7.1

(a). Suppose I know that when I have $n-2$ terms in $\langle F(t_1) F(t_2) \dots F(t_{n-2}) \rangle$ I get f_{n-2} different pairs in the expansion. Then I look at $\langle F(t_1) F(t_2) \dots F(t_n) \rangle$. I can first pair the '$t_1$' term with '$t_2$', and the remaining $(n-2)$ terms can be paired in f_{n-2} ways. I can then pair

't_1' term with 't_3' and the remaining $(n-2)$ pairs can be paired in f_{n-2} ways. Going like this, I pair 't_1' with 't_2', then with 't_3', then with 't_4', ..., and finally with 't_n' and in each case I can pair the remaining terms in f_{n-2} ways. So it must be that,

$$f_n = (n-1) f_{n-2}.$$

$$\Rightarrow \quad f_n = (n-1)(n-3) f_{n-5}$$

$$\Rightarrow \quad f_n = (n-1)(n-3)(n-5) \ldots\ldots 3.1$$

$$= \frac{n!}{(\frac{n}{2})!\, 2^{n/2}}$$

b) $\quad \frac{d\theta(t)}{dt} = W(t) \Rightarrow \theta(t_1) = \theta(t_2) + \int_{t_2}^{t_1} W(t)\, dt \quad \{\text{for } t_1 > t_2\}$

$$\Rightarrow \quad \theta(t_1) - \theta(t_2) = \int_{t_2}^{t_1} W(t)\, dt \Rightarrow \langle (\theta(t_1) - \theta(t_2))^2 \rangle = \int_{t_2}^{t_1} dt' \int_{t_2}^{t_1} dt'' \langle W(t') W(t'') \rangle$$

$$\Rightarrow \langle (\theta(t_1) - \theta(t_2))^2 \rangle = \int_{t_2}^{t_1} dt' \int_{t_2}^{t_1} dt''\, \gamma\, \delta(t' - t'') = \gamma(t_1 - t_2)$$

Similarly for $t_2 > t_1$ one can show $\langle (\theta(t_2) - \theta(t_1))^2 \rangle = \gamma(t_2 - t_1)$

$$\Rightarrow \langle [\theta(t_1) - \theta(t_2)]^2 \rangle = \gamma |t_1 - t_2|.$$

c) For n even:

$$\langle [\theta(t_1) - \theta(t_2)]^n \rangle = \int_{t_1}^{t_2} ds_1 \int_{t_1}^{t_2} ds_2 \int_{t_1}^{t_2} ds_3 \ldots \int_{t_1}^{t_2} ds_n \langle W(s_1) W(s_2) \ldots W(s_n) \rangle$$

there are $\frac{n!}{(n/2)!\, 2^{n/2}}$ different possible pairing schemes of the noise sources $W(\cdot)$. In each pairing scheme there will be $\frac{n}{2}$ pairs and each pair will give $\gamma|t_1-t_2|$ (as in part (a) above). Therefore, each pairing scheme will give $\gamma^{n/2}|t_1-t_2|^{n/2}$, and since there are $\frac{n!}{(n/2)!\,2^{n/2}}$ different pairing schemes, we get.

$$\langle[\theta(t_1)-\theta(t_2)]^n\rangle = \frac{n!}{(n/2)!\,2^{n/2}}\,\gamma^{n/2}\,|t_1-t_2|^{n/2}$$

For n odd:

$$\langle[\theta(t_1)-\theta(t_2)]^n\rangle = \int_{t_1}^{t_2} ds_1 \int_{t_1}^{t_2} ds_2 \cdots \int_{t_1}^{t_2} ds_n \,\langle W(s_1)\,W(s_2)\cdots W(s_n)\rangle$$

$$= 0 \quad \text{since } W(t) \text{ is a GWN.}$$

d) $$\langle \exp[i(\theta(t_1)-\theta(t_2))]\rangle = \sum_{n=0}^{\infty} \frac{(i)^n}{n!}\langle[\theta(t_1)-\theta(t_2)]^n\rangle$$

$$= \sum_{\substack{n=0\\ n \text{ even only}}}^{\infty} \frac{(i)^n}{(\frac{n}{2})!}\left(\frac{\gamma|t_1-t_2|}{2}\right)^{n/2} = \sum_{\substack{n=0\\ n \text{ even only}}}^{\infty} \frac{\left(i^2\,\frac{\gamma|t_1-t_2|}{2}\right)^{n/2}}{(n/2)!}$$

$$= \sum_{m=0}^{\infty} \frac{\left(-\frac{\gamma}{2}|t_1-t_2|\right)^m}{m!} \qquad \left\{\begin{array}{l}\text{where I defined}\\ m = n/2\end{array}\right\}$$

$$= \exp\left[-\frac{\gamma}{2}|t_1-t_2|\right]$$

e) $$I(t) = I_0 \cos[\omega_0 t + \theta(t)] = \frac{I_0}{2}\left\{e^{i\omega_0 t + i\theta(t)} + e^{-i\omega_0 t - i\theta(t)}\right\}$$

$$\Rightarrow \langle I(t_1)I(t_2)\rangle = \frac{I_0^2}{4}\left\langle e^{i\omega_0(t_1-t_2)+i[\theta(t_1)-\theta(t_2)]}\right\rangle + \frac{I_0^2}{4}\left\langle e^{-i\omega_0(t_1-t_2)-i[\theta(t_1)-\theta(t_2)]}\right\rangle$$

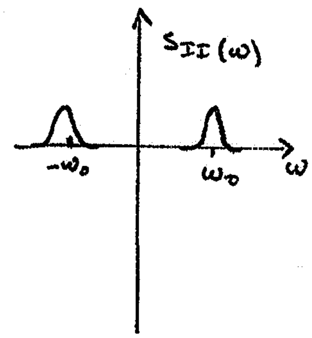
$$= \frac{I_0^2}{4} e^{i\omega_0(t_1-t_2)} e^{-\frac{\gamma}{2}|t_1-t_2|} + \frac{I_0^2}{4} e^{-i\omega_0(t_1-t_2)} e^{-\frac{\gamma}{2}|t_1-t_2|}$$

$$\Rightarrow R_{II}(\tau) = \frac{I_0^2}{4}\left\{ e^{i\omega_0\tau} + e^{-i\omega_0\tau}\right\} e^{-\frac{\gamma}{2}|\tau|}$$

f)

$$\Rightarrow S_{II}(\omega) = \int_{-\infty}^{\infty} d\tau \, e^{i\omega\tau} R_{II}(\tau)$$

$$= \frac{I_0^2}{4}\left\{ \frac{\gamma}{(\omega+\omega_0)^2+(\frac{\gamma}{2})^2} + \frac{\gamma}{(\omega-\omega_0)^2+(\frac{\gamma}{2})^2}\right\} =$$



$\Delta\omega$ = Line width $= 2\cdot\left(\frac{\gamma}{2}\right) = \gamma$

g)

$$S_{II}(\omega) = \frac{I_0^2}{4}\left\{ 2\pi\,\delta(\omega+\omega_0) + 2\pi\,\delta(\omega-\omega_0)\right\}.$$

if there is no phase noise

h)

$$\text{Power} = R\int_{-\infty}^{\infty} S_{II}(\omega)\,\frac{d\omega}{2\pi} = \frac{I_0^2}{2}R$$

makes sense since for $I(t) = I_0\cos(\omega_0 t + \theta(t))$ the power dissipated should not depend upon the phase $\theta(t)$. So $\text{power} = \langle I^2(t) R\rangle = I_0^2 R \langle \cos^2(\omega_0 t + \theta(t))\rangle$

$$= \frac{I_0^2 R}{2}$$

7.2

a) $|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \Rightarrow \hat{\rho} = |\alpha\rangle\langle\alpha| = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\alpha^n (\alpha^*)^m}{\sqrt{n!\,m!}} |n\rangle\langle m|\, e^{-|\alpha|^2}$

$P(n) =$

$$= \langle n|\hat{\rho}|n\rangle = \frac{(|\alpha|^2)^n}{n!} e^{-|\alpha|^2}$$

b) no part b!

c). $P(n) = \langle n|\hat{\rho}_A|n\rangle = \frac{(|\alpha|^2)^n}{n!} e^{-|\alpha|^2}$

d) Alice's density matrix has no off-diagonal terms in number state basis. Photon number measurements probe only the diagonal elements of the density matrix (in number state basis) and since $\hat{\rho}$ and $\hat{\rho}_A$ have the same diagonal components in number state basis, photon number measurements cannot distinguish between $\hat{\rho}$ and $\hat{\rho}_A$.

e) If one measures the field strengths $\hat{q}$ or $\hat{p}$ (where $\hat{a} = \frac{1}{\sqrt{2\hbar\omega_m}}(\omega_m \hat{q} + i\hat{p})$) then this will give different results for quantum states corresponding to $\hat{\rho}$ and $\hat{\rho}_A$.

e.g. $\mathrm{Tr}\{\hat{\rho}\,\hat{q}\} = \sqrt{\frac{\hbar}{2\omega_m}}(\alpha + \alpha^*)$ & $\mathrm{Tr}\{\hat{\rho}_A\,\hat{q}\} = 0$

Measurement results for $\hat{q}$ and $\hat{p}$ depend on the off-diagonal components of the density matrix in number state basis.

f) $\hat{\rho}_B = \int_0^{2\pi} \frac{d\phi}{2\pi} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\alpha^n (\alpha^*)^m}{\sqrt{n!\,m!}} |n\rangle\langle m|\, e^{-|\alpha|^2}$

$$\alpha = |\alpha| e^{i\phi}$$

$$\hat{\rho}_B = \int_0^{2\pi} \frac{d\phi}{2\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{|\alpha|^{n+m}}{\sqrt{n!\,m!}} e^{i\phi(n-m)} |n\rangle\langle m| \, e^{-|\alpha|^2}$$

$$= \sum_{n=0}^{\infty} \frac{(|\alpha|^2)^n}{n!} e^{-|\alpha|^2} |n\rangle\langle n|$$

g) $\hat{\rho}_B = \hat{\rho}_A$. No measurement can decide between the ensembles represented by the density matrices $\hat{\rho}_B$ and $\hat{\rho}_A$ since the density matrices are the same.

h) See (g).

7.3

a) Let $|\phi(t)\rangle = e^{\frac{i\hat{H}_0 t}{\hbar}} |\psi(t)\rangle \qquad \hat{H}_0 = \hbar\omega_0 \hat{a}^+\hat{a}$

Then $i\hbar \frac{\partial}{\partial t}|\phi(t)\rangle = e^{\frac{i\hat{H}_0 t}{\hbar}} \left\{ i\hbar \frac{|k|}{2}\left(e^{i2\omega_0 t - i\theta}\hat{a}^2 - e^{-i2\omega_0 t + i\theta}\hat{a}^{+2}\right)\right\} e^{\frac{-i\hat{H}_0 t}{\hbar}} |\phi(t)\rangle$

$$= i\hbar\frac{|k|}{2}\left(e^{-i\theta}\hat{a}^2 - e^{i\theta}\hat{a}^{+2}\right)|\phi(t)\rangle.$$

$$\Rightarrow \quad |\phi(t)\rangle = \exp\left[\frac{|k|t}{2}\left(e^{-i\theta}\hat{a}^2 - e^{i\theta}\hat{a}^{+2}\right)\right]|\phi(t=0)\rangle.$$

And

$$|\psi(t)\rangle = e^{\frac{-i\hat{H}_0 t}{\hbar}}|\phi(t)\rangle = e^{\frac{-i\hat{H}_0 t}{\hbar}} \exp\left[\frac{|k|t}{2}\left(e^{-i\theta}\hat{a}^2 - e^{i\theta}\hat{a}^{+2}\right)\right] e^{\frac{i\hat{H}_0 t}{\hbar}} e^{\frac{-i\hat{H}_0 t}{\hbar}}|0\rangle$$

$$= \exp\left[\frac{|k|t}{2}\left(e^{-i\theta + i2\omega_0 t}\hat{a}^2 - e^{i\theta - i2\omega_0 t}\hat{a}^{+2}\right)\right]|0\rangle.$$

b) $i\hbar \frac{d\hat{a}(t)}{dt} = \left[\hat{a}(t), \hat{H}(t)\right]$

$$\Rightarrow \quad \frac{d\hat{a}(t)}{dt} = -i\omega_0 \hat{a}(t) - |k|\, e^{-i2\omega_0 t + i\theta}\hat{a}^+(t)$$

$$\frac{d\hat{a}^+(t)}{dt} = i\omega_0 \hat{a}^+(t) - |k|\, e^{i2\omega_0 t - i\theta}\hat{a}(t)$$

Solution is:

$$\hat{a}(t) = \hat{a}(0)e^{-i\omega_0 t}\cosh|k|t - e^{i\theta}\hat{a}^+(0)e^{-i\omega_0 t}\sinh|k|t.$$

$$\hat{a}^+(t) = -e^{-i\theta}\hat{a}(0)e^{i\omega_0 t}\sinh|k|t + \hat{a}^+(0)e^{i\omega_0 t}\cosh|k|t.$$

c)

$$\hat{x}_\phi(t) = \frac{1}{2}\left[\hat{a}(t)e^{i\omega_0 t}e^{-i\phi} + \hat{a}^+(t)e^{-i\omega_0 t}e^{i\phi}\right]$$

$$\Rightarrow \quad \frac{d\hat{x}_\phi(t)}{dt} = -|k|\,\hat{x}_\phi(t) \qquad \text{for } \phi = \frac{\theta}{2}$$

& $$\frac{d\hat{x}_{\phi+\frac{\pi}{2}}(t)}{dt} = +|k|\,\hat{x}_{\phi+\frac{\pi}{2}}(t).$$

d) $\hat{X}_\phi(t) = e^{-|k|t}\,\hat{X}_\phi(0)$

$$\Rightarrow \quad \langle 0|\hat{X}^2_\phi(t)|0\rangle = \langle 0|e^{-2|k|t}\,\hat{X}^2_\phi(0)|0\rangle = e^{-2|k|t}\,\frac{1}{4}$$

$$\Rightarrow \quad \langle 0|\Delta\hat{X}^2_\phi(t)|0\rangle = e^{-2|k|t} \times \frac{1}{4} \longrightarrow \left\{\begin{array}{l}\text{Fluctuations decrease with}\\ \text{time}\end{array}\right.$$

Similarly $\langle 0|\Delta\hat{X}^2_{\phi+\frac{\pi}{2}}(t)|0\rangle = e^{+2|k|t} \times \frac{1}{4}. \longrightarrow \left\{\begin{array}{l}\text{Fluctuations increase}\\ \text{with time}\end{array}\right.$

e) There was a mistake in the question. I meant to as for $\langle\alpha|\hat{X}_\phi(t)|\alpha\rangle$ and not $\langle\alpha|\Delta\hat{X}_\phi(t)|\alpha\rangle$ which is trivially zero.

$$\langle\alpha|\hat{X}_\phi(t)|\alpha\rangle = e^{-|k|t}\langle\alpha|\hat{X}_\phi(0)|\alpha\rangle = e^{-|k|t}|\alpha|\cos(\eta-\phi)$$

$$\langle\alpha|\hat{X}_{\phi+\frac{\pi}{2}}(t)|\alpha\rangle = e^{|k|t}|\alpha|\sin(\eta-\phi)$$

The device can act as an amplifier, which amplifies one quadrature but reduces the other quadrature. This is an optical parametric amplifier. (or OPA).