ECE5310 Homework #9 Solutions (Farhan Rana)

9.1

a) $$\hat{b}_5(z,t) = \frac{1}{\sqrt{2}}\left[\hat{b}_4\left(z_1, t - \frac{(z-z_1)}{v_g}\right) + \hat{b}_3\left(z_1, t - \frac{(z-z_1)}{v_g} - \frac{\Delta L}{v_g}\right)e^{i\beta_o \Delta L}\right]$$

where $z \geq z_2$

$$= \frac{1}{2}\left[\left\{-\hat{b}_1\left(z_1, t - \frac{(z-z_1)}{v_g}\right) + \hat{b}_2\left(z_1, t - \frac{(z-z_1)}{v_g}\right)\right\} + \left\{\hat{b}_1\left(z_1, t - \frac{(z-z_1)}{v_g} - \frac{\Delta L}{v_g}\right) + \hat{b}_2\left(z_1, t - \frac{(z-z_1)}{v_g} - \frac{\Delta L}{v_g}\right)\right\}e^{i\beta_o \Delta L}\right]$$

$$= \frac{1}{2}\left[\hat{b}_1(z+\Delta L - v_g t, 0)\, e^{i\beta_o \Delta L} - \hat{b}_1(z - v_g t, 0)\right] + \frac{1}{2}\left[\hat{b}_2(z+\Delta L - v_g t, 0)\, e^{i\beta_o \Delta L} + \hat{b}_2(z - v_g t, 0)\right]$$

b) $$\langle \hat{F}_5(z,t)\rangle = v_g \langle \hat{b}_5^{+}(z,t)\, \hat{b}_5(z,t)\rangle$$

$$= \frac{\langle \hat{F}_1(z+\Delta L - v_g t, 0)\rangle}{4} + \frac{\langle \hat{F}_1(z - v_g t, 0)\rangle}{4} - \frac{v_g}{4}\langle \hat{b}_1^{+}(z+\Delta L - v_g t, 0)\, \hat{b}_1(z - v_g t, 0)\rangle e^{-i\beta_o \Delta L} - \frac{v_g}{4}\langle \hat{b}_1^{+}(z - v_g t, 0)\, \hat{b}_1(z+\Delta L - v_g t, 0)\rangle e^{i\beta_o \Delta L}$$

c) Average number of photons detected $= \langle \hat{N}(z)\rangle = \int_{-\infty}^{\infty} dt\, \langle \hat{F}_5(z,t)\rangle$

$$= \int_{-\infty}^{\infty} dt\, \frac{v_g}{4}\left\{|A(z+\Delta L - v_g t)|^2 + |A(z - v_g t, 0)|^2 - A^*(z+\Delta L - v_g t)\, A(z - v_g t)\, e^{-i\beta_o \Delta L} - A^*(z - v_g t)\, A(z+\Delta L - v_g t)\, e^{i\beta_o \Delta L}\right\}$$

$$= \frac{1}{4}\left\{1 + 1 - e^{-i\beta_o \Delta L} - e^{i\beta_o \Delta L}\right\} = \frac{1}{2}\left[1 - \cos \beta_o \Delta L\right]$$

9.2

a) $$\frac{\hat{I}_5(z,t)}{q} = \hat{F}_5(z,t) = v_g\, \hat{b}_5^{\dagger}(z,t)\, \hat{b}_5(z,t)$$

$$\hat{b}_5(z,t) = \frac{1}{2}\hat{b}_1(z-v_g t,0)\left[e^{i(\Delta\phi-\pi/2)} - 1\right] + \frac{1}{2}\hat{b}_2(z-v_g t,0)\left[e^{i(\Delta\phi+\pi/2)} + 1\right]$$

$$\Rightarrow \frac{\hat{I}_5(z,t)}{q v_g} = \hat{b}_1^{\dagger}(z-v_g t,0)\,\hat{b}_1(z-v_g t,0)\left[\frac{1-\sin\Delta\phi}{2}\right]$$
$$+ \hat{b}_2^{\dagger}(z-v_g t,0)\,\hat{b}_2(z-v_g t,0)\left[\frac{1+\sin\Delta\phi}{2}\right]$$
$$+ \frac{i}{2}\hat{b}_1^{\dagger}(z-v_g t,0)\,\hat{b}_2(z-v_g t,0)\cos(\Delta\phi)$$
$$- \frac{i}{2}\hat{b}_2^{\dagger}(z-v_g t,0)\,\hat{b}_1(z-v_g t,0)\cos(\Delta\phi)$$

b) Similarly,

$$\frac{\hat{I}_6(z,t)}{q v_g} = \hat{b}_1^{\dagger}(z-v_g t,0)\,\hat{b}_1(z-v_g t,0)\left[\frac{1+\sin\Delta\phi}{2}\right]$$
$$+ \hat{b}_2^{\dagger}(z-v_g t,0)\,\hat{b}_2(z-v_g t,0)\left[\frac{1-\sin\Delta\phi}{2}\right]$$
$$- \frac{i}{2}\hat{b}_1^{\dagger}(z-v_g t,0)\,\hat{b}_2(z-v_g t,0)\cos\Delta\phi$$
$$+ \frac{i}{2}\hat{b}_2^{\dagger}(z-v_g t,0)\,\hat{b}_1(z-v_g t,0)\cos\Delta\phi$$

c) $$\frac{\hat{I}(t)}{q v_g} = \frac{\hat{I}_6(t) - \hat{I}_5(t)}{q v_g} = \left[\hat{b}_1^{\dagger}(z-v_g t,0)\,\hat{b}_1(z-v_g t,0) - \hat{b}_2^{\dagger}(z-v_g t,0)\,\hat{b}_2(z-v_g t,0)\right]\sin$$
$$- i\left[\hat{b}_1^{\dagger}(z-v_g t,0)\,\hat{b}_2(z-v_g t,0) - \hat{b}_2^{\dagger}(z-v_g t,0)\,\hat{b}_1(z-v_g t,0)\right]\cos$$

d) $\langle \hat{I}(t)\rangle = q v_g \langle \hat{b}_1^{+}(z-v_g t,0)\,\hat{b}_1(z-v_g t,0)\rangle \sin(\Delta\phi)$

$\approx q v_g |\alpha|^2 \Delta\phi$.

e) $$\Delta\hat{I}(t) = 2q v_g |\alpha| \left[\frac{\hat{b}_2(z-v_g t,0)e^{-i\theta} - \hat{b}_2^{+}(z-v_g t,0)e^{i\theta}}{2i}\right]$$

Note that the expression in the brackets is just the $\theta+\pi/2$ quadrature of the quantum state coming in from input port 2.

$$\Delta\hat{I}(t) = 2q v_g |\alpha|\ \hat{X}_{\theta+\pi/2}(z-v_g t,0)$$

$$\Rightarrow \langle \Delta\hat{I}(t_1)\,\Delta\hat{I}(t_2)\rangle = 2q v_g |\alpha| \langle \hat{X}_{\theta+\pi/2}(z-v_g t_1,0)\hat{X}_{\theta+\pi/2}(z-v_g t_2,0)\rangle$$

For vacuum input into port 2,

$$\langle \Delta\hat{I}(t_1)\Delta\hat{I}(t_2)\rangle = (2q v_g |\alpha|)^2 \ \frac{1}{4}\frac{\delta(t_1-t_2)}{v_g}$$

$$= q^2 v_g |\alpha|^2 \delta(t_1-t_2)$$

$$\Rightarrow \ S_{\Delta I \Delta I}(\omega) = q^2 v_g |\alpha|^2$$

$$\Rightarrow \ \langle \Delta\hat{I}^2(t)\rangle = q^2 v_g \frac{|\alpha|^2}{T}$$

f) $$SNR = \frac{\langle\hat{I}(t)\rangle/\Delta\phi}{\sqrt{\langle\Delta\hat{I}^2(t)\rangle}} = \frac{q v_g |\alpha|^2}{\sqrt{q^2 v_g |\alpha|^2/T}} = \sqrt{v_g|\alpha|^2 T}$$

But $v_g|\alpha|^2$ is the input photon flux in port 1 and $v_g|\alpha|^2 T$ is the number of photons used in making the phase-change measurement in time T. Therefore,

$$SNR = \sqrt{\#\text{ of photons used}}$$

9) Instead of letting ordinary vacuum come into the input port 2, one can send in squeezed vacuum. Note that the current noise $\Delta\hat{I}(t)$ is related to the $\theta+\pi/2$ quadrature of the vacuum input state. A squeezed vacuum input with reduced fluctuations in the $\theta+\pi/2$ quadrature could therefore reduce the current noise fluctuations. Suppose,

$$\langle \hat{x}_{\theta+\pi/2}(z-v_g t_1,0)\, \hat{x}_{\theta+\pi/2}(z-v_g t_2,0)\rangle = \frac{e^{-2r}}{4v_g}\,\delta(t_1-t_2)$$

where r is the squeezing parameter. Then,

$$\langle \Delta\hat{I}^2(t)\rangle = q^2 v_g \frac{|\alpha|^2}{T} e^{-2r}$$

and $\text{SNR} = \sqrt{\#\text{ of photons used}}\; e^{r}$

9.3

a) $m\hat{\vec{v}}(t) = \hat{\vec{P}}(t) - q\hat{\vec{A}}(\hat{\vec{r}}(t), t)$.

$$\Rightarrow \left[m\hat{V}_k(t), m\hat{V}_j(t)\right] = \left[\hat{P}_k(t) - q\hat{A}_k(\hat{\vec{r}}(t), t), \hat{P}_j(t) - q\hat{A}_j(\hat{\vec{r}}(t), t)\right].$$

$$= -q\left[\hat{P}_k(t), \hat{A}_j(\hat{\vec{r}}(t), t)\right] - q\left[\hat{A}_k(\hat{\vec{r}}(t), t), \hat{P}_j(t)\right]$$

$$= qi\hbar\left[\partial_k \hat{A}_j(\hat{\vec{r}}(t), t) - \partial_j \hat{A}_k(\hat{\vec{r}}(t), t)\right] = qi\hbar\mu_0 \sum_r \varepsilon_{kjr} \hat{H}_r(\hat{\vec{r}}(t), t))$$

where $\mu_0 \hat{\vec{H}}(\vec{r}(t), t) = \nabla \times \hat{\vec{A}}(\vec{r}(t), t)$.

b) $i\hbar m \dfrac{d\hat{V}_k(t)}{dt} = \left[m\hat{V}_k(t), \hat{H}\right] = \left[\hat{P}_k(t) - q\hat{A}_k(\hat{\vec{r}}(t), t), \hat{H}\right]$.

i) The first term in the Hamiltonian is $\sum_j \frac{1}{2} m \hat{v}_j(t)\hat{v}_j(t)$.

From part (a) the result will be:

$$qi\hbar\mu_0 \sum_r \varepsilon_{kjr}\left\{\frac{\hat{v}_j(t)\hat{H}_r(\hat{\vec{r}}(t), t) + \hat{H}_r(\hat{\vec{r}}(t), t)\hat{v}_j(t)}{2}\right\}$$

The above is the k-th component of

$$qi\hbar\left\{\frac{\hat{\vec{v}}(t) \times \hat{\vec{B}}(\hat{\vec{r}}(t), t) - \hat{\vec{B}}(\hat{\vec{r}}(t), t) \times \hat{\vec{v}}(t)}{2}\right\}$$

since $\hat{\vec{v}}(t)$ does not commute with $\hat{\vec{r}}(t)$, the order is important.

ii) The second term in the Hamiltonian is $qV(\hat{\vec{r}}(t))$.

Since $\left[\hat{r}_k(t), \hat{P}_j(t)\right] = i\hbar\delta_{kj}$, the result will be:

$$\left[\hat{\vec{P}}(t), qV(\hat{\vec{r}}(t))\right] = -q\nabla V(\hat{\vec{r}}(t))\, i\hbar$$

iii) The last term in the Hamiltonian is the field energy. Note that $\hat{\vec{A}}(\vec{r},t)$ commutes with $\hat{\vec{H}}(\vec{r},t)$. We first find the commutation relation between $\hat{A}_K(\vec{r},t)$ and $\hat{E}_j(\vec{r},t)$.

Recall that:

$$\hat{A}_K(\vec{r},t) = \hat{e}_K \cdot \hat{\vec{A}}(\vec{r},t) = \sum_{\vec{k}}\sum_{\alpha} \sqrt{\frac{\hbar}{2\varepsilon_0 \omega_k}} \left(\hat{a}_\alpha(\vec{k},t) + \hat{a}_\alpha^{+}(-\vec{k},t)\right) \frac{e^{i\vec{k}\cdot\vec{r}}}{\sqrt{V}} \left[\hat{\varepsilon}_\alpha(\vec{k}) \cdot \hat{e}_K\right]$$

and

$$\hat{E}_j(\vec{r},t) = \hat{e}_j \cdot \hat{\vec{E}}(\vec{r},t) = i \sum_{\vec{k}}\sum_{\beta} \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0}} \left(\hat{a}_\beta(\vec{k},t) - \hat{a}_\beta^{+}(-\vec{k},t)\right) \frac{e^{i\vec{k}\cdot\vec{r}}}{\sqrt{V}} \left[\hat{\varepsilon}_\beta(\vec{k}) \cdot \hat{e}_j\right].$$

This gives

$$\left[\hat{A}_K(\vec{r},t), \hat{E}_j(\vec{r}',t)\right] = -\frac{i\hbar}{\varepsilon_0} \int \frac{d^3\vec{k}}{(2\pi)^3} \sum_\alpha \left[\hat{e}_K \cdot \hat{\varepsilon}_\alpha(\vec{k})\right]\left[\hat{\varepsilon}_\alpha(\vec{k}) \cdot \hat{e}_j\right] e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}$$

Recall that: $\sum_\alpha \left[\hat{e}_K \cdot \hat{\varepsilon}_\alpha(\vec{k})\right]\left[\hat{\varepsilon}_\alpha(\vec{k}) \cdot \hat{e}_j\right] = \hat{e}_K \cdot \left[1 - \hat{k}\otimes\hat{k}\right] \cdot \hat{e}_j$

$$= \delta_{Kj} - \frac{k_K k_j}{k^2}$$

$$\Rightarrow \left[\hat{A}_K(\vec{r},t), \hat{E}_j(\vec{r}',t)\right] = -\frac{i\hbar}{\varepsilon_0} \int \frac{d^3\vec{k}}{(2\pi)^3} \left(\delta_{Kj} - \frac{k_K k_j}{k^2}\right) e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}$$

$$= -\frac{i\hbar}{\varepsilon_0} \delta^{\perp}_{Kj}(\vec{r}-\vec{r}')$$

$\delta^{\perp}_{Kj}(\vec{r}-\vec{r}')$ is the transverse delta function. It "picks out" the the transverse component of field. For example,

$$\int d^3\vec{r}'\, \delta^{\perp}(\vec{r}-\vec{r}') \cdot \vec{E}(\vec{r}',t) = \vec{E}_T(\vec{r},t).$$

In the problem, the electric field is entirely transverse.

So finally we have

$$\left[q\hat{A}_k(\vec{r},t), \int d^3\vec{r}\,' \tfrac{1}{2}\varepsilon_0 \vec{E}(\vec{r}\,',t)\cdot\vec{E}(\vec{r}\,',t)\right]$$

$$= -i\hbar q\hat{E}_k(\vec{r},t)$$

Putting (i) and (ii) and (iii) together we get:

$$m\frac{d\hat{\vec{v}}(t)}{dt} = q\hat{\vec{E}}(\hat{\vec{r}}(t),t) - q\nabla V(\hat{\vec{r}}(t)) + q\left\{\frac{\hat{\vec{v}}(t)\times\hat{\vec{B}}(\hat{\vec{r}}(t),t) - \hat{\vec{B}}(\hat{\vec{r}}(t),t)\times\hat{\vec{v}}(t)}{2}\right\}$$