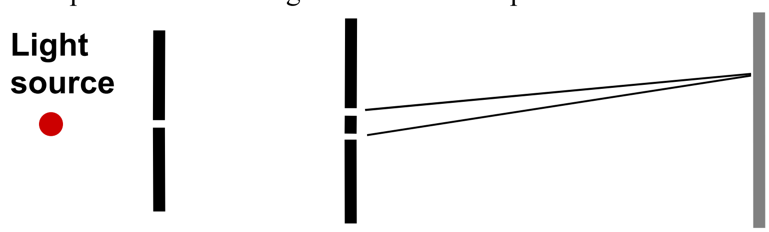
ECE 5310: Quantum Optics for Photonics and Optoelectronics

Fall 2013

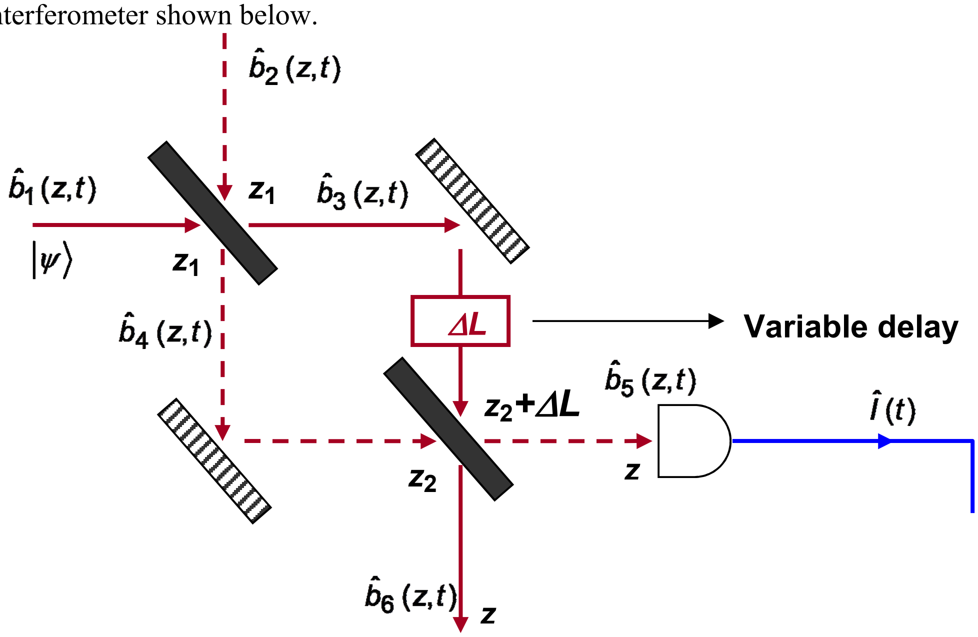
Homework 9 **Due on Nov. 19 at 5:00 PM (self-grade)**

Problem 9.1 (Single Photon Interference)

Young's double-slit experiment demonstrated the wave nature of photons. In the experiment, radiation from two slits was allowed to interfere on a photosensitive screen. The screen exhibited bright and dark fringes corresponding to whether the path difference from the slits to the observation point on the screen was an integral multiple of the wavelength or an odd multiple of half the wavelength.



There was some controversy in the beginning as many people mistakenly thought that photons, being waves, interfere with each other and this mutual interference was the cause of the bright and dark fringes on the screen. Later, it was shown that even when a single photon is sent through the slits, and this single photon experiment is repeated many times, fringes emerge on the screen. This self-interference property of a photon was puzzling to many but consistent with quantum electrodynamics. In this problem you will analyze a single photon interference experiment that uses an interferometer instead of slits. Consider the interferometer shown below.



The beam splitter relation for the first beam splitter is,

$$\begin{bmatrix} \hat{b}_3(z_1,t)\, e^{i\beta_o z_1} \\ b_4(z_1,t)\, e^{i\beta_o z_1} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \hat{b}_1(z_1,t)\, e^{i\beta_o z_1} \\ \hat{b}_2(z_1,t)\, e^{i\beta_o z_1} \end{bmatrix}$$

For the second beam splitter it is,

$$\begin{bmatrix} \hat{b}_5(z_2,t)\, e^{i\beta_o z_2} \\ b_6(z_2+\Delta L,t)\, e^{i\beta_o (z_2+\Delta L)} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \hat{b}_4(z_2,t)\, e^{i\beta_o z_2} \\ \hat{b}_3(z_2+\Delta L,t)\, e^{i\beta_o (z_2+\Delta L)} \end{bmatrix}$$

The average photocurrent from the detector placed right after the second beam splitter is measured as function of the path difference ΔL between the two arms of the interferometer. The path difference ΔL corresponds to a time delay τ of $\Delta L / v_g$.

a) Find an expression for the operator $\hat{b}_5(z,t)$ for $z \geq z_2$ and $z - v_g t < z_1$ in terms of the input operators $\hat{b}_1(z,t)$ and $\hat{b}_2(z,t)$.

b) Assuming that vacuum is incident from the input port 2 of the first beam splitter, find the average of the photon flux operator at the output port 5 and show that the result is,

$$\left\langle \hat{F}_5(z,t) \right\rangle = \frac{\left\langle \hat{F}_1(z+\Delta L - v_g t,0) \right\rangle}{4} + \frac{\left\langle \hat{F}_1(z - v_g t,0) \right\rangle}{4} - v_g \frac{\left\langle \hat{b}_1^+(z - v_g t,0)\, \hat{b}_1(z+\Delta L - v_g t,0) \right\rangle}{4} e^{i\beta_o \Delta L}$$
$$- v_g \frac{\left\langle \hat{b}_1^+(z+\Delta L - v_g t,0)\, \hat{b}_1(z - v_g t,0) \right\rangle}{4} e^{-i\beta_o \Delta L}$$

where the average above is taken with respect to the state going into the input port 1 of the first beam splitter.

c) Suppose the input state is a single photon packet incident from input port 1 and described by,

$$|\psi(t=0)\rangle = \int_{-\infty}^{\infty} dz'\, A(z')\, \hat{b}_1^+(z',0)\, |0\rangle$$

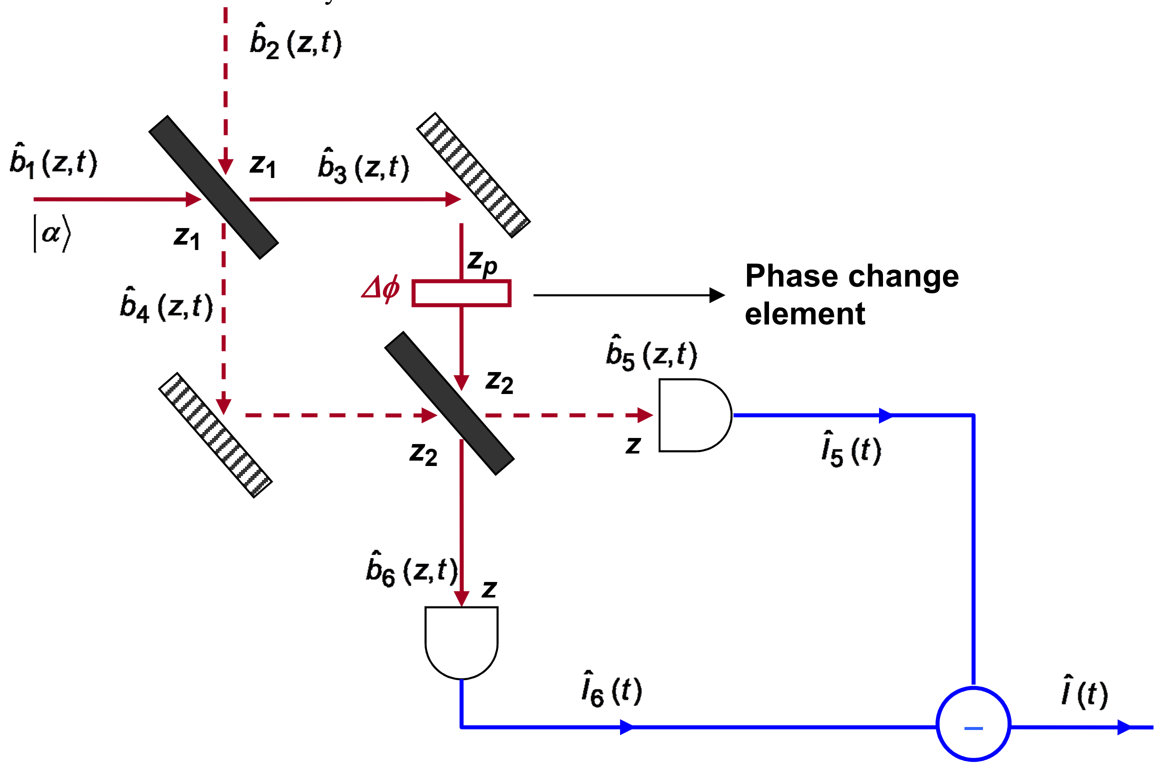
and is well localized in region $z < z_1$. We assume that the packet width D in space is much much longer than the wavelength and so for all essential purposes $|A(z)|^2 \approx 1/D$. Show that the average number of photons that are counted by the detector for a given path difference ΔL is,

$$\frac{1}{2}\left[1 - \cos(\beta_o \Delta L)\right]$$

Thus, as the path difference is varied the detector in output port 5 will either detect a photon with a high probability (bright fringe) or it will never detect a photon with a high probability (dark fringe). In case of a dark fringe, the photon leaves the interferometer from output port 6. If the experiment is repeated many times, using only a single photon at a time, the average number of photon detected for a given path difference ΔL will follow the expression above indicating that even a single photon interferes with itself.

Problem 11.2 (Quantum Limits on Phase Measurement)

Accurate phase measurements of light fields, or rather phase-change measurements, are important for many applications. In this problem you will analyze a gravitational-wave detection interferometer that is operated by CALTECH, MIT, NASA, JPL, and other institutions. The interferometer is shown below. A gravitational wave causes a phase-change in the light in only one arm of the interferometer. This phase-change is detected and from this information the strength of the gravitational wave can be found. Needless to say, the expected phase-change produced by a gravitational wave (from, say, a supernova explosion) is extremely small. The same technique can also be used in more practical applications. For example, the phase-change produced by a third order optical nonlinearity due to self-phase modulation can also be detected in exactly the same fashion.



The beam splitter relation for the first beam splitter is,

$$\begin{bmatrix} \hat{b}_3(z_1,t)\, e^{i\beta_o z_1} \\ b_4(z_1,t)\, e^{i\beta_o z_1} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \hat{b}_1(z_1,t)\, e^{i\beta_o z_1} \\ \hat{b}_2(z_1,t)\, e^{i\beta_o z_1} \end{bmatrix}$$

For the second beam splitter it is,

$$\begin{bmatrix} \hat{b}_5(z_2,t)\, e^{i\beta_o z_2} \\ b_6(z_2,t)\, e^{i\beta_o z_2} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \begin{bmatrix} \hat{b}_4(z_2,t)\, e^{i\beta_o z_2} \\ \hat{b}_3(z_2,t)\, e^{i\beta_o z_2} \end{bmatrix}$$

We assume that all phase change happens in location z_p such that,

$$\hat{b}_3(z > z_p, t) = \hat{b}_3(z < z_p, t)\, e^{i\Delta\phi}$$

a) Find an expression for the current operator $\hat{I}_5(t)$ in terms of the operators in input ports 1 and 2.

b) Find an expression for the current operator $\hat{I}_6(t)$ in terms of the operators in input ports 1 and 2.

c) Find an expression for the current operator $\hat{I}(t) = \hat{I}_6(t) - \hat{I}_5(t)$ in terms of the operators in input ports 1 and 2. The answer should be,

$$\begin{aligned} \hat{I}(t) = \hat{I}_6(t) - \hat{I}_5(t) = q v_g \left[\hat{b}_1^+(z - v_g t, 0)\hat{b}_1(z - v_g t, 0) - \hat{b}_2^+(z - v_g t, 0)\hat{b}_2(z - v_g t, 0)\right] \sin\Delta\phi \\ - i\, q v_g \left[\hat{b}_1^+(z - v_g t, 0)\hat{b}_2(z - v_g t, 0) - \hat{b}_2^+(z - v_g t, 0)\hat{b}_1(z - v_g t, 0)\right] \cos\Delta\phi \end{aligned}$$

Assume that a strong continuous wave coherent state (a laser beam for example) is coming in on input port 1 and vacuum is coming in on input port 2. We have, $\left\langle \hat{b}_1(z,t) \right\rangle = \alpha = |\alpha|\, e^{i\theta}$.

d) Find the average current $\left\langle \hat{I}(t) \right\rangle$ and show that its proportional to $\Delta\phi$ when $\Delta\phi << 1$.

The ultimate sensitivity of the measurement for detecting very small values of $\Delta\phi$ is set by the detector current noise when $\Delta\phi = 0$. Suppose $\Delta\phi = 0$. The current noise is entirely contributed by the term,

$$\Delta\hat{I}(t) = -i\, q v_g \left[\hat{b}_1^+(z - v_g t, 0)\hat{b}_2(z - v_g t, 0) - \hat{b}_2^+(z - v_g t, 0)\hat{b}_1(z - v_g t, 0) \right]$$

Interestingly, the shot noise from the coherent state input (laser) is completely cancelled in the balanced detection scheme. It is convenient to replace all instances of the operator $\hat{b}_1(z - v_g t, 0)$ by its average value $\left\langle \hat{b}_1(z - v_g t, 0) \right\rangle = \alpha = |\alpha|\, e^{i\theta}$ in the expression for the current noise to get,

$$\Delta\hat{I}(t) = 2\, q v_g\, |\alpha| \left[\frac{\hat{b}_2(z - v_g t, 0)\, e^{-i\theta} - \hat{b}_2^+(z - v_g t, 0)\, e^{i\theta}}{2i} \right]$$

e) Suppose the integration time (or averaging time) for phase-change measurement is T. This means that the RF electrical bandwidth of the measurement is $\Delta\omega = 2\pi/T$. Find the spectral density of the current noise, and integrate it over a bandwidth equal to $\Delta\omega = 2\pi/T$ to get the mean square current fluctuations,

$$\left\langle \Delta\hat{I}^2(t) \right\rangle = \int_{-\pi/T}^{\pi/T} \frac{d\omega}{2\pi}\, S_{\Delta I\, \Delta I}(\omega)$$

f) The signal to noise ratio is defined as the ratio of the differential change in the average current due to a small change in phase to the root mean square current fluctuations,

$$SNR = \frac{\left\langle \hat{I}(t) \right\rangle / \Delta\phi}{\sqrt{\left\langle \Delta\hat{I}^2(t) \right\rangle}}$$

Show that the SNR is equal to the square root of the total number of photons that are used in the measurement during the averaging time T. The lesson is that a stronger laser will yield a better measurement (thanks to the balanced detection scheme that gets rid of the laser shot noise).

g) The sensitivity limit for phase-change measurements, as found in part (f) is not the ultimate limit and can be beaten. If you have a light source that can produce any quantum state of light how, would you use it to increase the sensitivity of the phase measurement and get a SNR value better than that found in part (f) above.

Problem 9.3 (Lorentz Force Law)

Consider the complete Hamiltonian of a single free particle interacting with quantized electromagnetic field,

$$\hat{H} = \frac{\left[\hat{\vec{p}}(t) - q\hat{\vec{A}}(\hat{\vec{r}}(t), t) \right]^2}{2m} + qV(\hat{\vec{r}}(t)) + \int d^3\vec{r} \left\{ \frac{1}{2}\, \varepsilon_o\, \hat{\vec{E}}(\vec{r}, t).\hat{\vec{E}}(\vec{r}, t) + \frac{1}{2}\, \mu_o \hat{\vec{H}}(\vec{r}, t).\hat{\vec{H}}(\vec{r}, t) \right\}$$

Assume Coulomb gauge in which $\hat{\vec{A}}_L(\vec{r}, t) = 0$, $\hat{\vec{A}}(\vec{r}, t) = \hat{\vec{A}}_T(\vec{r}, t)$, and,

$$\hat{\vec{E}}_T(\vec{r},t) = -\frac{\partial \hat{\vec{A}}(\vec{r},t)}{\partial t}$$

a) Prove the commutation relation for the components of the particle kinetic momentum,

$$\left[m\hat{v}_k(t), m\hat{v}_j(t)\right] = i\hbar q\mu_o \sum_r \varepsilon_{kjr}\ \hat{H}_r\left(\hat{\vec{r}}(t),t\right)$$

b) Find the rate of change of the particle kinetic momentum,

$$m\frac{d\hat{\vec{v}}(t)}{dt} = ?$$

starting from the Heisenberg equation.