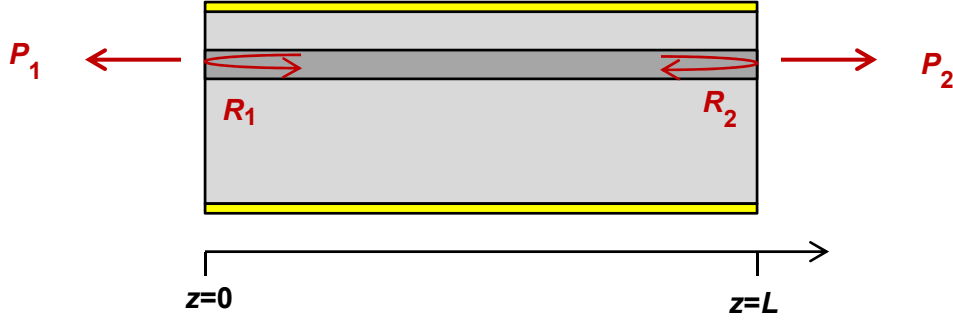
12.1.2 Mirror Reflectivities and Output Power:

Consider a Fabry-Perot optical cavity with mirror reflectivities R_1 and R_2 . The optical power comes from out both mirrors of the laser cavity and the total output power is,

$$P = \left(\frac{\tilde{\alpha}_m}{\tilde{\alpha}_m + \tilde{\alpha}} \right) \hbar\omega \frac{n_p V_p}{\tau_p} = \eta_o \hbar\omega \frac{n_p V_p}{\tau_p}$$

The output power is usually collected form only one mirror of the laser and the question is how the output power is distributed among the two mirrors. Consider the Fabry-Perot laser cavity shown below. If the optical power travelling in the cavity in the +z-direction at $z = L$ is $P^+(L)$, then the optical power coming out of the right mirror of the cavity is $P_2 = P^+(L)(1 - R_2)$. Similarly, if the power travelling in the cavity in the -z-direction at $z = 0$ is $P^-(0)$, then the optical power coming out of the right mirror of the cavity is $P_1 = P^-(0)(1 - R_1)$. We must have,

$$P_1 + P_2 = P = \left(\frac{\tilde{\alpha}_m}{\tilde{\alpha}_m + \tilde{\alpha}} \right) \hbar\omega \frac{n_p V_p}{\tau_p} = \eta_o \hbar\omega \frac{n_p V_p}{\tau_p}$$



In order to uniquely determine P_1 and P_2 we must obtain one more expression relating them. We have,

$$P^-(0) = P^+(L)R_2 e^{(\Gamma_a \tilde{g} - \tilde{\alpha})L}$$

Using the lasing condition,

$$R_1 R_2 e^{(\Gamma_a \tilde{g} - \tilde{\alpha})2L} = 1$$

we get,

$$P^-(0) = P^+(L) \frac{\sqrt{R_2}}{\sqrt{R_1}}$$

$$\Rightarrow P_1 = P^-(0)(1 - R_1) = P^+(L) \frac{(1 - R_1)\sqrt{R_2}}{\sqrt{R_1}} = P_2 \frac{(1 - R_1)\sqrt{R_2}}{(1 - R_2)\sqrt{R_1}}$$

$$\Rightarrow \frac{P_1}{P_2} = \frac{(1 - R_1)\sqrt{R_2}}{(1 - R_2)\sqrt{R_1}}$$

The above expression shows that more power comes out of the less reflective mirror. We can now write,

$$P_2 = \frac{(1 - R_2)\sqrt{R_1}}{(1 - R_2)\sqrt{R_1} + (1 - R_1)\sqrt{R_2}} \eta_0 \hbar \omega \frac{n_p V_p}{\tau_p}$$

$$P_1 = \frac{(1 - R_1)\sqrt{R_2}}{(1 - R_2)\sqrt{R_1} + (1 - R_1)\sqrt{R_2}} \eta_0 \hbar \omega \frac{n_p V_p}{\tau_p}$$

If one wants most of the power to come out from any one of the two mirrors, then that mirror must have a low reflectivity and the other mirror must have a high reflectivity.

12.2 The S-Matrix and the T-Matrix Techniques

12.2.1 Introduction:

Consider a dielectric interface with two normally incident waves with transverse components,

$$\vec{E}_{a1} = \hat{n}_t a_1 e^{i\beta_1 z} \quad \vec{E}_{b2} = \hat{n}_t b_2 e^{-i\beta_2 z}$$

and two outgoing plane waves,

$$\vec{E}_{a2} = \hat{n}_t a_2 e^{i\beta_2 z} \quad \vec{E}_{b1} = \hat{n}_t b_1 e^{-i\beta_1 z}$$

The amplitudes of the outgoing waves can be related to those of the incoming waves through the S-matrix,

$$\begin{bmatrix} b_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ b_2 \end{bmatrix}$$

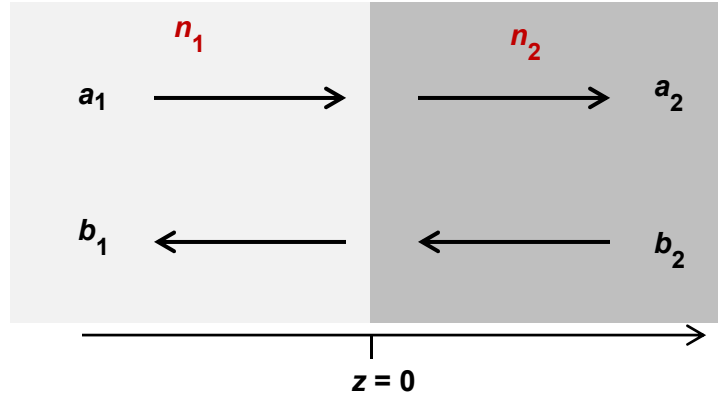
where,

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$$S_{11} = r = \frac{n_1 - n_2}{n_1 + n_2} \quad S_{22} = -r = \frac{n_2 - n_1}{n_1 + n_2}$$

$$S_{12} = t' = \frac{2n_2}{n_1 + n_2} \quad S_{21} = t = \frac{2n_1}{n_1 + n_2}$$

$$\Rightarrow S = \begin{bmatrix} r & t' \\ t & -r \end{bmatrix}$$



Note that $|S_{11}|^2$ and $|S_{21}|^2 n_2/n_1$ represent the reflectivity and the transmittivity, respectively, for a wave incident from the left, and $|S_{22}|^2$ and $|S_{12}|^2 n_1/n_2$ is the represent the reflectivity and the transmittivity, respectively, for a wave incident from the right. Whereas the S-matrix relates the amplitudes of the outgoing waves to that of the incoming waves, the T-matrix relates the amplitudes of the waves on the left side of the interface to that of the waves on the right side of the interface,

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$$

The relation between T-matrix and S-matrix coefficients is,

$$T_{11} = \frac{1}{S_{21}} \quad T_{12} = -\frac{S_{22}}{S_{21}} \quad T_{21} = \frac{S_{11}}{S_{21}} \quad T_{22} = S_{12} - \frac{S_{11}S_{22}}{S_{21}}$$

or,

$$S_{21} = \frac{1}{T_{11}} \quad S_{22} = -\frac{T_{12}}{T_{11}} \quad S_{11} = \frac{T_{21}}{T_{11}} \quad S_{12} = T_{22} - \frac{T_{12}T_{21}}{T_{11}}$$

So for the dielectric interface considered above, the T-matrix is,

$$T = \begin{bmatrix} 1/t & r/t \\ r/t & 1/t \end{bmatrix}$$

Now consider free propagation of waves (without reflections), as shown below. The S- and T-matrices are,

$$\text{T-matrix: } \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} e^{-i\beta L} & 0 \\ 0 & e^{i\beta L} \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$$

$$\text{S-matrix: } \begin{bmatrix} b_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 & e^{i\beta L} \\ e^{i\beta L} & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ b_2 \end{bmatrix}$$

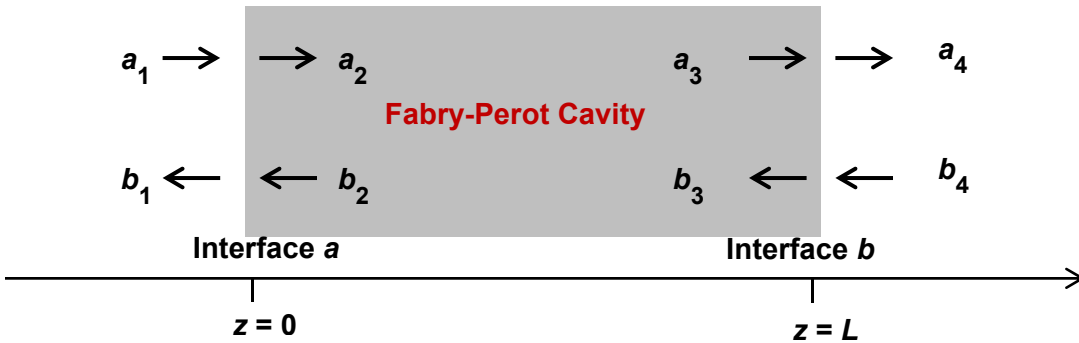


If the medium has gain/loss then the S- and T-matrices become,

$$\begin{aligned}
 \text{T-matrix: } & \begin{bmatrix} e^{-i\beta L} e^{-\left(\Gamma_a \frac{\tilde{g}}{2} - \frac{\tilde{\alpha}}{2}\right)L} & 0 \\ 0 & e^{i\beta L} e^{\left(\Gamma_a \frac{\tilde{g}}{2} - \frac{\tilde{\alpha}}{2}\right)L} \end{bmatrix} \\
 \text{S-matrix: } & \begin{bmatrix} 0 & e^{i\beta L} e^{\left(\Gamma_a \frac{\tilde{g}}{2} - \frac{\tilde{\alpha}}{2}\right)L} \\ e^{i\beta L} e^{\left(\Gamma_a \frac{\tilde{g}}{2} - \frac{\tilde{\alpha}}{2}\right)L} & 0 \end{bmatrix}
 \end{aligned}$$

12.2.2 Fabry-Perot Cavity Treated via S-Matrix and T-Matrix:

Now we have all the ingredients needed to analyze a simple Fabry-Perot Laser cavity using S- and T-matrices. We shall see below how the S-matrix and the T-matrix can be used to calculate the frequencies of the cavity longitudinal modes as well as the threshold gain for the laser. Consider the Cavity shown below.



The T-matrices can be concatenated as shown below enabling one to obtain the T-matrix of the entire cavity,

$$\begin{aligned}
 \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} &= \begin{bmatrix} 1 & r_a \\ t_1 & t_a \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 & r_a \\ r_a & 1 \\ t_a & t_a \end{bmatrix} \begin{bmatrix} e^{-i\beta L} e^{-\left(\Gamma_a \frac{\tilde{g}}{2} - \frac{\tilde{\alpha}}{2}\right)L} & 0 \\ 0 & e^{i\beta L} e^{\left(\Gamma_a \frac{\tilde{g}}{2} - \frac{\tilde{\alpha}}{2}\right)L} \end{bmatrix} \begin{bmatrix} a_3 \\ b_3 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & r_a \\ t_a & t_a \\ r_a & 1 \\ t_a & t_a \end{bmatrix} e^{-i\beta L} e^{-\left(\Gamma_a \frac{\tilde{g}}{2} - \frac{\tilde{\alpha}}{2}\right)L} \begin{bmatrix} 0 & 1 \\ e^{i\beta L} e^{\left(\Gamma_a \frac{\tilde{g}}{2} - \frac{\tilde{\alpha}}{2}\right)L} & 0 \end{bmatrix} \begin{bmatrix} 1 & r_b \\ t_b & t_b \\ r_b & 1 \\ t_b & t_b \end{bmatrix} \begin{bmatrix} a_4 \\ b_4 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} a_4 \\ b_4 \end{bmatrix}
 \end{aligned}$$

Once the T-matrix of the structure has been calculated this way, the S-matrix of the whole structure can be obtained by using the formulas described earlier,

$$\begin{bmatrix} b_1 \\ a_4 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ b_4 \end{bmatrix}$$

Lasing means light coming out of the cavity even when no light is going into the cavity, i.e. $b_1 \neq 0$ and $a_1 \neq 0$ even when $a_1 = 0$ and $b_4 = 0$. How can this be stated in the language of the S-matrix? Looking at the equation above, one can have $b_1 \neq 0$ and $a_4 \neq 0$ even when $a_1 = b_4 = 0$ provided the coefficients of the S-matrix are infinite. For example, for the cavity under consideration, S_{21} is,

$$S_{21} = \frac{t_a t_b e^{i\beta L} e^{(\Gamma_a \tilde{g} - \tilde{\alpha}) \frac{L}{2}}}{1 + r_a r_b e^{i\beta 2L} e^{(\Gamma_a \tilde{g} - \tilde{\alpha}) L}}$$

S_{21} is infinite if the denominator is zero. The denominator is zero when,

$$r_a r_b e^{(\Gamma_a \tilde{g} - \tilde{\alpha}) L} e^{i\beta 2L} = -1$$

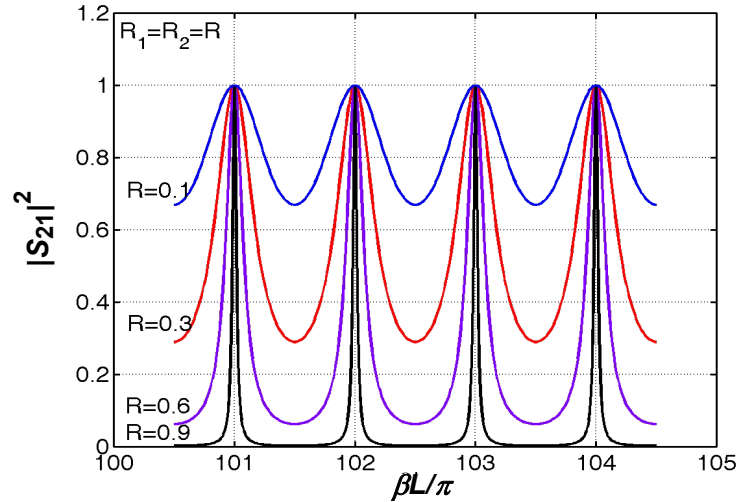
Note that r_a is the reflection coefficient for a wave incident from outside the cavity on the left mirror of the cavity. The lasing condition given above is the same as the one obtained earlier if we realize that,

$$\begin{aligned} r_a &= \sqrt{R_1} e^{i\pi} & r_b &= \sqrt{R_2} \\ \Rightarrow \sqrt{R_1 R_2} e^{(\Gamma_a \tilde{g} - \tilde{\alpha}) L} e^{i\beta 2L} &= 1 \end{aligned}$$

One can write,

$$S_{21} = \frac{t_a t_b e^{i\beta L} e^{(\Gamma_a \tilde{g} - \tilde{\alpha}) \frac{L}{2}}}{1 - \sqrt{R_1 R_2} e^{i\beta 2L} e^{(\Gamma_a \tilde{g} - \tilde{\alpha}) L}}$$

Information about the cavity modes and the threshold gain can be obtained from the S-matrix (and in particular from the coefficient S_{21}). Below we discuss how to extract this information step by step.



Step 1: Assume $\tilde{g} = 0$ (passive cavity) and plot $|S_{21}|^2$ as a function of β . The longitudinal modes of the cavity correspond to those values of β for which $|S_{21}|^2$ has a maximum (i.e. has a resonance). We can see from the expression above that $|S_{21}|^2$ has a maximum when,

$$\beta L = n\pi \quad \Rightarrow \quad \beta = n \frac{\pi}{L} \quad \{n = 1, 2, 3, \dots\}$$

From the knowledge of β as a function of ω , the frequencies of the cavity modes can be computed. The Figure above shows $|S_{21}|^2$ plotted as a function of β for different values of the cavity facet (mirror) reflectivities assuming $\tilde{\alpha} = 0$. The resonances in $|S_{21}|^2$ correspond to the cavity longitudinal modes. Resonances in $|S_{21}|^2$ become narrower when the cavity mirrors have higher reflectivities and/or the waveguide losses are smaller. If the resonances are sharp and narrow, one can obtain the width of a resonance by expanding β around the resonance. Suppose,

$$\beta = n \frac{\pi}{L} + \Delta\beta$$

$$|S_{21}|^2 \approx \left| \frac{t_a t_b e^{-\tilde{\alpha}L/2}}{1 - \sqrt{R_1 R_2} (1 + i\Delta\beta 2L) e^{-\tilde{\alpha}L}} \right|^2 = \frac{|t_a t_b|^2 e^{-\tilde{\alpha}L}}{\left(1 - \sqrt{R_1 R_2} e^{-\tilde{\alpha}L}\right)^2 + \left(\sqrt{R_1 R_2} \Delta\beta 2L e^{-\tilde{\alpha}L}\right)^2}$$

The full width at half maximum of the resonance is,

$$\Delta\beta_{\text{FWHM}} = \frac{1}{L} \left[\frac{1 - \sqrt{R_1 R_2} e^{-\tilde{\alpha}L}}{\sqrt{R_1 R_2} e^{-\tilde{\alpha}L}} \right]$$

The width of the resonance in frequency is therefore,

$$\Delta\omega_{\text{FWHM}} = \frac{v_g}{L} \left[\frac{1 - \sqrt{R_1 R_2} e^{-\tilde{\alpha}L}}{\sqrt{R_1 R_2} e^{-\tilde{\alpha}L}} \right]$$

Photon Lifetime and Cavity Quality Factor: The term ‘‘cavity mode photon lifetime’’ is usually associated with the inverse of the width of the resonance in frequency,

$$\frac{1}{\tau_p} = \Delta\omega_{\text{FWHM}} = \frac{v_g}{L} \left[\frac{1 - \sqrt{R_1 R_2} e^{-\tilde{\alpha}L}}{\sqrt{R_1 R_2} e^{-\tilde{\alpha}L}} \right]$$

Earlier, we had derived the following expression for the photon lifetime in a Fabry-Perot cavity,

$$\frac{1}{\tau_p} = \frac{v_g}{L} \log\left(\frac{1}{\sqrt{R_1 R_2}}\right) + v_g \tilde{\alpha} = v_g \tilde{\alpha}_m + v_g \tilde{\alpha}$$

The above two expression for the photon lifetime are not too different. The difference is due to the fact that the photon density in a Fabry-Perot laser cavity in the presence of gain or internal loss is not uniform along the length of the cavity (as was also seen in our analysis of the SOAs). The difference disappears if the cavity losses are small,

$$\begin{aligned} \frac{v_g}{L} \log\left(\frac{1}{\sqrt{R_1 R_2}}\right) + v_g \tilde{\alpha} &= \frac{v_g}{L} \log\left(\frac{1}{\sqrt{R_1 R_2}}\right) + \frac{v_g}{L} \log(e^{\tilde{\alpha}L}) = \frac{v_g}{L} \log\left(\frac{1}{\sqrt{R_1 R_2} e^{-\tilde{\alpha}L}}\right) \\ &= \frac{v_g}{L} \log\left(1 + \frac{1}{\sqrt{R_1 R_2} e^{-\tilde{\alpha}L}} - 1\right) = \frac{v_g}{L} \log\left(1 + \frac{1 - \sqrt{R_1 R_2} e^{-\tilde{\alpha}L}}{\sqrt{R_1 R_2} e^{-\tilde{\alpha}L}}\right) \\ &\approx \frac{v_g}{L} \left[\frac{1 - \sqrt{R_1 R_2} e^{-\tilde{\alpha}L}}{\sqrt{R_1 R_2} e^{-\tilde{\alpha}L}} \right] \quad \text{if : } 1 - \sqrt{R_1 R_2} e^{-\tilde{\alpha}L} \ll 1 \end{aligned}$$

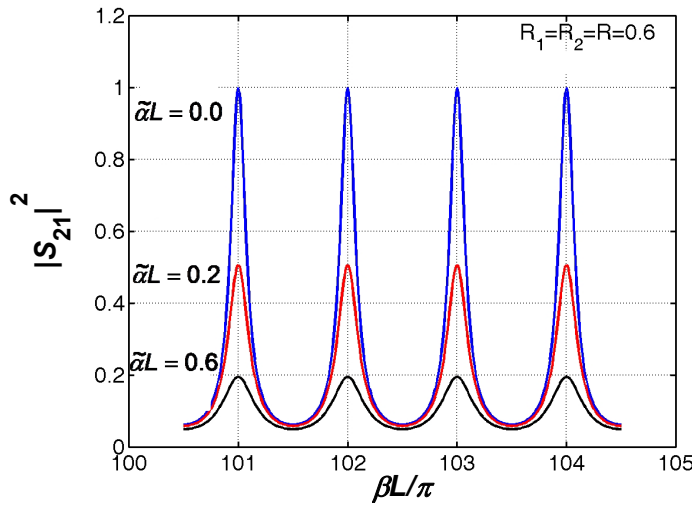
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A cavity with smaller losses requires smaller gain to achieve lasing threshold and, therefore, the photon density distribution long the length of the cavity is also more uniform during laser operation. The photon lifetime for any optical cavity can be (with certain rare exceptions) put in the form,

$$\frac{1}{\tau_p} = \frac{1}{\tau_p^{\text{ext}}} + \frac{1}{\tau_p^{\text{int}}}$$

Here, the first term on the right hand side describes cavity loss rate due to photons escaping from the cavity into the external world (e.g., the mirror loss) and the second term on the right hand side describes cavity loss rate due to photons getting absorbed inside the cavity. The cavity quality factor Q is defined as,

$$Q = \frac{\omega}{\Delta\omega_{\text{FWHM}}} = \omega\tau_p \quad \Rightarrow \quad \frac{1}{Q} = \frac{1}{Q_{\text{text}}} + \frac{1}{Q_{\text{int}}}$$

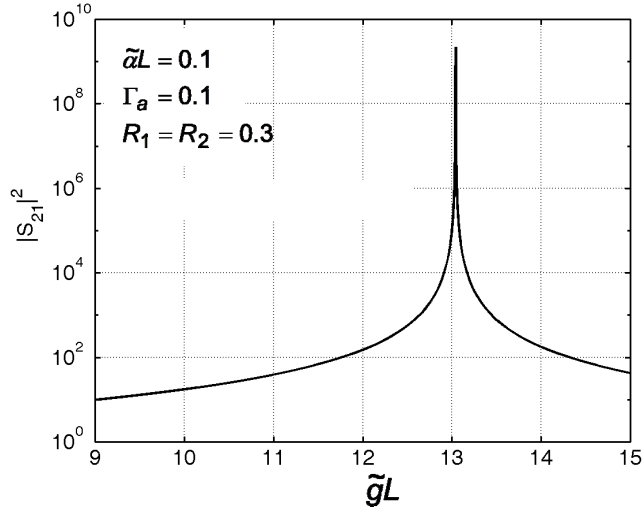


Since $|S_{21}|^2$ is the transmittivity through the cavity, the maximum value of $|S_{21}|^2$ at a resonance is unity when $\tilde{\alpha} = 0$. When $\tilde{\alpha} \neq 0$, the maximum value of $|S_{21}|^2$ is less than unity, as shown in the Figure above.

Step 2: Now suppose the material gain \tilde{g} is not zero. As \tilde{g} is increased from zero, the cavity resonances become very sharp and the maximum values of $|S_{21}|^2$ (which occur when $\beta L = n\pi$) become very large and exceed unity. When the gain reaches the threshold gain \tilde{g}_{th} the maximum values of $|S_{21}|^2$ become infinite. The recipe for obtaining the threshold gain \tilde{g} is as follows. Choose one of the allowed values of β found in step 1 above, and using this value of β plot $|S_{21}|^2$ as a function of the gain \tilde{g} . The value of \tilde{g} for which $|S_{21}|^2 \rightarrow \infty$ corresponds to the threshold gain \tilde{g}_{th} . Consider a Fabry-Perot cavity with $R_1 = R_2 = 0.3$, $\tilde{\alpha}L = 0.1$, and $\Gamma_a = 0.1$. Using the expression,

$$\Gamma_a \tilde{g}_{th} = \frac{1}{L} \log \left(\frac{1}{\sqrt{R_1 R_2}} \right) + \tilde{\alpha}$$

the threshold gain $\tilde{g}_{th}L$ is found to be 13.04. The plot of $|S_{21}|^2$ versus the material gain is shown below (the value of β is chosen to be an integral multiple of π). The value of $\tilde{g}L$ for which $|S_{21}|^2 \rightarrow \infty$ is the threshold gain $\tilde{g}_{th}L$. This technique gives the same value of the threshold gain.



12.2.3 Distributed Bragg Reflectors (DBRs):

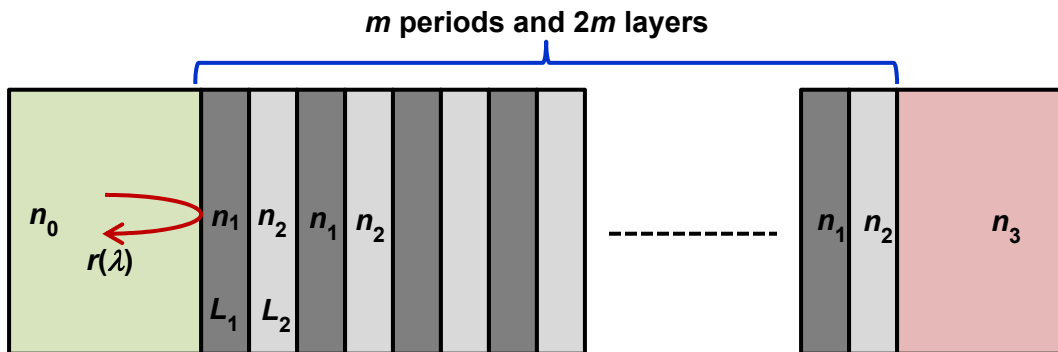
Stacks of multiple dielectric layers are frequently used to perform a variety of tasks in optics. One important use of such a stack is in the realization of high reflectivity mirrors. A DBR mirror consists of alternating dielectric layers of indices n_1 and n_2 forming a periodic structure. One period consists of two layers; one layer with index n_1 and one layer with index n_2 . In a DBR mirror with a peak reflectivity at the (free-space) wavelength λ , the phase accumulated by a wave propagating through one complete period must be π .

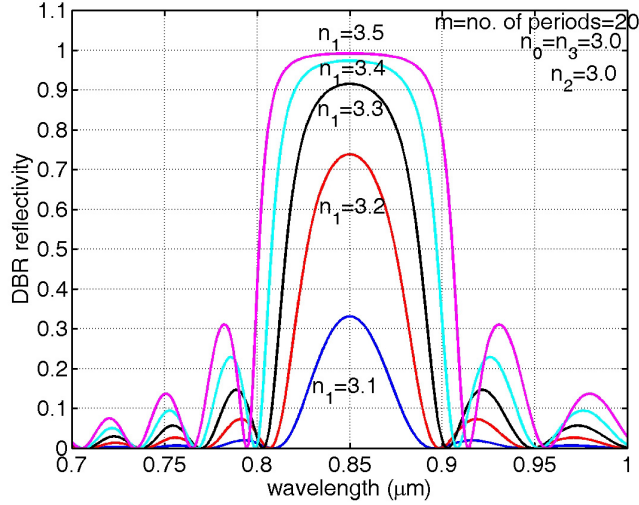
$$\beta_1 L_1 + \beta_2 L_2 = \pi$$

In most cases, in order to meet the above condition, the thickness of each layer is chosen such that the phase accumulated by a wave propagating through the layer is $\pi/2$,

$$\beta_1 L_1 = \frac{\pi}{2} \quad \beta_2 L_2 = \frac{\pi}{2} \quad \Rightarrow \quad L_1 = \frac{\lambda}{4n_1} \quad L_2 = \frac{\lambda}{4n_2}$$

The thickness of each layer is therefore one-quarter of the wavelength of light in that layer. The reflectivity of a DBR mirror is wavelength dependent. The reflectivity $|r(\lambda)|^2$ as a function of the wavelength λ can be obtained by calculating the S-matrix coefficient $S_{11}(\lambda)$ as a function of λ .





This Figure above shows a plot of the reflectivity $R(\lambda) = |r(\lambda)|^2$ as a function of the wavelength λ for $n_0 = n_3 = 3.0$ and $n_2 = 3.0$, and the value of n_1 is varied from 3.1 to 3.5. Suppose the maximum reflectivity occurs at the (free-space) wavelength λ_B . For the m -period DBR structure shown in the Figure above, the reflection coefficient at $\lambda = \lambda_B$ can be obtained by using the S-matrix analysis and comes out to be,

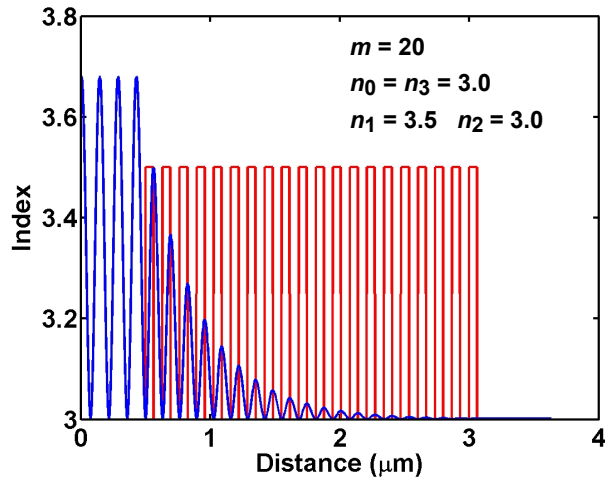
$$r(\lambda_B) = S_{11}(\lambda_B) = \frac{\left(\frac{n_2}{n_1}\right)^{2m} \frac{n_0}{n_3} - 1}{\left(\frac{n_2}{n_1}\right)^{2m} \frac{n_0}{n_3} + 1}$$

The peak reflectivity $|r(\lambda_B)|^2$ depends on the ratio n_2/n_1 as well as on the number of periods m . If $n_2 > n_1$, then $r(\lambda_B)$ approaches +1 as the number of periods is increased, and if $n_2 < n_1$ then $r(\lambda_B)$ approaches -1 as the number of periods is increased. As the wavelength is changed from λ_B , the reflectivity $|r(\lambda)|^2$ decreases.

The DBR is the simplest example of a 1D photonic crystal structure. In the limit $m \rightarrow \infty$, the DBR stack develops frequency bands, called the stop bands or bandgaps, such that light within these bands is unable to propagate within the stack and is therefore completely reflected from the ends. These bandgaps emerge as a result of Bragg scattering of light from the periodic dielectric structure much like the bandgaps that appear in the energy spectrum of electrons in crystals because of Bragg scattering of the electrons from the periodic potential of the atoms. The peak reflectivity occurs when the wavelength of the incident light equals the Bragg wavelength λ_B of the DBR mirror. When the wavelength of the incident light is equal to the Bragg wavelength of the DBR mirror, the reflected waves from every period of the structure add constructively in the backward direction thereby enhancing the reflectivity.

The Figure below shows the intensity of a wave incident on a DBR mirror with wavelength equal to the Bragg wavelength of the DBR mirror. The reflectivity of the mirror is near unity and the envelope of the wave intensity decays exponentially within the mirror. Note that the wavelength of the incident wave is within the bandgap of the DBR mirror and that is why the wave intensity envelope decays exponentially within the mirror. For the same reason, in a quantum well the envelope function of the electron

wavefunction in the effective mass approximation decays exponentially inside the barrier region. If the number of periods in a DBR mirror is small, incident light with wavelength within the bandgap can tunnel through the mirror, similar to how electrons tunnel through thin heterostructure barriers, and the DBR reflectivity will be small.

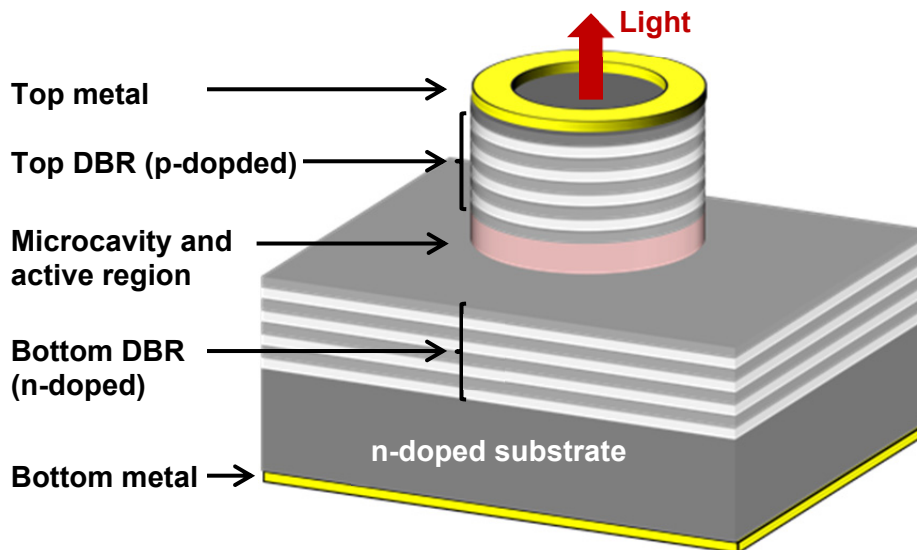


12.3 Vertical Cavity Surface Emitting Laser (VCSEL)

12.3.1 Introduction:

The VCSEL has become the work horse of modern fiber optic communication links. The structure of a VCSEL is shown in the Figure below. VCSEL uses two high reflectivity DBR mirrors to make an optical microcavity and the gain region (which usually consists of few quantum wells) is placed in the middle of the microcavity. The details of the microcavity and the active region are shown below. The mode field inside the VCSEL microcavity propagating in the forward direction can be written as,

$$\vec{E}(x, y, z) = [\hat{x}E_x(x, y) + \hat{y}E_y(x, y) + \hat{z}E_z(x, y)]e^{i\beta z}$$



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The transverse mode profile, and the number of supported transverse modes, are both determined by the transverse dimensions of the cavity. Assuming weak transverse mode confinement (large area device), the propagation vector in each layer is related to the index in that layer by the simple relation,

$$\beta = \frac{\omega}{c} n$$

The group velocity of the mode in each layer is then equal to the material group velocity,

$$v_g = \frac{c}{n_g^M} = v_g^M$$

Suppose the top and bottom DBRs have reflection coefficients r_1 and r_2 and both are approximately equal to -1. Consider a wave inside the microcavity bouncing back and forth between the two mirrors (ignore the quantum wells for now). The roundtrip condition for the phase is,

$$2\beta L + \pi + \pi = 2\pi m \quad \{m = 1, 2, 3, \dots\}$$

$$\Rightarrow \beta L = \pi m \quad \{m = 1, 2, 3, \dots\}$$

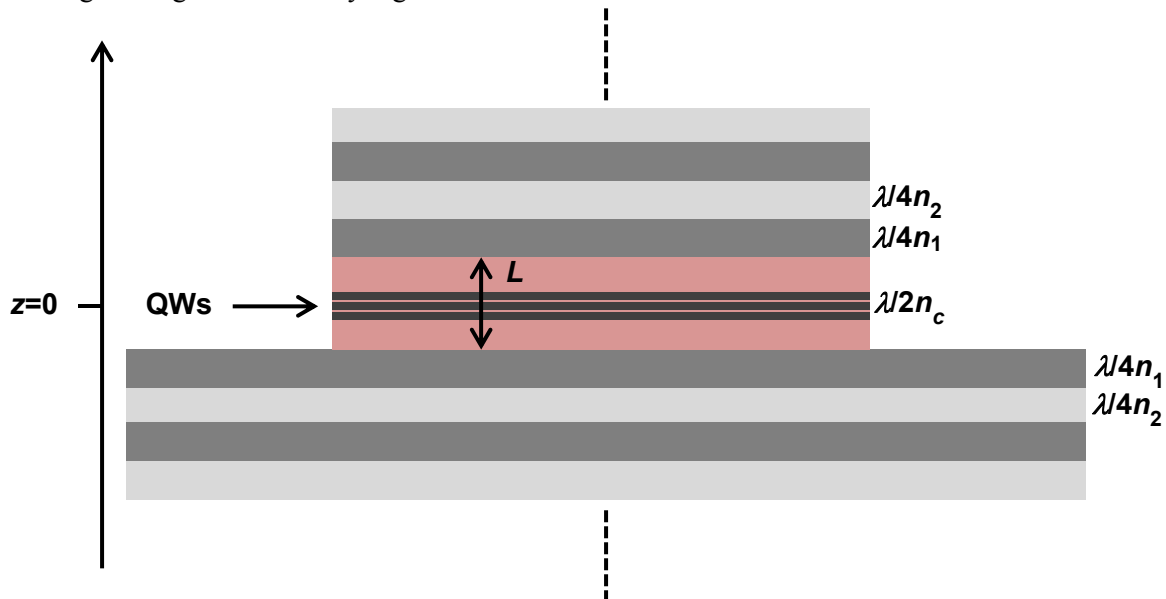
Different values of β correspond to different longitudinal modes of the cavity. Since the mirror reflectivities are close to unity, the forward and backward waves must have the same magnitudes and therefore the field amplitude inside the cavity must be proportional to either $\cos(\beta z)$ or $\sin(\beta z)$. Since the mirror reflection coefficients are approximately -1, the field amplitude at the mirrors must be close to zero as a result of destructive interference between the incident and the reflected waves. If the field amplitude inside the cavity is proportional to $\cos(\beta z)$, then,

$$\cos\left(\beta \frac{L}{2}\right) = 0$$

$$\Rightarrow \beta L = p\pi \quad \{p = 1, 3, 5, \dots\}$$

$$\Rightarrow L = p \frac{\lambda}{2n_c}$$

Here, n_c is the index of the cavity region. The smallest cavity would have a length equal to half the wavelength of light in the cavity region.



If the field amplitude is proportional to $\sin(\beta z)$ then,

$$\begin{aligned} \sin\left(\beta \frac{L}{2}\right) &= 0 \\ \Rightarrow \beta L &= p\pi \quad \{ p = 2, 4, 6, \dots \} \\ \Rightarrow L &= p \frac{\lambda}{2n_c} \end{aligned}$$

The smallest cavity would then have a length equal to the wavelength of light in the cavity region. If the field amplitude is proportional to $\sin(\beta z)$ then the field intensity will be zero or very small in the middle of the cavity where the quantum wells are usually located. Therefore, the cavity lengths are chosen such that the sine modes are eliminated. In most cases, the cavity length is half the wavelength of light in the cavity region, as shown in the Figure above, and therefore the cavity supports only a single longitudinal mode (the lowest one). A single transverse mode and a single longitudinal mode imply that the VCSEL would have only a single cavity mode.

The lowest cavity longitudinal mode satisfies the condition,

$$\beta L = \pi$$

If the cavity region consists of layers with different indices, such as multiple quantum wells, then the lowest cavity longitudinal mode is given by the condition,

$$\sum_k \beta_k L_k = \pi$$

where, β_k is the propagation vector in the k -th region of thickness L_k inside the cavity.

In the presence of material gain or loss, the index and the propagation vector in each layer acquire imaginary parts,

$$\begin{aligned} \text{index} = n &= n' + in'' = n' - i \frac{g}{2\omega} + j \frac{\alpha}{2\omega} \\ \beta &= \frac{\omega}{c} n = \frac{\omega}{c} n' - i \frac{g}{2} + j \frac{\alpha}{2} \end{aligned}$$

Note that since the mode group velocity in each layer is assumed to be equal to the material group velocity (weak transverse mode confinement assumption), $\tilde{g} = g$ and $\tilde{\alpha} = \alpha$.

12.3.2 S-Matrix and T-Matrix Analysis of VCSELs:

S- and T-matrices provide the simplest computational technique to determine the wavelength of the cavity mode as well as the threshold gain. To illustrate the technique, we consider a 850 nm **GaAs/AlGaAs** VCSEL shown in the Figure below. The bottom DBR is made of 30 periods of **Al_{0.2}Ga_{0.8}As/Al_{0.9}Ga_{0.1}As** layers whose indices at 850 nm are 3.49/3.06, respectively. The top DBR is made of 25 periods of **Al_{0.2}Ga_{0.8}As/Al_{0.9}Ga_{0.1}As** layers. The optical cavity is made of **Al_{0.2}Ga_{0.8}As** with three 80 Å **GaAs** quantum wells embedded in the center of the cavity separated by 80 Å **Al_{0.2}Ga_{0.8}As** barriers. The index of **GaAs** at 850 nm is 3.65. The substrate is **GaAs** (index 3.65), and the top most layer (also called the cap layer) is assumed to be **Al_{0.2}Ga_{0.8}As**.

DBR Mirrors:

If the DBRs are to have a maximum reflectivity at 850 nm, the Bragg wavelength λ_B , then the thicknesses of the **Al_{0.2}Ga_{0.8}As** and **Al_{0.9}Ga_{0.1}As** layers in the DBR need to be $\lambda_B/4(3.49)$ and $\lambda_B/4(3.06)$, respectively. The S-matrix and the T-matrix techniques can be used to calculate the

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reflectivities of the two DBRs. Assuming $\alpha = 0$ for all DBR layers, the reflectivities, R_1 and R_2 , of the bottom and top DBRs, respectively, at 850 nm are found to be,

$$R_1 = 0.998562$$

$$R_2 = 0.994433$$

VCSEL mirror reflectivities are typically close to unity. Material losses in DBRs, if present, can decrease the DBR reflectivities. For lossless DBR mirrors one must have,

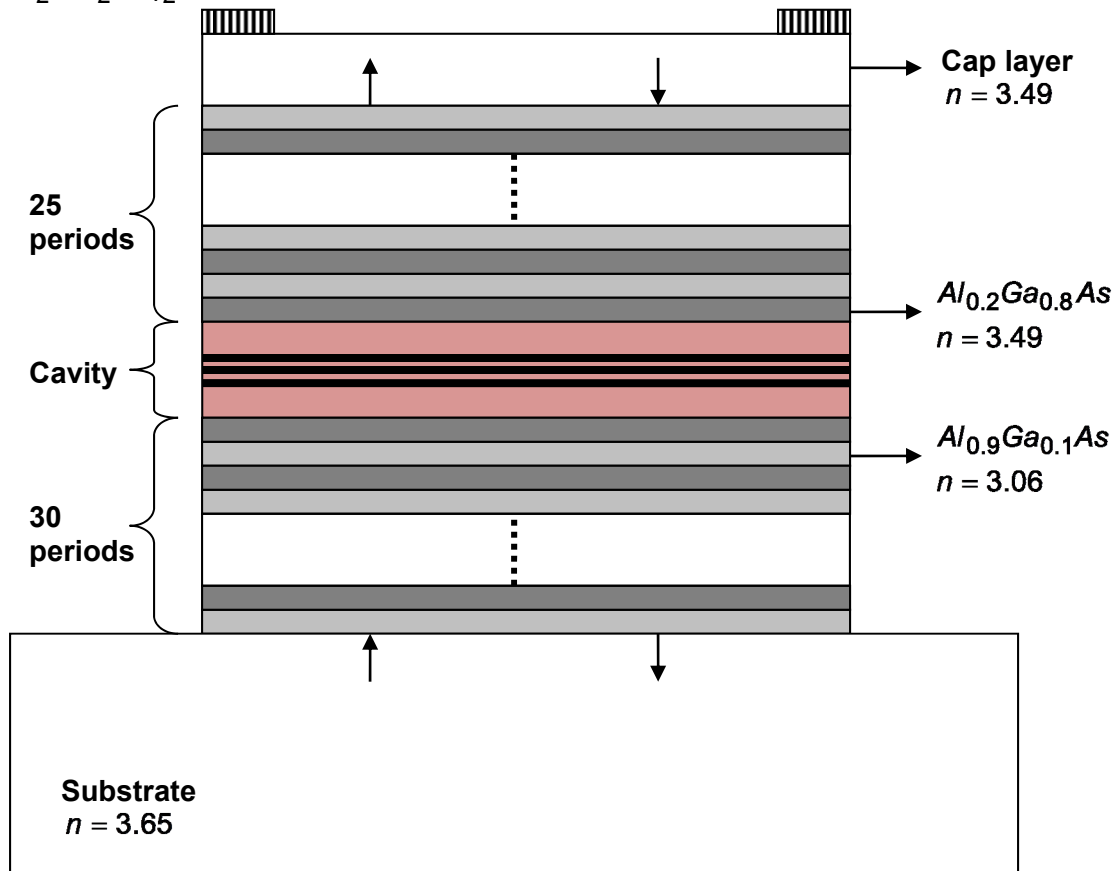
$$R_1 + T_1 = 1$$

$$R_2 + T_2 = 1$$

If the DBR mirrors have internal losses (due to free-carrier absorption, for example) then,

$$R_1 + T_1 = \eta_1 < 1$$

$$R_2 + T_2 = \eta_2 < 1$$



Microcavity Mode:

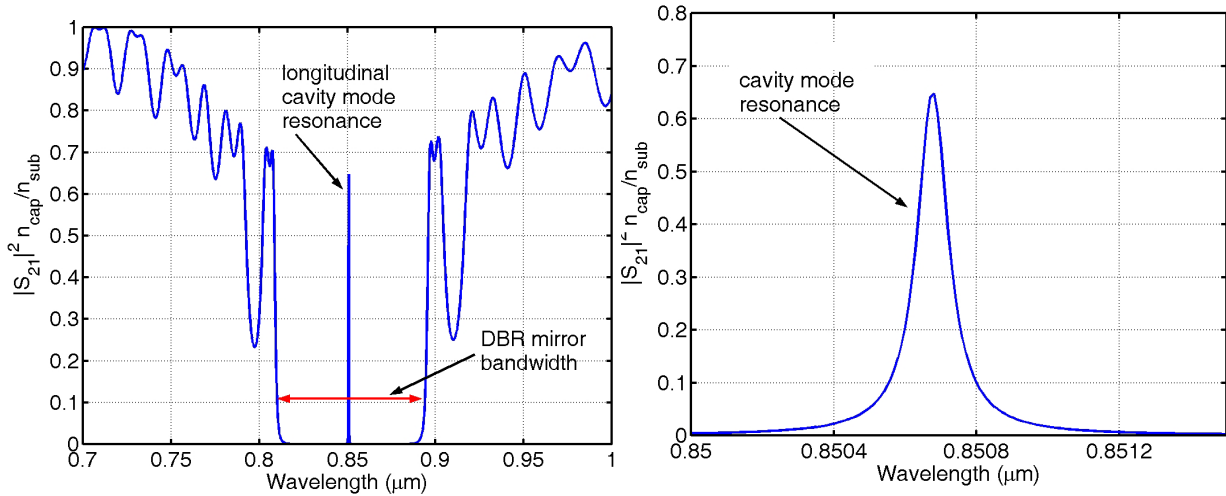
The thicknesses of the layers in the cavity region must satisfy the following condition in order to allow only the lowest longitudinal mode,

$$\sum_k \beta_k L_k = \pi$$

Using the S- and T-matrix techniques, we calculate the transmittivity, given by $|S_{21}|^2 n_{cap}/n_{sub}$, of a wave going from the substrate into the cap layer as a function of the wavelength while assuming that the values of gain g and the loss α are zero for all layers. The results are shown in the Figure below. The transmittivity is very small for wavelengths within the bandgap of the two DBRs. However, at the wavelength corresponding to the cavity mode, the transmittivity is very high. This behavior is similar to

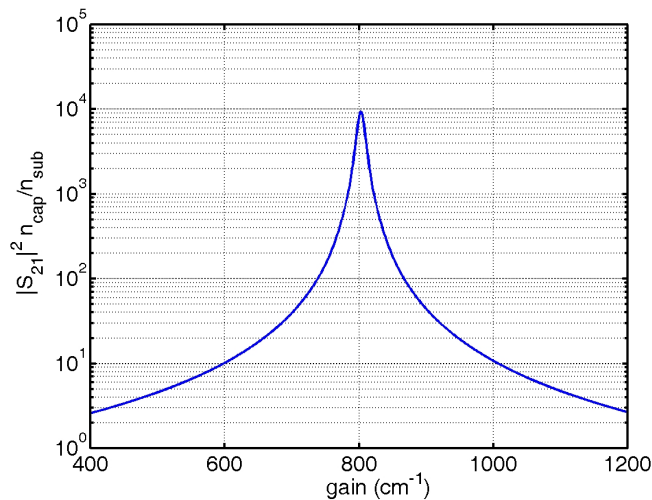
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what was seen in the case of the Fabry-Perot cavity where resonances in $|S_{21}|^2$ corresponded to the cavity longitudinal modes. The cavity mode wavelength obtained from the S-matrix analysis is, $\lambda_0 = 0.850677 \mu\text{m}$ which is close to the intended design wavelength. One needs the wavelength of the cavity mode to this much accuracy in order to calculate the threshold gain, as discussed below.

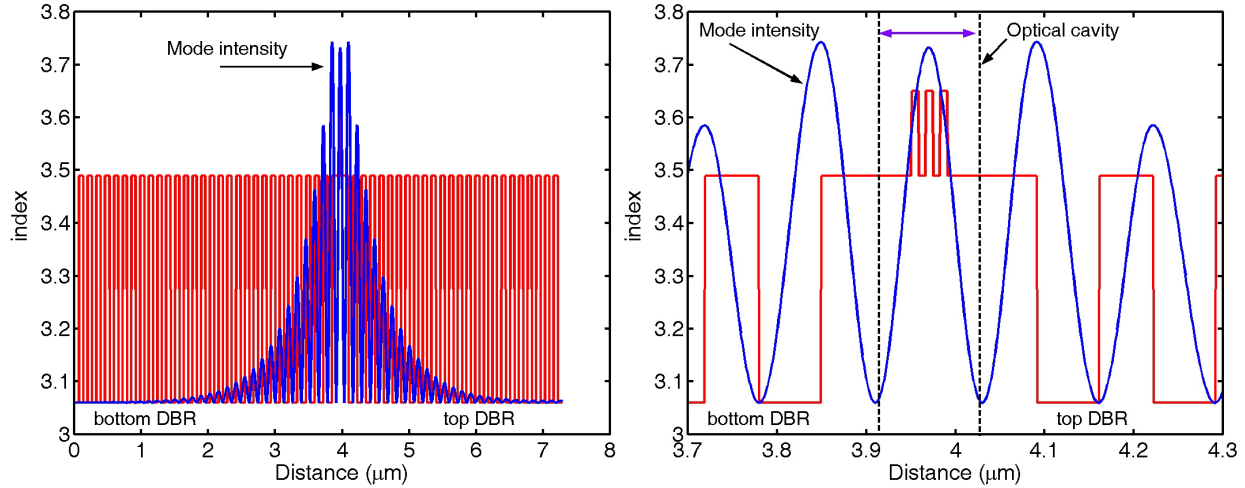


Threshold Gain and Cavity Photon Lifetime:

Once the wavelength of the cavity mode has been determined, the next step is to find the threshold gain. The total cavity loss consists of the cavity internal loss, described by the value of α for each layer, and the mirror loss due to photons escaping from the cavity from the two DBR mirrors into the outside world. We will first assume that $\alpha = 0$ for all layers and the only source of photon loss is the cavity mirror loss. In order to calculate the threshold gain, we assume that the wavelength is equal to the computed cavity mode wavelength λ_0 , and plot $|S_{21}|^2$ as a function of the material gain g and see which value of the gain makes $|S_{21}|^2$ infinite. Numerical errors will not let you see $|S_{21}|^2$ actually become infinite at the threshold gain, but it is easy to identify the gain value for which $|S_{21}|^2$ peaks. The Figure below shows $|S_{21}|^2$ plotted as a function of the active region material gain. $|S_{21}|^2$ peaks when $g = g_{th} = 803 \text{ 1/cm}$.



To proceed further, we need to have more knowledge about the cavity mode. More specifically, we need to find the active region mode confinement factor Γ_a . A slightly modified version of the T-matrix analysis, explored in detail in the homework set, allows the computation of the cavity mode. The results are displayed in the Figure below which shows the longitudinal mode intensity in the entire device.



The Figure above shows that the mode intensity has the expected $\cos^2(\beta z)$ dependence inside the cavity (where \mathbf{z} is measured from the center of the cavity). A considerable amount of mode energy is also present within both the top DBR and the bottom DBR. The envelope of the mode intensity decays exponentially inside the DBRs. Once the optical mode has been obtained, the mode confinement factor for the active region can be obtained using,

$$\Gamma_a = \frac{\int_{\text{active}} \epsilon_0 n n_g^M \bar{\mathbf{E}} \cdot \bar{\mathbf{E}}^* d^3\bar{\mathbf{r}}}{\int \epsilon_0 n n_g^M \bar{\mathbf{E}} \cdot \bar{\mathbf{E}}^* d^3\bar{\mathbf{r}}}$$

For the mode shown in the above Figure, the active region mode confinement Γ_a is .0619. Knowing the threshold gain g_{th} and Γ_a , one can calculate the mirror loss α_m using the expression,

$$\Gamma_a g_{th} = \alpha_m$$

The value of α_m comes out to be approximately 50 1/cm for the cavity under consideration. Once α_m has been determined this way, one can repeat the calculation of the threshold gain but this time let the loss α in each layer have its actual value. The value of the threshold gain thus obtained would correspond to the total cavity loss,

$$\Gamma_a g_{th} = \alpha_m + \alpha_i$$

Here, α_i is the total cavity internal loss and its value can be obtained from the above equation. There is an important assumption being made here; the mirror loss and the cavity internal loss are additive. This assumption will hold if the cavity internal loss, especially the contribution to it from the loss in the DBR mirrors, is not too large. Cavity photon lifetime is related to the total cavity loss as follows,

$$\Gamma_a v_g^M g_{th} = v_g^M (\alpha_m + \alpha_i) = \frac{1}{\tau_p}$$

where v_g^M is the material group velocity in the active region.

Output Power:

The output coupling efficiency is,

$$\eta_o = \frac{\alpha_m}{\alpha_m + \alpha_i}$$

Suppose the output power is only collected from the top DBR mirror. If the DBR mirrors are lossless then,

$$\eta_o = \frac{(1-R_2)\sqrt{R_1}}{(1-R_2)\sqrt{R_1} + (1-R_1)\sqrt{R_2}} \left(\frac{\alpha_m}{\alpha_m + \alpha_i} \right)$$

The output coupling efficiency in the case of lossy mirrors becomes,

$$\eta_o = \frac{(\eta_2 - R_2)\sqrt{R_1}}{(\eta_2 - R_2)\sqrt{R_1} + (\eta_1 - R_1)\sqrt{R_2}} \left(\frac{\alpha_m}{\alpha_m + \alpha_i} \right)$$

The expression that works in both cases is,

$$\eta_o = \frac{T_2\sqrt{R_1}}{T_2\sqrt{R_1} + T_1\sqrt{R_2}} \left(\frac{\alpha_m}{\alpha_m + \alpha_i} \right)$$

12.3.3 VCSEL Rate Equations:

The VCSEL rate equations can be written as follows,

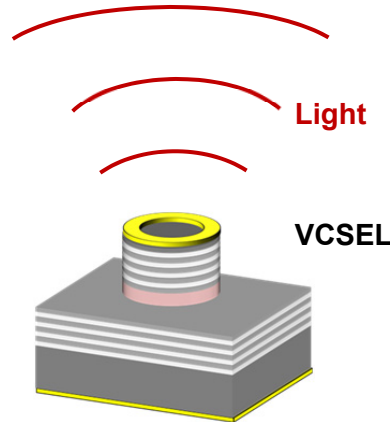
$$\frac{dn_p}{dt} = \left[\Gamma_a v_g^M g - \frac{1}{\tau_p} \right] n_p + \Gamma_a v_g^M g \frac{n_{sp}}{V_p}$$

$$\frac{dn}{dt} = \frac{\eta_i I}{qV_a} - [R_{nr}(n) - G_{nr}(n)] - [R_r(n) - G_r(n)] - v_g^M g n_p$$

Here, v_g^M is the material group velocity in the active region.

12.3.4 Another Look at VCSELs:

With the benefit and availability of sophisticated numerical techniques for solving complex eigenvalue problems, the analysis of optical micro- and nanocavities can be performed in an altogether different way. We consider here a technique that can be applied to any optical cavity but will consider the VCSEL cavity as an example.



The time-dependent field in the complex time-harmonic notation is,

$$\vec{E}(\vec{r}, t) = \text{Re} \left\{ \vec{E}(\vec{r}) e^{-i\omega t} \right\}$$

The complex wave equation,

$$\nabla \times \nabla \times \vec{E}(\vec{r}) = \frac{\omega^2}{c^2} n^2(\omega, \vec{r}) \vec{E}(\vec{r})$$

can be considered a generalized eigenvalue equation of the form,

$$\vec{A} \vec{v} = \lambda \vec{B} \vec{v}$$

where the frequency ω^2 is the eigenvalue, as long as the dispersion of the index is ignored. In the first step, one assumes that the gain and the loss in every layer is zero. The wave equation can be solved numerically with outgoing waves as a boundary condition to yield complex eigenvalues and the corresponding eigenvectors. Since the assumed time dependence of the field is $e^{-i\omega t}$, the real part of the eigenvalue corresponds to the cavity mode frequency and the imaginary part corresponds to the cavity photon lifetime due to photon loss from the cavity,

$$\omega = \omega' + i\omega'' = \omega_0 - \frac{i}{2\tau_p^{\text{ext}}}$$

The corresponding eigenvector is the field of the cavity mode. In what follows we assume that the cavity supports only a single mode. One can now repeat the procedure above and this time include the cavity internal losses. The imaginary part of the frequency will now correspond to the cavity photon lifetime due to photon loss from the cavity as well as due to the cavity internal losses,

$$\omega = \omega' + i\omega'' = \omega_0 - \frac{i}{2} \left(\frac{1}{\tau_p^{\text{ext}}} + \frac{1}{\tau_p^{\text{int}}} \right) = \omega_0 - \frac{i}{2} \left(\frac{1}{\tau_p} \right)$$

Since,

$$\frac{1}{\tau_p^{\text{ext}}} = v_g^M \alpha_m \quad \frac{1}{\tau_p^{\text{int}}} = v_g^M \alpha_i$$

the above technique allows the computation of the cavity mirror and internal losses. In the third and last step, one can include the cavity gain in a perturbative way. Suppose the gain is included as an index perturbation and the change in the index is $\Delta n(\vec{r})$,

$$\Delta n(\omega, \vec{r}) = -i \frac{g(\vec{r})c}{2\omega}$$

Using the complex cavity eigenmode obtained in the second step above, the change in the cavity frequency due to an index change can be calculated using first order perturbation theory,

$$\begin{aligned} \Delta\omega &= -\omega \frac{\int n(\omega, \vec{r}) \Delta n(\omega, \vec{r}) \vec{E}^*(\vec{r}) \cdot \vec{E}(\vec{r}) d^3\vec{r}}{\int n(\omega, \vec{r}) n_g^M(\omega, \vec{r}) \vec{E}^*(\vec{r}) \cdot \vec{E}(\vec{r}) d^3\vec{r}} \\ \Rightarrow \Delta\omega &= \frac{ic}{2} \frac{\int n(\omega, \vec{r}) g(\vec{r}) \vec{E}^*(\vec{r}) \cdot \vec{E}(\vec{r}) d^3\vec{r}}{\int n(\omega, \vec{r}) n_g^M(\omega, \vec{r}) \vec{E}^*(\vec{r}) \cdot \vec{E}(\vec{r}) d^3\vec{r}} = \frac{icg}{2n_g^M} \frac{\int n(\omega, \vec{r}) \vec{E}^*(\vec{r}) \cdot \vec{E}(\vec{r}) d^3\vec{r}}{\int n(\omega, \vec{r}) n_g^M(\omega, \vec{r}) \vec{E}^*(\vec{r}) \cdot \vec{E}(\vec{r}) d^3\vec{r}} \end{aligned}$$

Here, n_g^M without any spatial dependence is assumed to be the material group index of the active region. The ratio of the integral in the above expression is the familiar active region mode confinement factor Γ_a . Therefore,

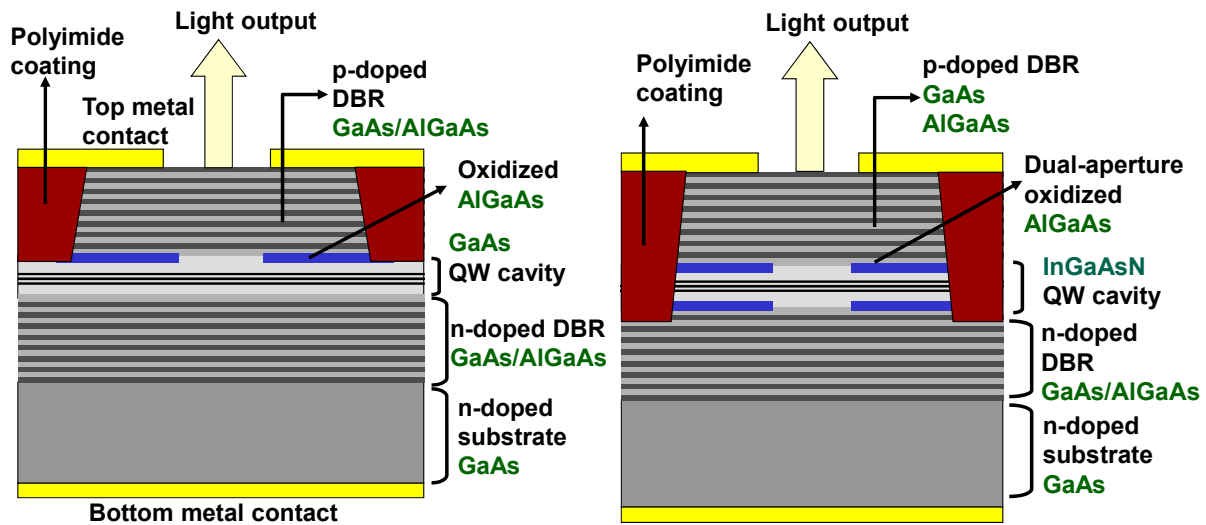
$$\Delta\omega = \frac{i}{2} \Gamma_a v_g^M g$$

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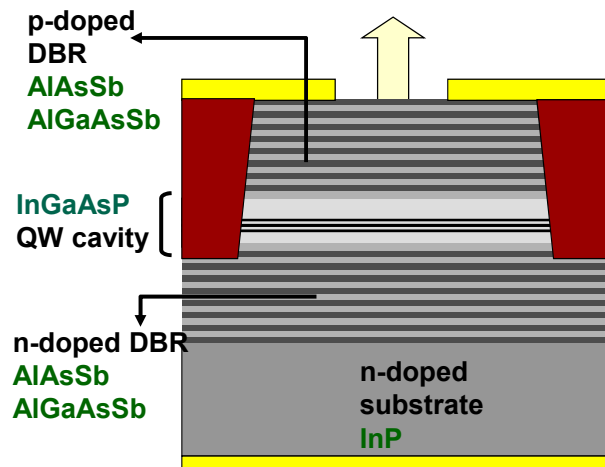
A positive imaginary part of the complex frequency implies a field amplitude gain of $\Gamma_a v_g^M g/2$ and a field energy gain of $\Gamma_a v_g^M g$, as expected.

12.3.4 Some State of the Art VCSEL Structures:

VCSELs are popular in applications requiring 850 nm light. These applications include short distance single-mode and multi-mode optical fiber links, such as those used in local area network (LANs), board-to-board communication in optical routers (CISCO), and in supercomputers. One device commonly used for single cavity mode applications is the oxide apertured VCSEL, shown in the Figure below. In this device, a layer of AlGaAs is oxidized to give an aluminium oxide layer, which is an insulator and has a small refractive index compared to III-V semiconductors. The oxide aperture helps confine the optical mode in the transverse direction and allows only a single transverse mode to lase. In addition, the oxide aperture helps to confine the current as well and enables current pumping of only that part of the active region where the lowest order lasing optical mode resides thereby reducing the laser threshold current. Higher order transverse modes are more spread out in the transverse dimensions, and see more of the unpumped active region and, therefore, experience more loss and find it difficult to lase.



Single cavity mode VCSELs operating at 1300 nm can be realized, for example, by using InGaAsN quantum well in place of GaAs quantum wells, as shown in the Figure above. 1300 nm VCSELs are used for longer distance optical fiber links, such as those used in metro area networks (MANs).



1550 nm VCSELs are more challenging because they require InGaAsP quantum wells on InP substrates and for a long time there were no good dielectric layers that could be used for DBRs and that are also lattice matched to the InP substrate. Wafer fusion, a relatively crude process in which two semiconductor wafers are thermally fused under pressure, was used to realize the first 1550 nm VCSELs with GaAs/AlGaAs DBR layers and InGaAsP active region. With the improvement in material growth technology, AlAsSb/AlGaAsSb DBR mirrors were grown on InP substrates and integrated with InGaAsP active regions to realize a 1550 nm VCSELs. The structure of such a device is shown in the Figure above.

12.4 Purcell Enhancement and Suppression of Spontaneous Emission Rates in Optical Microcavities

12.4.1 Introduction:

The spontaneous emission rate R_{Tsp} (units: number of photons emitted per unit volume of the gain material) for photon emission into all radiation modes for a bulk semiconductor is given by the expression,

$$R_{Tsp} = \int_0^{\infty} v_g^M(\omega) g(\omega) n_{sp}(\omega) g_p(\omega) d\omega$$

The spontaneous emission rate depends on the photon density of states given by $g_p(\omega)$ which for bulk medium equals,

$$g_p(\omega) = \left(\frac{\omega n(\omega)}{\pi c} \right)^2 \frac{1}{v_g^M(\omega)}$$

Note that the total number of radiation modes in any frequency interval of interest between ω_1 and ω_2 is given by the integral,

$$\int_{\omega_1}^{\omega_2} V_p g_p(\omega) d\omega$$

The photon density of states can be modified drastically in optical cavities specially those of very small dimensions. For example, an optical cavity like the VCSEL can be designed to support just a single cavity mode (in the frequency range of interest). Compared to spontaneous emission rates in bulk media, spontaneous emission rates in optical cavities can be both enhanced or suppressed significantly and this phenomenon is called the Purcell Effect.

12.4.2 Photon Density of States in Optical Microcavities

Consider an optical microcavity with only a single cavity mode of frequency ω_0 (in the frequency range of interest between ω_1 and ω_2), cavity photon lifetime τ_p , and modal volume V_p . We need to find the photon density of states $g_p(\omega)$ which must satisfy,

$$\int_{\omega_1}^{\omega_2} V_p g_p(\omega) d\omega = 1$$

We assume that the cavity frequency spectrum is Lorentzian (as in the case of the VCSEL) and the full width (at half maximum) of the cavity resonance is $1/\tau_p$. This implies that,

$$g_p(\omega) = \frac{1}{\pi V_p} \frac{1/2\tau_p}{(\omega - \omega_0)^2 + (1/2\tau_p)^2}$$

12.4.3 Spontaneous Emission Rates in Microcavities:

Consider now the spontaneous emission rate (per unit volume of the gain material) in an optical microcavity. Using the expression obtained above for the photon density of states we get,

There are two cases of interest in which the above integral can be evaluated analytically. First consider the situation where the gain bandwidth is much broader than the width of the cavity resonance. This is the situation in most semiconductor devices. All terms in the integrand other than the Lorentzian can be evaluated at the peak of the Lorentzian and the integral can then be performed easily to get,

$$R_{Tsp} = v_g^M(\omega_o) g(\omega_o) \frac{n_{sp}(\omega_o)}{V_p}$$

The expression above is the familiar result we had derived earlier. If the mode volume V_p is small enough, the spontaneous emission rate in a microcavity can exceed the spontaneous emission rate in a bulk semiconductor despite the fact that the spontaneous emission in a microcavity is going into just a single mode.

Now consider the case where the gain bandwidth is smaller than or comparable to the width of the cavity resonance. This situation can arise, for example, when the gain medium consists of semiconductor quantum dots. In this case, we can write the frequency dependence of the gain as a Lorentzian function,

$$v_g^M(\omega) g(\omega) n_{sp}(\omega) = v_g^M g_o n_{sp} \frac{\gamma^2}{(\omega - \omega_g)^2 + (\gamma)^2}$$

Here, g_o is the peak gain. The spontaneous emission rate is,

$$\begin{aligned} R_{Tsp} &= v_g^M g_o \frac{n_{sp}}{V_p} \int_0^\infty \frac{\gamma^2}{(\omega - \omega_g)^2 + (\gamma)^2} \frac{1/2\pi\tau_p}{(\omega - \omega_o)^2 + (1/2\tau_p)^2} d\omega \\ &= v_g^M g_o \frac{n_{sp}}{V_p} \frac{\gamma(\gamma + 1/2\tau_p)}{(\omega_g - \omega_o)^2 + (\gamma + 1/2\tau_p)^2} \end{aligned}$$

If the center frequency of the gain spectrum and the cavity mode frequency are far away from each other, then the spontaneous emission rate is completely suppressed since there is no cavity mode within the gain bandwidth into which photons can be emitted. The spontaneous emission rate is maximum when $\omega_g = \omega_o$ and the cavity resonance is aligned with the gain spectrum. In this case,

$$R_{Tsp} = v_g^M g_o \frac{n_{sp}}{V_p} \frac{\gamma}{(\gamma + 1/2\tau_p)}$$

If the width of the cavity resonance is much wider than the gain spectrum then,

$$R_{Tsp} = v_g^M g_o \frac{n_{sp}}{V_p} 2\gamma\tau_p = v_g^M g_o \frac{n_{sp}}{V_p} \frac{2\gamma}{\omega_o} Q$$

In this limit, the spontaneous emission rate increases with the quality factor of the optical mode.

