8.1

a) i) $\hat{S}^{\dagger}(\varepsilon)\hat{b}(z,0)\hat{S}(\varepsilon) = \hat{b}(z,0)\cosh r(z) - \hat{b}^{\dagger}(z,0)\, e^{2i\phi(z)} \sinh r(z)$

ii) $\hat{S}^{\dagger}(\varepsilon)\hat{b}^{\dagger}(z,0)\hat{S}(\varepsilon) = \hat{b}^{\dagger}(z,0)\cosh r(z) - \hat{b}(z,0)\, e^{-2i\phi(z)} \sinh r(z)$

b) $|\psi(t=0)\rangle = \hat{T}(\alpha)\hat{S}(\varepsilon)|0\rangle = |\alpha(z),\varepsilon(z)\rangle$

$|\psi(t)\rangle = e^{-i\frac{\hat{H}t}{\hbar}}|\psi(t=0)\rangle = e^{-i\frac{\hat{H}t}{\hbar}}\hat{T}(\alpha)e^{i\frac{\hat{H}t}{\hbar}}\, e^{-i\frac{\hat{H}t}{\hbar}}\hat{S}(\varepsilon)e^{i\frac{\hat{H}t}{\hbar}}\, e^{-i\frac{\hat{H}t}{\hbar}}|0\rangle$

Assuming vacuum energy is zero,

$$|\psi(t)\rangle = \hat{T}\left(\alpha(z-v_g t)e^{-i\omega_o t}\right)\hat{S}\left(\varepsilon(z-v_g t)e^{-2i\omega_o t}\right)|0\rangle$$

$$= \left|\alpha(z-v_g t)e^{-i\omega_o t},\ \varepsilon(z-v_g t)e^{-2i\omega_o t}\right\rangle$$

$$= \left||\alpha|e^{i(\phi+\frac{\pi}{2})}e^{-i\omega_o t},\ re^{i2\phi}e^{-2i\omega_o t}\right\rangle$$

c) $\langle\psi(t=0)|\hat{X}_{\phi}(z,t)|\psi(t=0)\rangle = \langle\psi(t=0)|\dfrac{\hat{b}(z-v_g t,0)e^{-i\phi} + \hat{b}^{\dagger}(z-v_g t,0)e^{i\phi}}{2}|\psi(t=0)\rangle$

$$= \langle 0|S^{\dagger}(re^{2i\phi})\hat{T}^{\dagger}(|\alpha|e^{i(\phi+\pi/2)})\left\{\frac{\hat{b}(z-v_g t,0)e^{-i\phi} + \hat{b}^{\dagger}(z-v_g t,0)e^{i\phi}}{2}\right\}\hat{T}(|\alpha|e^{i(\phi+\pi/2)})$$

$$\times\ \hat{S}(re^{2i\phi})|0\rangle$$

$$= \langle 0|\hat{S}^{\dagger}\left\{\frac{\hat{T}^{\dagger}\hat{b}\hat{T}e^{-i\phi} + \hat{T}^{\dagger}\hat{b}^{\dagger}\hat{T}e^{i\phi}}{2}\right\}\hat{S}|0\rangle$$

$$= \langle 0|\hat{S}^{\dagger}\left\{\frac{\left[\hat{b}(z-v_g t,0)+|\alpha|e^{i(\phi+\pi/2)}\right]e^{-i\phi} + \left[\hat{b}^{\dagger}(z-v_g t,0)+|\alpha|e^{-i(\phi+\pi/2)}\right]e^{i\phi}}{2}\right\}\hat{S}|0\rangle$$

$$= \langle 0|\frac{\hat{S}^{\dagger}\hat{b}\hat{S}e^{-i\phi}+|\alpha|e^{i\pi/2} + \hat{S}^{\dagger}\hat{b}^{\dagger}\hat{S}e^{i\phi}+|\alpha|e^{-i\pi/2}}{2}|0\rangle = 0.$$

Similarly:

$$\langle\psi(t=0)|\hat{X}_{\phi+\pi/2}(z,t)|\psi(t=0)\rangle = |\alpha|$$

d) $\langle\psi(t=0)|\ \Delta\hat{X}_\phi(z,t_1)\ \Delta\hat{X}_\phi(z,t_2)\,|\psi(t=0)\rangle$

$= \langle\psi(t=0)|\ \hat{X}_\phi(z,t_1)\hat{X}_\phi(z,t_2)\,|\psi(t=0)\rangle$ $\left\{ \text{since } \langle\psi(t=0)|\,\hat{X}_\phi(z,t)\,|\psi(t=0)\rangle = 0 \right\}$

$= \frac{1}{4}\langle 0|\left\{\hat{S}^+\,\hat{b}(z-v_gt_1,0)\,\hat{S}\,e^{-i\phi} + \hat{S}^+\hat{b}^+(z-v_gt_1,0)\hat{S}\,e^{i\phi}\right\}\left\{\hat{S}^+\,\hat{b}(z-v_gt_2,0)\hat{S}\,e^{-i\phi} + \hat{S}^+\hat{b}^+(z-v_gt_2,0)\hat{S}\,e^{i\phi}\right\}|0\rangle$

$= (\cosh r - \sinh r)^2\,\frac{1}{4}\langle 0|\,\hat{b}(z-v_gt_1,0)\,\hat{b}^+(z-v_gt_2,0)\,|0\rangle$

$= \frac{e^{-2r}}{4}\ \frac{\delta(t_1-t_2)}{v_g}$

Similarly, $\langle\psi(t=0)|\ \Delta\hat{X}_{\phi+\pi/2}(z,t_1)\ \Delta\hat{X}_{\phi+\pi/2}(z,t_2)\,|\psi(t=0)\rangle$

$= \frac{e^{2r}}{4}\ \frac{\delta(t_1-t_2)}{v_g}$

e). $\langle\psi(t=0)|\ \hat{F}(z,t)\,|\psi(t=0)\rangle = v_g\langle\psi(t=0)|\,\hat{b}^+(z-v_gt,0)\,b(z-v_gt,0)\,|\psi(t=0)\rangle$

$= v_g\,\langle 0|\,\hat{S}^+\hat{T}^+\hat{b}^+ b\,\hat{T}\hat{S}\,|0\rangle = v_g\,\langle 0|\,\hat{S}^+\{\hat{T}^+\hat{b}^+\hat{T}\}\{\hat{T}^+\hat{b}\,\hat{T}\}\hat{S}\,|0\rangle$

$= v_g\,\langle 0|\,\hat{S}^+\{\hat{b}^+(z-v_gt,0) + |\alpha|e^{-i(\phi+\pi/2)}\}\{\hat{b}(z-v_gt,0) + |\alpha|e^{i(\phi+\pi/2)}\}\hat{S}\,|0\rangle$

$= v_g\left\{|\alpha|^2 + \langle 0|\,\hat{S}^+\hat{b}^+\hat{S}\,\hat{S}^+\hat{b}\,\hat{S}\,|0\rangle\right\}$

$= v_g|\alpha|^2 + v_g\sinh^2 r\,\langle 0|\,\hat{b}(z-v_gt,0)\,\hat{b}^+(z-v_gt,0)\,|0\rangle$

$= v_g|\alpha|^2 + v_g\sinh^2 r\ \delta(z-z')\Big|_{z=z'} = v_g|\alpha|^2 + v_g\sinh^2 r\left(\frac{\Delta\beta}{2\pi}\right).$

f) $\langle \psi(t=0) | \hat{F}(z,t_1) \hat{F}(z,t_2) | \psi(t=0) \rangle$

$$= \left[v_g |\alpha|^2 + v_g \sinh^2 r \left(\frac{\Delta\beta}{2\pi} \right) \right]^2 + v_g |\alpha|^2 e^{2r} \delta(t_1 - t_2)$$

$$+ 2 v_g \sinh^2 r \cosh^2 r \left(\frac{\Delta\beta}{2\pi} \right).$$

$$\Rightarrow \langle \psi(t=0) | \Delta\hat{F}(z,t_1) \Delta\hat{F}(z,t_2) | \psi(t=0) \rangle = v_g |\alpha|^2 e^{2r} \delta(t_1 - t_2) + 2 v_g \sinh^2 r \cosh^2 r \left(\frac{\Delta\beta}{2\pi} \right) \delta(t_1 - t_2)$$

$$\approx v_g |\alpha|^2 e^{2r} \delta(t_1 - t_2) \quad \left\{ \text{if } |\alpha|^2 \gg \frac{\Delta\beta}{2\pi} \right.$$

For a coherent state $\hat{T}(|\alpha| e^{i(\phi + \pi/2)}) |0\rangle$ we would have obtained

$$\langle \psi(t=0) | \Delta\hat{F}(z,t_1) \Delta\hat{F}(z,t_2) | \psi(t=0) \rangle = v_g |\alpha|^2 \delta(t_1 - t_2)$$

Therefore, the squeezed state has larger photon flux fluctuations.

However, if $\alpha = |\alpha| e^{i\phi}$ instead of $\alpha = |\alpha| e^{i(\phi + \pi/2)}$ then for the squeezed state we would have obtained,

$$\langle \psi(t=0) | \Delta\hat{F}(z,t_1) \Delta\hat{F}(z,t_2) | \psi(t=0) \rangle \approx v_g |\alpha|^2 e^{-2r} \delta(t_1 - t_2)$$

In this case, the squeezed state has reduced photon flux fluctuations compared to the coherent state.

8.2

a) $$\langle \hat{F}(z,0)\rangle = v_g \sum_{\beta}\sum_{\beta'} \langle \hat{a}^{+}(\beta)\,\hat{a}(\beta')\rangle \frac{e^{i(\beta'-\beta)z}}{L}$$

$$\langle \hat{a}^{+}(\beta)\,\hat{a}(\beta')\rangle = \delta_{\beta\beta'}\, n_{th}(\omega_0)$$

$$\Rightarrow \langle \hat{F}(z,0)\rangle = v_g \sum_{\beta} n_{th}(\omega_0)\frac{1}{L} \approx v_g\, L\int_{\beta_0-\Delta\beta/2}^{\beta_0+\Delta\beta/2}\frac{d\beta}{2\pi}\, n_{th}(\omega_0)\frac{1}{L}$$

$$= n_{th}(\omega_0)\, v_g \frac{\Delta\beta}{2\pi} = n_{th}(\omega_0)\frac{\Delta\omega}{2\pi}$$

b) $$\langle \hat{F}(z,t_1)\hat{F}(z,t_2)\rangle = v_g^2 \sum_{\beta_1}\sum_{\beta_2}\sum_{\beta_3}\sum_{\beta_4}\Big\{\langle \hat{a}^{+}(\beta_1)\,\hat{a}(\beta_2)\,\hat{a}^{+}(\beta_3)\,\hat{a}(\beta_4)\rangle \frac{e^{i(\beta_2-\beta_1)z}}{L}\frac{e^{i(\beta_4-\beta_3)z}}{L} e^{-i[\omega(\beta_2)-\omega(\beta_1)]t_1} e^{-i[\omega(\beta_4)-\omega(\beta_3)]t_2}\Big\}$$

$$\langle \hat{a}^{+}(\beta_1)\hat{a}(\beta_2)\hat{a}^{+}(\beta_3)\hat{a}(\beta_4)\rangle = \delta_{\beta_1\beta_2}\delta_{\beta_3\beta_4}\, n_{th}^2(\omega_0) + \delta_{\beta_1\beta_4}\delta_{\beta_2\beta_3}\, n_{th}(\omega_0)\times[1+n_{th}(\omega_0)]$$

$$\Rightarrow \langle \hat{F}(z,t_1)\hat{F}(z,t_2)\rangle = \left[n_{th}(\omega_0)\frac{\Delta\omega}{2\pi}\right]^2 + v_g^2\int_{\beta_0-\Delta\beta/2}^{\beta_0+\Delta\beta/2}\frac{d\beta_1}{2\pi}\int_{\beta_0-\Delta\beta/2}^{\beta_0+\Delta\beta/2}\frac{d\beta_2}{2\pi}\, n_{th}(\omega_0)[1+n_{th}(\omega_0)]\, e^{i\omega(\beta_1)(t_1-t_2)} e^{-i\omega(\beta_2)(t_1-t_2)}$$

$$= \langle \hat{F}(z,t_1)\rangle^2 + n_{th}(\omega_0)[1+n_{th}(\omega_0)]\left\{\frac{\Delta\omega}{2\pi}\frac{\sin\frac{\Delta\omega}{2}(t_1-t_2)}{\frac{\Delta\omega}{2}(t_1-t_2)}\right\}^2$$

$$\approx \langle \hat{F}(z,t_1)\rangle^2 + n_{th}(\omega_0)[1+n_{th}(\omega_0)]\frac{\Delta\omega}{2\pi}\delta(t_1-t_2)$$

c) $$\langle \Delta\hat{F}(z,t_1)\Delta\hat{F}(z,t_2)\rangle \approx n_{th}(\omega_0)[1+n_{th}(\omega_0)]\frac{\Delta\omega}{2\pi}\delta(t_1-t_2)$$

$\Rightarrow$ $S_{\Delta F\Delta F}(\omega) = n_{th}(\omega_0)[1+n_{th}(\omega_0)]\frac{\Delta\omega}{2\pi}$ = flat – independent of frequency ω

when $n_{th}(\omega_0) \ll 1$!

d) $$S_{\Delta F\Delta F}(\omega) \approx n_{th}(\omega_0)\frac{\Delta\omega}{2\pi}\cancel{\delta(t_1-t_2)} = \langle \hat{F}(z,t_1)\rangle \cancel{\delta(t_1-t_2)}$$

$\Rightarrow$ Shot noise !

8.3

a) $$|\psi(t)\rangle = e^{-\frac{i\hat{H}t}{\hbar}} |\psi(t=0)\rangle = e^{-\frac{i\hat{H}t}{\hbar}} \frac{\left[\int dz'\, A(z')\, \hat{b}^{+}(z',0)\right]^n}{\sqrt{n!}} |0\rangle$$

$$= e^{-in\omega_0 t} \frac{\left[\int dz'\, A(z'-v_g t) \frac{1}{\sqrt{2}}\left[\hat{b}_3(z',0) + \hat{b}_4(z',0)\right]\right]^n}{\sqrt{n!}} |0\rangle$$

Let $\hat{A}_3^{+} = \int dz'\, A(z'-v_g t)\, \hat{b}_3^{+}(z',0)$ $\quad \hat{A}_4^{+} = \int dz'\, A(z'-v_g t)\, \hat{b}_4^{+}(z',0)$

$$\Rightarrow |\psi(t)\rangle = e^{-in\omega_0 t} \frac{\left[\frac{1}{\sqrt{2}}\hat{A}_3^{+} + \frac{1}{\sqrt{2}}\hat{A}_4^{+}\right]^n}{\sqrt{n!}} |0\rangle$$

$$= \frac{e^{-in\omega_0 t}}{\sqrt{n!}} \frac{1}{2^{n/2}} \sum_{m=0}^{n} \frac{n!}{(n-m)!\, m!} (\hat{A}_3^{+})^m (\hat{A}_4^{+})^{n-m} |0\rangle$$

$$= \frac{e^{-in\omega_0 t}}{2^{n/2}} \sum_{m=0}^{\infty} \frac{\sqrt{n!}}{\sqrt{(n-m)!}\sqrt{m!}} |m\rangle_3 \otimes |n-m\rangle_4 \qquad \left\{ \begin{array}{l} |m\rangle_3 = \frac{(\hat{A}_3^{+})^m}{\sqrt{m!}} |0\rangle \\ |m\rangle_4 = \frac{(\hat{A}_4^{+})^m}{\sqrt{m!}} |0\rangle \end{array} \right.$$

b) $P(m) = \left| \begin{array}{l} \text{Coefficient of the state } |m\rangle_3 \otimes |n-m\rangle_4 \\ \text{in the superposition above} \end{array} \right|^2 = \frac{n!}{m!(n-m)!} \left(\frac{1}{2}\right)^m \left(\frac{1}{2}\right)^{n-m}$

$$= \frac{n!}{m!(n-m)!} \frac{1}{2^n} \qquad \left\{ \begin{array}{l} P(m) = 0 \\ \text{for } m > n \end{array} \right\}$$

c) Average number of photons going out in channel 3 $= \sum_{m=0}^{n} m\, P(m)$

$= \sum_{m=0}^{n} m \frac{n!}{m!(n-m)!} \frac{1}{2^n} = \frac{n}{2} \rightarrow$ This makes sense for a 50-50 beam splitter.

d) State immediately after detection of "p" photons in channel 4 is $= |n-p\rangle_3 \otimes |0\rangle_4$. Now if a measurement of photon number is made in output channel 3 the result "n-p" will be obtained with probability 1. The photons are entangled in the two output channels and measurement of photon number in one channel will effect the results of a subsequent measurement on the other channel.

8.5

a) The time development is given by the Heisenberg equations,

$$\frac{d\hat{a}(t)}{dt} = (-i\omega_o + g)\hat{a}(t)$$

$$\frac{d\hat{a}^+(t)}{dt} = (i\omega_o + g)\hat{a}^+(t)$$

The creation and destruction operators at time t are,

$$\hat{a}(t) = \hat{a}(t=0)e^{(-i\,\omega_o + g)\,t}$$

$$\hat{a}^+(t) = \hat{a}^+(t=0)e^{(i\,\omega_o + g)\,t}$$

It follows that the equal time commutation relation at time t is,

$$\left[\hat{a}(t), \hat{a}^+(t)\right] = \left[\hat{a}(t=0), \hat{a}^+(t=0)\right]e^{2g\,t}$$

If at time $t = 0$,

$$\left[\hat{a}(t=0), \hat{a}^+(t=0)\right] = 1$$

then at time t,

$$\left[\hat{a}(t), \hat{a}^+(t)\right] = e^{2g\,t}$$

The commutation relations increase with time.

b)

$$\frac{d\hat{a}(t)}{dt} = -i\omega_o\,\hat{a}(t) + g\hat{a}(t) + \sqrt{A}\,\hat{F}_{in}(t)$$

$$\frac{d\hat{a}^+(t)}{dt} = i\omega_o\,\hat{a}^+(t) + g\hat{a}^+(t) + \sqrt{A}\,\hat{F}_{in}^+(t)$$

$\hat{F}_{in}(t)$ and $\hat{F}_{in}^+(t)$ are Langevin noise operators and the requirement we impose on them is that they satisfy,

$$\left[\hat{F}_{in}(t), \hat{F}_{in}^+(t')\right] = -\delta(t-t')$$

Solving the Equations above one obtains,

$$\hat{a}(t) = \hat{a}(t=0)e^{(-i\omega_o + g)t} + \sqrt{A}\int_o^t dt'\,e^{(-i\,\omega_o - +g)(t-t')}\hat{F}_{in}(t')$$

$$\hat{a}^+(t) = \hat{a}^+(t=0)e^{(+i\omega_o + g)t} + \sqrt{A}\int_o^t dt'\,e^{(i\,\omega_o + g)(t-t')}\hat{F}_{in}^+(t')$$

The equal-time commutation relation is,

$$\left[\hat{a}(t), \hat{a}^+(t)\right] = \left[\hat{a}(t=0), \hat{a}^+(t=0)\right]e^{2g\,t} + A\int_o^t dt_1\int_o^t dt_2\,e^{(-i\,\omega_o + g)(t-t_1)}\,e^{(i\,\omega_o - \gamma)(t-t_2)} - \delta(t_1 - t_2)$$

$$= e^{2g\,t} - A\int_o^t dt_1\,e^{2g\,(t-t_1)}$$

$$= e^{2g\,t} - \frac{A}{2g}\left(e^{2g\,t} - 1\right)$$

If $A = 2g$, then $\left[\hat{a}(t), \hat{a}^+(t)\right] = 1$ for all time t.

8.4

a) $X_\theta(t) = \frac{1}{2}\left[\hat{a}(t)e^{-i\theta}e^{i\omega_0 t} + \hat{a}^+(t)e^{i\theta}e^{-i\omega_0 t}\right]$

$$\Rightarrow \frac{d\hat{X}_\theta(t)}{dt} = -\gamma\,\hat{X}_\theta(t) + \sqrt{2\gamma}\left\{\frac{\hat{S}_{in}(t)\,e^{-i\theta}e^{i\omega_0 t} + \hat{S}^+_{in}(t)\,e^{i\theta}e^{-i\omega_0 t}}{2}\right\}$$

$$\frac{d\hat{X}_{\theta+\frac{\pi}{2}}(t)}{dt} = -\gamma\,\hat{X}_{\theta+\frac{\pi}{2}}(t) + \sqrt{2\gamma}\left\{\frac{\hat{S}_{in}(t)e^{-i\theta}e^{i\omega_0 t} - \hat{S}^+_{in}(t)\,e^{i\theta}\,e^{-i\omega_0 t}}{2i}\right\}$$

b) $$\hat{X}_\theta(t) = \hat{X}_\theta\,e^{-\gamma t} + \sqrt{2\gamma}\int_0^t dt'\left\{\frac{\hat{S}_{in}(t')\,e^{-i\theta}e^{i\omega_0 t'} + \hat{S}^+_{in}(t')e^{i\theta}e^{-i\omega_0 t'}}{2}\right\}e^{-\gamma(t-t')}$$

$$\hat{X}_{\theta+\frac{\pi}{2}}(t) = \hat{X}_{\theta+\frac{\pi}{2}}\,e^{-\gamma t} + \sqrt{2\gamma}\int_0^t dt'\left\{\frac{\hat{S}_{in}(t')e^{-i\theta}e^{i\omega_0 t'} + \hat{S}^+_{in}(t')e^{i\theta}e^{-i\omega_0 t'}}{2}\right\}e^{-\gamma(t-t')}$$

$$\Rightarrow \langle\hat{X}_\theta(t)\rangle = \langle\hat{X}_\theta\rangle e^{-\gamma t} \quad \& \quad \langle\hat{X}_{\theta+\frac{\pi}{2}}(t)\rangle = \langle\hat{X}_{\theta+\frac{\pi}{2}}\rangle e^{-\gamma t}$$

Since noise averages are zero.

c)

$$\langle\hat{X}^2_\theta(t)\rangle = \langle\hat{X}^2_\theta\rangle e^{-2\gamma t} + \left\langle\sqrt{2\gamma}\,\hat{X}_\theta\,e^{-\gamma t}\int_0^t dt'\left\{\frac{\hat{S}_{in}(t')e^{-i\theta}e^{i\omega_0 t'} + \hat{S}^+_{in}(t')e^{i\theta}e^{-i\omega_0 t'}}{2}\right\}e^{-\gamma(t-t')}\right\rangle$$

$$+ \left\langle\sqrt{2\gamma}\int_0^t dt'\left\{\frac{\hat{S}_{in}(t')e^{-i\theta}e^{i\omega_0 t} + \hat{S}^+_{in}(t')e^{i\theta}e^{-i\omega_0 t'}}{2}\right\}e^{-\gamma(t-t')}\,\hat{X}_\theta\,e^{-\gamma t}\right\rangle$$

$$+ \frac{2\gamma}{4}\int_0^t dt'\int_0^t dt''\left\langle\hat{S}_{in}(t')\hat{S}^+_{in}(t'')\right\rangle e^{-\gamma(t-t')}e^{-\gamma(t-t'')}$$

$$\Rightarrow \langle\hat{X}^2_\theta(t)\rangle = \langle\hat{X}^2_\theta\rangle e^{-2\gamma t} + 0 + 0 + \frac{\gamma}{2}\int_0^t dt'\,e^{-2\gamma(t-t')}$$

$$\langle\hat{X}^2_\theta(t)\rangle = \langle\hat{X}^2_\theta\rangle e^{-2\gamma t} + \frac{1}{4}\left(1 - e^{-2\gamma t}\right)$$

$$\langle \Delta\hat{X}_\theta^2(t)\rangle = \langle \hat{X}_\theta^2(t)\rangle - \langle \hat{X}_\theta(t)\rangle^2$$

$$= \left\{ \langle \hat{X}_\theta^2\rangle - \langle \hat{X}_\theta\rangle^2 \right\} e^{-2\gamma t} + \frac{1}{4}\left(1-e^{-2\gamma t}\right)$$

$$\langle \Delta\hat{X}_\theta^2(t)\rangle = \langle \Delta\hat{X}_\theta^2\rangle e^{-2\gamma t} + \frac{1}{4}\left(1-e^{-2\gamma t}\right)$$

Since the above eq. is valid for any value of θ, it is valid also if one replaces θ by $\theta+\frac{\pi}{2}$.

$$\Rightarrow \langle \Delta\hat{X}_{\theta+\frac{\pi}{2}}^2(t)\rangle = \langle \Delta\hat{X}_{\theta+\frac{\pi}{2}}^2\rangle e^{-2\gamma t} + \frac{1}{4}\left(1-e^{-2\gamma t}\right)$$

d) Note $\hat{X}_1 = \hat{X}_\theta\big|_{\theta=0}$ $\hat{X}_2 = \hat{X}_\theta\big|_{\theta=\frac{\pi}{2}}$ $\therefore$ Part (c) solution implies:

$$\langle \Delta\hat{X}_1^2(t)\rangle = \langle \Delta\hat{X}_1^2\rangle e^{-2\gamma t} + \frac{1}{4}\left(1-e^{-2\gamma t}\right)$$

$$\& \quad \langle \Delta\hat{X}_2^2(t)\rangle = \langle \Delta\hat{X}_2^2\rangle e^{-2\gamma t} + \frac{1}{4}\left(1-e^{-2\gamma t}\right).$$

So irrespective of the initial mean square quadrature noise, the quadrature noise approaches $\frac{1}{4}$ as $t\to\infty$. In particular, if the initial state were a squeezed state with $\langle\Delta\hat{X}_1^2\rangle = \frac{1}{4}e^{-2r}$ and $\langle\Delta\hat{X}_2^2\rangle = \frac{1}{4}e^{2r}$, the squeezing will diminish as time progresses.

e)
$$\hat{a}(t) = \hat{a}\, e^{(-i\omega_0-\gamma)t} + \sqrt{2\gamma}\int_0^t dt'\, e^{(-i\omega_0-\gamma)(t-t')}\hat{S}_{in}(t')$$

$$\hat{a}^+(t) = \hat{a}^+ e^{(i\omega_0-\gamma)t} + \sqrt{2\gamma}\int_0^t dt'\, e^{(i\omega_0-\gamma)(t-t')}\hat{S}_{in}^+(t')$$

$$\Rightarrow \hat{n}(t) = \hat{a}^+(t)\hat{a}(t)$$

f) $\langle \hat{n}(t) \rangle = \langle \hat{a}^{+}\hat{a} \rangle e^{-2rt}$ $\left\{ \begin{array}{l} \text{all other terms go to zero on} \\ \text{averaging.} \end{array} \right.$

$= \langle \hat{n} \rangle e^{-2rt}$.

g) $\langle \hat{n}^2(t) \rangle = \langle \hat{n}^2 \rangle e^{-4rt} + \langle \hat{a}^{+}\hat{a}\, e^{-2rt}\, 2r \int_0^t dt' \int_0^t dt'' \left\{ \hat{S}_{in}(t')\, \hat{S}^{+}_{in}(t'')\, e^{(-i\omega_0 - r)(t-t')}\, e^{(i\omega_0 - r)(t-t'')} \right\}$

Since, the field is assumed to be uncorrelated with noise of the future the averages in the second term factor out.

$$\langle \hat{n}^2(t) \rangle = \langle \hat{n}^2 \rangle e^{-4rt} + \langle \hat{n} \rangle e^{-2rt} \left(1 - e^{-2rt}\right)$$

$$\Rightarrow \langle \Delta\hat{n}^2(t) \rangle = \langle \hat{n}^2(t) \rangle - \langle \hat{n}(t) \rangle^2$$

$$= \left\{ \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2 \right\} e^{-4rt} + \underbrace{\langle \hat{n} \rangle e^{-2rt}} \left(1 - e^{-2rt}\right)$$

$$\langle \Delta\hat{n}^2(t) \rangle = \langle \Delta\hat{n}^2 \rangle e^{-4rt} + \langle \hat{n}(t) \rangle \left(1 - e^{-2rt}\right)$$

$\Rightarrow$ irrespective of the initial mean square photon number noise, the mean square photon number noise approaches the average value as $t \to \infty$.

$$\Rightarrow \frac{\langle \Delta\hat{n}^2(t) \rangle}{\langle \hat{n}(t) \rangle} = \frac{\langle \Delta\hat{n}^2 \rangle}{\langle \hat{n} \rangle} e^{-2rt} + \left(1 - e^{-2rt}\right)$$

h) $|\psi(t=0)\rangle = |n\rangle \Rightarrow \langle \Delta\hat{n}^2 \rangle = 0$.

$\Rightarrow \dfrac{\langle \Delta\hat{n}^2(t) \rangle}{\langle \hat{n}(t) \rangle} = \left(1 - e^{-2rt}\right)$ which approaches 1 as $t \to \infty$

i) $|\psi(t=0)\rangle = |\alpha\rangle \Rightarrow \langle \Delta\hat{n}^2\rangle = |\alpha|^2 = \langle \hat{n}\rangle$.

$$\Rightarrow \frac{\langle \Delta\hat{n}^2(t)\rangle}{\langle \hat{n}(t)\rangle} = \frac{\langle\hat{n}\rangle}{\langle\hat{n}\rangle} e^{-2\gamma t} + (1-e^{-2\gamma t})$$

$$= e^{-2\gamma t} + (1-e^{-2\gamma t})$$

$$= 1 \quad \text{for all time.}$$

Coherent states remain coherent (with poisson statistics when undergoing loss).

j) $$\hat{S}_{out}(t) = -\hat{S}_{in}(t) + \sqrt{2\gamma}\, \hat{a}(t)$$

$$= -\hat{S}_{in}(t) + \sqrt{2\gamma}\left\{ \hat{a}\, e^{(-i\omega_0-\gamma)t} + \sqrt{2\gamma}\int_0^t dt'\, e^{(-i\omega_0-\gamma)(t-t')}\hat{S}_{in}(t') \right\}.$$

$$\langle \hat{S}^{\dagger}_{out}(t)\, \hat{S}_{out}(t)\rangle = 2\gamma\langle \hat{a}^{\dagger}(t)\, \hat{a}(t)\rangle$$

$$= 2\gamma\langle \hat{n}(t)\rangle.$$ $\Rightarrow$ avg. output flux is related to the mean photon number in the cavity. This is always true.