ECE 5310: Quantum Optics for Photonics and Optoelectronics

Fall 2013

Homework 8 **Due on Nov. 05, 2013 (self-grade)**

Problem 8.1 (Propagating squeezed states of light)

In this problem you will look at propagating squeezed states. Define a squeezing operator as follows,

$$\hat{S}(\varepsilon) = e^{\int_{-\infty}^{\infty} dz' \left(\frac{\varepsilon^*(z')}{2} \left(\hat{b}(z',0)\right)^2 - \frac{\varepsilon(z')}{2} \left(\hat{b}^+(z',0)\right)^2 \right)}$$

where,

$$\varepsilon(z) = r(z) e^{i2\phi(z)}$$

and $r(z)$ and $\phi(z)$ are real functions.

a) Find,

i) $\hat{S}^+(\varepsilon)\hat{b}(z,0)\hat{S}(\varepsilon)$

ii) $\hat{S}^+(\varepsilon)\hat{b}^+(z,0)\hat{S}(\varepsilon)$

The displacement operator is,

$$\hat{T}(\alpha) = e^{\int_{-\infty}^{\infty} dz' \left(\alpha(z')\hat{b}^+(z',0) - \alpha^*(z')\hat{b}(z',0) \right)}$$

A propagating squeezed state is written as,

$$|\alpha(z), \varepsilon(z)\rangle = \hat{T}(\alpha)\,\hat{S}(\varepsilon)|0\rangle$$

A propagating squeezed vacuum state is written as,

$$\hat{S}(\varepsilon)|0\rangle$$

For all the remaining parts assume, $\varepsilon(z) = r\, e^{i2\phi} =$ constant , and $\alpha(z) = |\alpha| e^{i(\phi+\pi/2)} =$ constant . Also, assume free fields (no interactions) and no dispersion. Suppose, at time $t = 0$,

$$|\psi(t=0)\rangle = |\alpha(z), \varepsilon(z)\rangle = \hat{T}(\alpha)\,\hat{S}(\varepsilon)|0\rangle$$

b) Find the state $|\psi(t)\rangle$ at time t .

c) Find the average values of the quadrature operators $\hat{x}_{\phi}(z,t)$ and $\hat{x}_{\phi+\pi/2}(z,t)$ at time t . You need to find,

$$\langle\psi(t=0)|\hat{x}_{\phi}(z,t)|\psi(t=0)\rangle \text{ and } \langle\psi(t=0)|\hat{x}_{\phi+\pi/2}(z,t)|\psi(t=0)\rangle$$

d) Find the quadrature noise correlation functions,

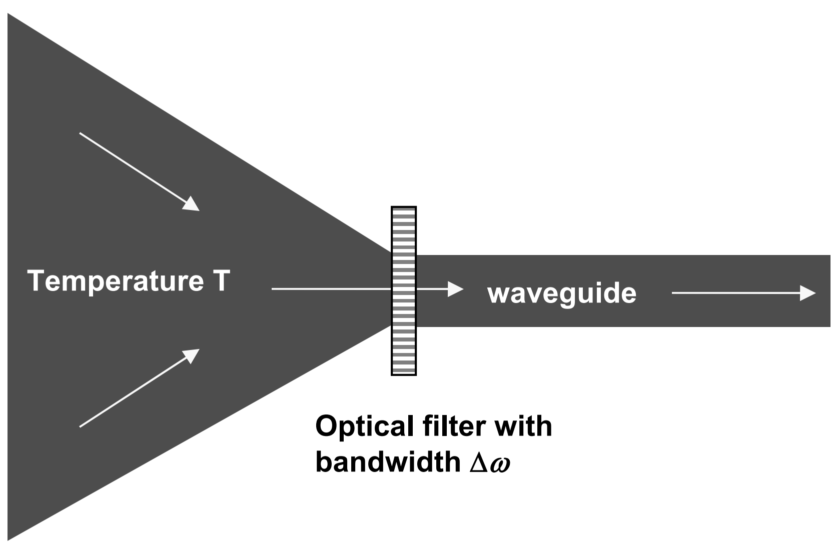
$$\langle \psi(t=0) | \Delta\hat{x}_{\phi}(z,t_1)\, \Delta x_{\phi}(z,t_2) | \psi(t=0) \rangle$$
$$\langle \psi(t=0) | \Delta\hat{x}_{\phi+\pi/2}(z,t_1)\, \Delta x_{\phi+\pi/2}(z,t_2) | \psi(t=0) \rangle$$

e) Find the average value of the photon flux,
$$\langle \psi(t=0) | \hat{F}(z,t) | \psi(t=0) \rangle$$

f) Find the photon flux noise correlation function,
$$\langle \psi(t=0) | \hat{F}(z,t_1)\, \hat{F}(z,t_2) | \psi(t=0) \rangle$$
and compare your results with the case when the quantum state is a coherent state with the same value of $\alpha(z) = |\alpha| e^{i(\phi+\pi/2)}$ = constant .

Problem 8.2 (Thermal radiation and noise)

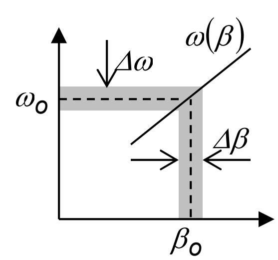
In many cases of practical interest, one is interested in detecting and characterizing thermal radiation (e.g. light from arc lamps, electric bulbs, infrared imaging, night vision, satellite remote sensing etc). In this problem, you will look at a model for thermal radiation. Consider the figure shown below,



Radiation from some thermal source at temperature T is collected and channeled into a **non-interacting non-dispersive** waveguide. An optical filter with transmission bandwidth $\Delta\omega$ centered at ω_o is placed in front of the waveguide. The filter only allows incoming photons that have frequencies within this transmission band to pass. It therefore makes sense to expand the field operator inside the waveguide in only this bandwidth $\Delta\omega$, as shown below,

$$\hat{b}(z,t) = L \int_{\beta_o - \Delta\beta/2}^{\beta_o + \Delta\beta/2} \frac{d\beta}{2\pi}\; \hat{a}(\beta)\, \frac{\exp[i(\beta - \beta_o)\, z]}{\sqrt{L}}\, \exp[-i(\omega(\beta) - \omega(\beta_o))\, t]$$

where $\Delta\beta = \frac{d\beta}{d\omega}\Delta\omega = \frac{\Delta\omega}{v_g}$, and $\omega_o = \omega(\beta_o)$.



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NOTES ADDED:

In a multi-mode cavity, the density operator corresponding to a thermal state is given by the factored form,

$$\hat{\rho} = \hat{\rho}_1 \otimes \hat{\rho}_2 \otimes \hat{\rho}_3 \otimes \ldots\ldots\ldots\ldots$$

where the thermal density operator $\hat{\rho}_k$ for the k-th mode is,

$$\hat{\rho}_k = \sum_{n=0}^{\infty} P_k(n) |n\rangle_k \, {}_k\langle n| \qquad \text{and} \qquad P_k(n) = \left[1 - \exp\left(-\frac{\hbar\omega_k}{K_B\, T}\right)\right] \exp\left(-\frac{n\, \hbar\omega_k}{K_B\, T}\right)$$

The following relations follow from the form of the thermal density matrix $\hat{\rho}$,

$$\langle \hat{a}_k \rangle = \mathrm{Tr}\{\hat{\rho}\ \hat{a}_k\} = 0 \qquad \text{and} \qquad \langle \hat{a}_k^+ \rangle = \mathrm{Tr}\{\hat{\rho}\ \hat{a}_k^+\} = 0$$

$$\left\langle \hat{a}_k^+\ \hat{a}_q \right\rangle = \delta_{kq}\ n_{th}(\omega_k) = \frac{\delta_{kq}}{\exp\left(\dfrac{\hbar\omega_k}{K_B\, T}\right) - 1} \qquad \text{and} \qquad \left\langle \hat{a}_q\ \hat{a}_k^+ \right\rangle = \delta_{kq}\ [n_{th}(\omega_k) + 1]$$

So the averages are uncorrelated among different modes.

What about more complex averages? For example, $\left\langle \hat{a}_r\ \hat{a}_s^+\ \hat{a}_q\ \hat{a}_k^+ \right\rangle$. These can be evaluated by using the following easy-to-remember rules which work for the thermal density operator $\hat{\rho}$ (this does not mean that these rules will work for any other density operator):

i) If the total number of creation and destruction operators is odd the result is zero.

ii) If the total number of creation operators is not equal to the total number of destruction operators, the result is zero.

iii) If (i) and (ii) are satisfied, then the result is the sum of products of all possible pairings of the creation and destruction operators without disturbing the order of the operators, as shown below. The two possible pairings schemes are shown graphically with solid lines at the top and bottom.

$$\left\langle \hat{a}_r^+\ \hat{a}_s\ \hat{a}_q^+\ \hat{a}_k \right\rangle = \left\langle \hat{a}_r^+\ \hat{a}_s \right\rangle \left\langle \hat{a}_q^+\ \hat{a}_k \right\rangle + \left\langle \hat{a}_r^+\ \hat{a}_k \right\rangle \left\langle \hat{a}_s\ \hat{a}_q^+ \right\rangle$$

You are going to use the ideas presented above for the waveguide modes. Since the photons traveling in the waveguide in the rightward direction are coming from the thermal source, their state is described by a thermal density operator. Generalizing from what we saw for the case of the cavity, we can write the following expressions for the operator averages,

$$\langle \hat{a}(\beta) \rangle = 0 \qquad \text{and} \qquad \langle \hat{a}^+(\beta) \rangle = 0$$

$$\left\langle \hat{a}^+(\beta)\, \hat{a}(\beta') \right\rangle = n_{th}(\omega(\beta))\, \delta_{\beta\,\beta'} = \frac{\delta_{\beta\,\beta'}}{\exp\left(\dfrac{\hbar\omega(\beta)}{K_B\, T}\right) - 1} \approx \frac{\delta_{\beta\,\beta'}}{\exp\left(\dfrac{\hbar\omega_o}{K_B\, T}\right) - 1} = n_{th}(\omega_o)\, \delta_{\beta\,\beta'}$$

$$\left\langle \hat{a}(\beta)\, \hat{a}^+(\beta') \right\rangle = [n_{th}(\omega_o) + 1]\, \delta_{\beta\,\beta'}$$

$$\langle \hat{a}^{+}(\beta_1)\,\hat{a}(\beta_2)\,\hat{a}^{+}(\beta_3)\,\hat{a}(\beta_4)\rangle = \langle \hat{a}^{+}(\beta_1)\,\hat{a}(\beta_2)\rangle\langle \hat{a}^{+}(\beta_3)\,\hat{a}(\beta_4)\rangle + \langle \hat{a}^{+}(\beta_1)\,\hat{a}(\beta_4)\rangle\langle \hat{a}(\beta_2)\,\hat{a}^{+}(\beta_3)\rangle$$

a) Calculate the average photon flux in the waveguide, $\langle \hat{F}(z,0)\rangle$. The average is with respect to a statistical mixture corresponding to a thermal state for each mode.

Hint: You can also assume that $\omega(\beta)$ is not a strong function of the wavevector β, and so every occurrence of the thermal occupation number $n_{th}(\omega(\beta))$ can be replaced by its value $n_{th}(\omega_o)$ at the center of the observation bandwidth. Your answer should be: $\langle \hat{F}(z,0)\rangle = \frac{\Delta\omega}{2\pi} n_{th}(\omega_o)$.

INTERPRETATION:
The number of rightward propagating modes in the waveguide of length L in a **wavevector** bandwidth $\Delta\beta$ is equal to $\frac{L}{2\pi}\Delta\beta = \frac{L}{2\pi}\frac{\Delta\omega}{v_g}$. Each mode has an average photon occupation equal to $n_{th}(\omega_o)$. So the number of photons per unit length is $\frac{1}{2\pi}\frac{\Delta\omega}{v_g} n_{th}(\omega_o)$. Photons move at the group velocity v_g. So the average photon flux at any point in the waveguide in a **frequency** bandwidth $\Delta\omega$ is equal to,

$$\frac{\Delta\omega}{2\pi\ v_g} \times n_{th}(\omega_o) \times v_g = \frac{\Delta\omega}{2\pi} n_{th}(\omega_o)$$

b) Calculate the photon flux correlation function, $\langle \hat{F}(z,t_1)\hat{F}(z,t_2)\rangle$.

Hint: Your answer should be,

$$\langle \hat{F}(z,t_1)\hat{F}(z,t_2)\rangle = \langle \hat{F}(z,t_1)\rangle^2 + n_{th}(\omega_o)\left[n_{th}(\omega_o)+1\right]\left[\frac{\Delta\omega}{2\pi}\frac{\sin\left(\frac{\Delta\omega\,(t_1-t_2)}{2}\right)}{\frac{\Delta\omega\,(t_1-t_2)}{2}}\right]^2$$

c) Using your results from parts (a) and (b), calculate the flux noise correlation function,

$$\langle \Delta\hat{F}(z,t_1)\Delta\hat{F}(z,t_2)\rangle$$

and also the flux noise spectral density $S_{\Delta F\ \Delta F}(\omega)$. You may make the following approximation,

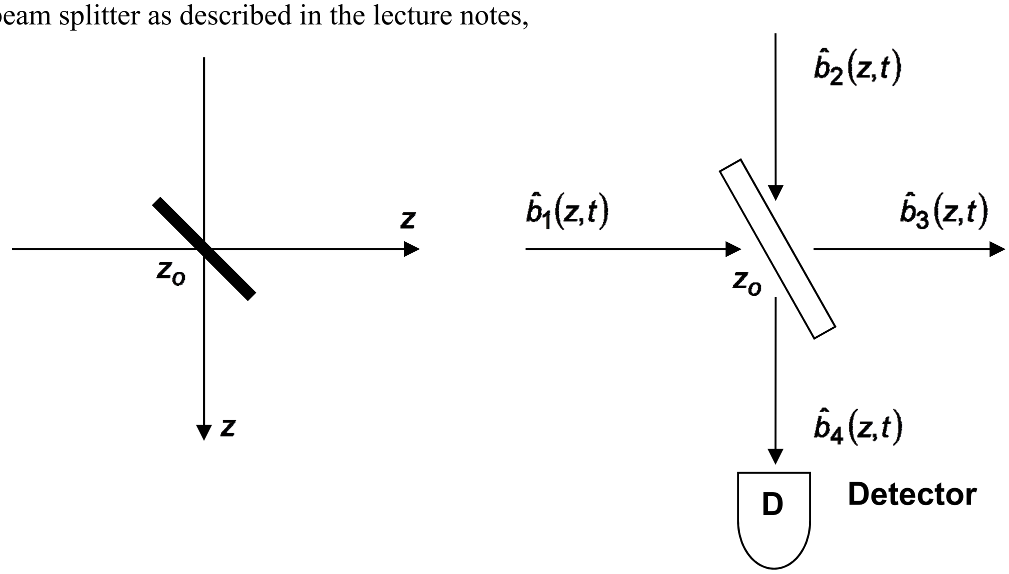
$$\left[\frac{\Delta\omega}{2\pi}\frac{\sin\left(\frac{\Delta\omega\,(t_1-t_2)}{2}\right)}{\frac{\Delta\omega\,(t_1-t_2)}{2}}\right]^2 \approx \frac{\Delta\omega}{2\pi}\,\delta(t_1-t_2)$$

d) Argue that when $n_{th}(\omega_o) << 1$, the photon flux has shot noise.

NOTE ADDED: For visible and near infrared frequencies (λ < 5 μm) $n_{th}(\omega_o) << 1$ (unless the temperature of the thermal source is more than few thousand degrees), and the photon flux has shot noise. However, when $n_{th}(\omega_o) >> 1$, the flux noise is more than shot noise.

Problem 8.3 (Beam splitters and transformation of quantum states of light under optical loss)

In this problem you will see how photon number states transform under photon loss. Consider a 50-50 beam splitter as described in the lecture notes,



for which the input-output relation are given at all times by,

$$\begin{bmatrix} \hat{b}_3(z_o,t) \\ \hat{b}_4(z_o,t) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \hat{b}_1(z_o,t) \\ \hat{b}_2(z_o,t) \end{bmatrix}$$

and the inverse relation is,

$$\begin{bmatrix} \hat{b}_1(z_o,t) \\ \hat{b}_2(z_o,t) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \hat{b}_3(z_o,t) \\ \hat{b}_4(z_o,t) \end{bmatrix}$$

Part I: For parts (a) through (c) the incoming photon packet state on channel 1 is described at time $t = 0$ by the expression,

$$|\psi(t=0)\rangle = |n, A(z)\rangle_1 = \frac{\left[\int_{-\infty}^{\infty} dz' \;\; A(z')\,\hat{b}_1^{+}(z',0)\right]^n}{\sqrt{n!}} |0\rangle$$

a) What is the state $|\psi(t)\rangle$ for time sufficiently large that the packet has for sure gone through the beam splitter? You will see that the photons states in the output channels are **entangled**.

b) If a photodetector is inserted in the output channel 3, what is the probability $P(m)$ that it will detect "m" photons?

Hint: The probability for detecting a certain number of photons follows a binomial distribution.

c) What is the average number of photons that will be detected by a photodetector placed in the output channel 3. Does your result confirm to your intuitive expectation?

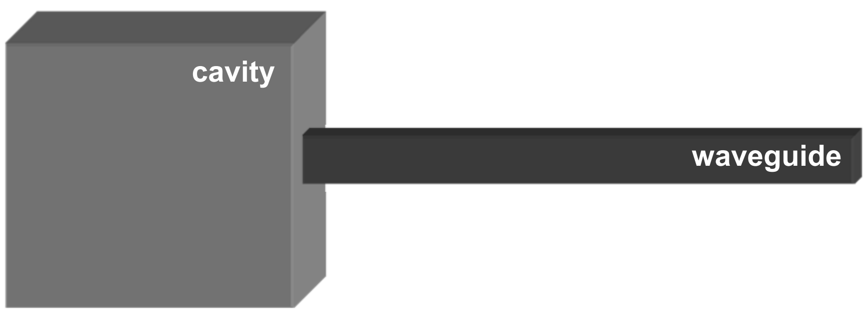
d) **Quantum state collapse:** For this part assume that the detector is taken from channel 3 and placed in channel 4. If at time t the detector placed in channel 4 detected (or measured) "p" photons, what is the quantum state of **all the channels** immediately after this detection? NOTE: You can assume that the state describing channel 4 immediately after the measurement is the vacuum state $|0\rangle_4$, since photodetection, during the detection (or counting) process, usually destroys the photons that were originally in the state. If after the detection in channel 4, a measurement is subsequently made to count the number of photons in channel 3, what are the possible results and with what probabilities? Did the act of photodetection in channel 4 changed the photodetection results for channel 3 that you had obtained earlier in part (b). The reason this happens is because the photon states in the output channels are **entangled**.

Problem 8.4 (Lossy optical cavities: destruction of squeezing and increased photon number fluctuations)

Consider a single mode lossy optical cavity with the Hamiltonian,

$$\hat{H} = \hbar\omega_o \, \hat{a}^+ \hat{a}$$

The loss is due the photons escaping from the cavity into the waveguide.



The time development equations for the Heisenberg operators $\hat{a}(t)$ and $\hat{a}^+(t)$ are as in the lecture notes,

$$\frac{d}{dt}\begin{bmatrix} \hat{a}(t) \\ \hat{a}^+(t) \end{bmatrix} = \begin{bmatrix} -i\omega_o - \gamma & 0 \\ 0 & i\omega_o - \gamma \end{bmatrix} \begin{bmatrix} \hat{a}(t) \\ \hat{a}^+(t) \end{bmatrix} + \sqrt{2\gamma} \begin{bmatrix} \hat{S}_{in}(t) \\ \hat{S}_{in}^+(t) \end{bmatrix}$$

where, $\left[\hat{S}_{in}(t), \hat{S}_{in}^+(t')\right] = \delta(t - t')$ and we define,

$$\hat{S}_{in}(t) = \sqrt{v_g}\, \hat{b}_L(0,t) e^{-i\omega_o t}$$

$$\hat{S}_{out}(t) = \sqrt{v_g}\, \hat{b}_R(0,t) e^{-i\omega_o t}$$

a) Find the time differential equations for the quadrature operators $\hat{X}_\theta(t)$ and $\hat{X}_{\theta+\frac{\pi}{2}}(t)$, where the time dependent quadrature operators are defined as in the lecture notes,

$$\hat{X}_\theta(t) = \frac{\hat{a}(t)\exp(i\omega_o t)\exp(-i\theta) + \hat{a}^+(t)\exp(-i\omega_o t)\exp(i\theta)}{2}$$

$$\hat{X}_{\theta+\frac{\pi}{2}}(t) = \frac{\hat{a}(t)\exp(i\omega_o t)\exp(-i\theta) - \hat{a}^+(t)\exp(-i\omega_o t)\exp(i\theta)}{2i}$$

Note: the angle θ is completely arbitrary.

b) Solve the differential equations derived in part (a), and find the average values of the quadrature operators at time t in terms of their average values at time $t=0$ for any arbitrary initial quantum state of the cavity. In other words, find $\langle \hat{X}_\theta(t) \rangle$ and $\langle \hat{X}_{\theta+\pi/2}(t) \rangle$ in terms of $\langle \hat{X}_\theta \rangle$ and $\langle \hat{X}_{\theta+\pi/2} \rangle$.

NOTE ADDED: The averages denoted by the angled brackets above mean averages w.r.t. the initial state of the cavity as well as the initial state of the noise causing system. The initial state of the noise causing system is assumed to be the "vacuum" entering through the waveguide so that the following relations hold,

$$\langle \hat{S}_{in}^+(t) \rangle = \langle \hat{S}_{in}(t) \rangle = 0 \qquad \langle \hat{S}_{in}^+(t)\, \hat{S}_{in}(t') \rangle = 0 \qquad \langle \hat{S}_{in}(t)\, \hat{S}_{in}^+(t') \rangle = \delta(t-t')$$

$\langle \ldots\ldots \hat{S}_{in}(t) \rangle = \langle \hat{S}_{in}^+(t) \ldots\ldots \rangle = 0$ (the dots stand for any sequence of operators)

$\langle \hat{S}_{in}^a(t_1) \hat{S}_{in}^b(t_2) \ldots\ldots \hat{S}_{in}^c(t_n) \rangle = 0$ for n odd (the alphabets 'a', 'b', and 'c' mean that the operator is an adjoint if the alphabet is '+', or not an adjoint if the alphabet is ' ').

c) Find variances of the quadraure operators at time t in terms of their values at time $t=0$, i.e. find $\langle \Delta\hat{X}_\theta^2(t) \rangle$ and $\langle \Delta\hat{X}_{\theta+\pi/2}^2(t) \rangle$ in terms of $\langle \Delta\hat{X}_\theta^2 \rangle$ and $\langle \Delta\hat{X}_{\theta+\pi/2}^2 \rangle$.

d) Conclude from your result in part (c) that if the initial state at time $t=0$ in the cavity were a squeezed state with $\langle \Delta\hat{X}_1^2 \rangle = \frac{1}{4}\exp(-2r)$ and $\langle \Delta\hat{X}_2^2 \rangle = \frac{1}{4}\exp(2r)$, then the degree of squeezing will diminish as photons are lost from the cavity. The degree of squeezing is the degree by which the maximum and minimum standard deviations of the quadrature operators differ from the coherent state value of $\frac{1}{4}$.

e) Find the photon number operator $\hat{n}(t)$, where $\hat{n}(t) = \hat{a}^+(t)\hat{a}(t)$, by first solving for $\hat{a}^+(t)$ and $\hat{a}(t)$. You can use the results from the lecture notes directly.

f) From your answer in part (e), express the average value of the photon number operator at time t in terms of its average value at time $t=0$, find $\langle \hat{n}(t) \rangle$ in terms of $\langle \hat{n} \rangle$.

g) Find the standard deviation in the photon number at time t in terms of the standard deviation at time $t=0$ and the average value at time t, i.e. find $\langle \Delta\hat{n}^2(t) \rangle$ in terms of $\langle \Delta\hat{n}^2 \rangle$ and $\langle \hat{n}(t) \rangle$.

h) Conclude from your result in part (g) that **if the initial state at time $t=0$ in the cavity were a photon number state $|n\rangle$ with no photon number fluctuations**, then as time progresses the standard deviation in the photon number at time t, normalized to the average photon number at time t, increases with time and approaches unity, i.e. show that $\frac{\langle \Delta\hat{n}^2(t) \rangle}{\langle \hat{n}(t) \rangle}$ is an increasing function of time and approaches unity for large times.

i) Conclude from your result in part (g) that **if the initial state at time $t = 0$ in the cavity were a coherent state $|\alpha\rangle$**, then $\frac{\langle \Delta\hat{n}^2(t)\rangle}{\langle \hat{n}(t)\rangle}$ is constant and equal to unity for all time.

NOTE ADDED: Just like in the beam splitter problem, a coherent state in a cavity remains a coherent state when undergoing loss.

j) Given that, $\hat{S}_{out}(t) = -\hat{S}_{in}(t) + \sqrt{2\gamma}\;\hat{a}(t)$, find the average output photon flux from the cavity at time t **if the initial state at time $t = 0$ in the cavity were a coherent state $|\alpha\rangle$.**

Hint: You need to evaluate $\left\langle \hat{S}^{+}_{out}(t)\,\hat{S}_{out}(t)\right\rangle$

Problem 8.5 (Optical cavities with gain)

In this problem you will explore the noise sources for a linear optical amplifier. You have seen how optical gain is produced in a semi-classical treatment in an earlier problem set. Here we build a quantum model of a cavity with gain (as opposed to loss). We will discuss a more detailed microscopic model for gain in a week or two in the class. This is a warm up exercise. The amplifier considered below is an example of a phase insensitive amplifier since it will amplify all field quadratures in the same way.

a) Suppose one writes equations for the field operators in a cavity with optical gain g as follows,

$$\frac{d}{dt}\begin{bmatrix}\hat{a}(t)\\ \hat{a}^{+}(t)\end{bmatrix} = \begin{bmatrix}-\mathrm{i}\omega_o + g & 0\\ 0 & \mathrm{i}\omega_o + g\end{bmatrix}\begin{bmatrix}\hat{a}(t)\\ \hat{a}^{+}(t)\end{bmatrix}$$

Show that the equal-time commutation relation for the cavity field operators is violated because of the gain (i.e. $\left[\hat{a}(t),\, \hat{a}^{+}(t)\right] \neq 1$).

b) If one tries to satisfy the equal-time commutation relation by introducing phenomenological noise sources, as shown below,

$$\frac{d}{dt}\begin{bmatrix}\hat{a}(t)\\ \hat{a}^{+}(t)\end{bmatrix} = \begin{bmatrix}-\mathrm{i}\omega_o + g & 0\\ 0 & \mathrm{i}\omega_o + g\end{bmatrix}\begin{bmatrix}\hat{a}(t)\\ \hat{a}^{+}(t)\end{bmatrix} + \sqrt{A}\begin{bmatrix}\hat{F}_{in}(t)\\ \hat{F}^{+}_{in}(t)\end{bmatrix}$$

then show that this recipe works provided,

$$\left[\hat{F}_{in}(t),\, \hat{F}^{+}_{in}(t')\right] = -\,\delta(t-t') \qquad \text{(notice the strange minus sign!)}$$

and

$$A = 2g$$

Hint: Follow exactly the same steps as in the lecture notes for the case of loss.