6.1

a) $|\Psi(t=0)\rangle = |1\rangle_1 \otimes |0\rangle_2$ Let $|\Psi(t)\rangle = C_0(t)\,|1\rangle_1\otimes|0\rangle_2 + C_1(t)\,|0\rangle_1\otimes|1\rangle_2$

Plug in $i\hbar \frac{\partial}{\partial t}|\Psi(t)\rangle = \hat{H}\,|\Psi(t)\rangle$ to get

$$i\hbar \frac{d}{dt}C_0\; |1\rangle_1\otimes|0\rangle_2 + i\hbar\frac{d}{dt}C_1\; |0\rangle_1\otimes|1\rangle_2 = \left\{2\hbar\omega_0\, C_0(t) - U\,C_1(t)\right\}|1\rangle_1\otimes|0\rangle_2 + \left\{2\hbar\omega_0\, C_1(t) - U\,C_0(t)\right\}|0\rangle_1\otimes|1\rangle_2$$

$\Rightarrow \frac{d}{dt}C_0(t) = -i2\omega_0\, C_0(t) + i\frac{U}{\hbar}C_1(t)$ & $\frac{d}{dt}C_1(t) = -i2\omega_0\, C_1(t) + i\frac{U}{\hbar}C_0(t)$

Let $C_0(t) = b_0(t)\,e^{-i2\omega_0 t}$ $C_1(t) = b_1(t)\,e^{-i2\omega_0 t}$

$\Rightarrow \frac{db_0(t)}{dt} = i\frac{U}{\hbar}b_1(t)$ & $\frac{db_1(t)}{dt} = i\frac{U}{\hbar}b_0(t)$ $\Rightarrow \begin{cases} \frac{d^2b_0}{dt^2} = -\left(\frac{U}{\hbar}\right)^2 b_0(t) \\ \frac{d^2b_1}{dt^2} = -\left(\frac{U}{\hbar}\right)^2 b_1(t) \end{cases}$

Since $C_0(t=0)=1$ & $C_1(t=0)=0$ $\Rightarrow$ $b_0(t=0)=1$ & $b_1(t=0)=0$

$\Rightarrow$ Solution is $b_0(t) = \cos\left(\frac{U}{\hbar}t\right)$ $b_1(t) = i\sin\left(\frac{Ut}{\hbar}\right)$.

$$\Rightarrow |\Psi(t)\rangle = e^{-i2\omega_0 t}\left\{\cos\left(\frac{Ut}{\hbar}\right)|1\rangle_1\otimes|0\rangle_2 + i\sin\left(\frac{Ut}{\hbar}\right)|0\rangle_1\otimes|1\rangle_2\right\}$$

b) $\hat{H} = \hbar\omega_0\left(\hat{n}_1+\frac{1}{2}\right) + \hbar\omega_0\left(\hat{n}_2+\frac{1}{2}\right) - U\left(\hat{S}_+ + \hat{S}_-\right)$

$$i\hbar\frac{d}{dt}\hat{n}_1(t) = \left[\hat{n}_1(t), \hat{H}(t)\right] = U\left[\hat{S}_+(t) - \hat{S}_-(t)\right]$$

$$i\hbar\frac{d}{dt}\hat{n}_2(t) = \left[\hat{n}_2(t), \hat{H}(t)\right] = -U\left[\hat{S}_+(t) - \hat{S}_-(t)\right] = -i\hbar\frac{d}{dt}\hat{n}_1(t)$$

$$i\hbar\frac{d}{dt}\hat{S}_+(t) = -U\left[\hat{n}_2(t) - \hat{n}_1(t)\right] \Rightarrow i\hbar\frac{d\hat{S}_-(t)}{dt} = U\left[\hat{n}_2(t) - \hat{n}_1(t)\right]$$

c) Let $\Delta\hat{n}(t) = \hat{n}_2(t) - \hat{n}_1(t)$ $\Delta\hat{S}(t) = \hat{S}_+(t) - \hat{S}_-(t)$

$$\Rightarrow \frac{d^2}{dt^2}\Delta\hat{n}(t) = -\left(\frac{2U}{\hbar}\right)^2\Delta\hat{n}(t)$$

These look exactly like the coupled quantum well problem. So,

$$\hat{n}_1(t) = \hat{n}_1\cos^2\left(\frac{Ut}{\hbar}\right) + \hat{n}_2\sin^2\left(\frac{Ut}{\hbar}\right) - \frac{i}{2}\left(\hat{S}_+ - \hat{S}_-\right)\sin\left(\frac{2U}{\hbar}t\right)$$

Since $\hat{n}_1(t) + \hat{n}_2(t) = \hat{n}_1 + \hat{n}_2 \Rightarrow \hat{n}_2(t) = \hat{n}_1 + \hat{n}_2 - \hat{n}_1(t)$

$$\hat{n}_2(t) = \hat{n}_1 \sin^2\left(\frac{Ut}{\hbar}\right) + \hat{n}_2 \cos^2\left(\frac{Ut}{\hbar}\right) + \frac{i}{2}\left(\hat{S}_+ - \hat{S}_-\right) \sin\left(\frac{2U}{\hbar}t\right).$$

d) Again use the principle of photon number conservation

$$|\psi(t)\rangle = \sum_{j=0}^{n} c_j(t)\ |n-j\rangle_1 \otimes |j\rangle_2$$ and plug into Shrodinger equation

$$\sum_{j=0}^{n} i\hbar \frac{d}{dt} c_j(t)\ |n-j\rangle_1 \otimes |j\rangle_2 = \sum_{j=0}^{n} (n+1)\hbar\omega_0\ c_j(t)\ |n-j\rangle_1 \otimes |j\rangle_2$$
$$- U \sum_{j=1}^{n} c_j(t)\ \sqrt{n-j+1}\sqrt{j}\ |n-j+1\rangle_1 \otimes |j-1\rangle_2$$
$$- U \sum_{j=0}^{n-1} c_j(t)\ \sqrt{n-j}\sqrt{j+1}\ |n-j-1\rangle_1 \otimes |j+1\rangle_2$$

$\Rightarrow$

$$i\hbar \frac{d}{dt} c_0(t) = (n+1)\hbar\omega_0\, c_0(t) - U\, c_1(t)\sqrt{n}$$
$$i\hbar \frac{d}{dt} c_n(t) = (n+1)\hbar\omega_0\, c_n(t) - U\, c_{n-1}(t)\sqrt{n}$$

& for $0 < k < n$

$$i\hbar \frac{d}{dt} c_k(t) = (n+1)\hbar\omega_0\, c_k(t) - U\, c_{k+1}(t)\sqrt{n-k}\sqrt{k+1}$$
$$- U\, c_{k-1}(t)\sqrt{n-k+1}\sqrt{k}$$

e) In Shrodinger picture $\langle \hat{n}_1\rangle(t) = \sum_{j=0}^{n} (n-j)\ |c_j(t)|^2$
$\langle \hat{n}_2\rangle(t) = \sum_{j=0}^{n} j\ |c_j(t)|^2$. Very difficult to solve for the $c_j(t)$ coefficients. In Heisenberg picture:

$$\langle \hat{n}_1\rangle(t) = \langle \psi(t=0)|\ \hat{n}_1(t)\ |\psi(t=0)\rangle = {}_2\langle 0| \otimes {}_1\langle n|\ \hat{n}_1(t)\ |n\rangle_1 \otimes |0\rangle_2$$
$$= n \cos^2\left(\frac{Ut}{\hbar}\right)$$

$$\langle \hat{n}_2\rangle(t) = {}_2\langle 0| \otimes {}_1\langle n|\ \hat{n}_2(t)\ |n\rangle_1 \otimes |0\rangle_2 = n \sin^2\left(\frac{Ut}{\hbar}\right).$$

a) Detectors B and C have equal probability of detecting a photon. Detector A will never detect any photon. See the detailed sol. below.

b) $i\hbar \frac{d\hat{a}_1}{dt} = \hbar\omega_0 \hat{a}_1 - U a_2$ $\quad$ $i\hbar \frac{d\hat{a}_2}{dt} = \hbar\omega_0 \hat{a}_2 - U a_1$

$\Rightarrow M = \begin{bmatrix} \hbar\omega_0 & -U \\ -U & \hbar\omega_0 \end{bmatrix}$ eigenvalues are $\begin{cases} \lambda_+ = \hbar\omega_0 + U \\ \lambda_- = \hbar\omega_0 - U \end{cases}$

c) Let $\hat{d}_- = \frac{\hat{a}_1 + \hat{a}_2}{\sqrt{2}}$ $\quad$ $\hat{d}_+ = \frac{\hat{a}_1 - \hat{a}_2}{\sqrt{2}}$

$$\Rightarrow i\hbar \frac{d\hat{d}_-(t)}{dt} = \frac{1}{\sqrt{2}}\left[i\hbar\frac{d\hat{a}_1}{dt} + i\hbar\frac{d\hat{a}_2}{dt}\right] = \hbar\omega_0\left(\frac{\hat{a}_1(t)+\hat{a}_2(t)}{\sqrt{2}}\right) - U\left(\frac{\hat{a}_1(t)+\hat{a}_2(t)}{\sqrt{2}}\right)$$

$$= (\hbar\omega_0 - U)\left(\frac{\hat{a}_1(t)+\hat{a}_2(t)}{\sqrt{2}}\right) = (\hbar\omega_0 - U)\,\hat{d}_-(t)$$

and similarly $i\hbar \frac{d\hat{d}_+(t)}{dt} = (\hbar\omega_0 + U)\,\hat{d}_+(t)$.

Also $[\hat{d}_-, \hat{d}_-^+] = \frac{1}{2}\left\{[\hat{a}_1, \hat{a}_1^+] + [\hat{a}_2, \hat{a}_2^+]\right\} = 1.$

& $[\hat{d}_+, \hat{d}_+^+] = \frac{1}{2}\left\{[\hat{a}_1, \hat{a}_1^+] + [\hat{a}_2, \hat{a}_2^+]\right\} = 1$

and $[\hat{d}_-, \hat{d}_+^+] = \frac{1}{2}\left\{[\hat{a}_1, \hat{a}_1^+] - [\hat{a}_2, \hat{a}_2^+]\right\} = 0$

and $[\hat{d}_+, \hat{d}_-^+] = 0$.

d) $\hat{a}_1 = \frac{\hat{d}_- + \hat{d}_+}{\sqrt{2}}$ $\quad$ $\hat{a}_2 = \frac{\hat{d}_- - \hat{d}_+}{\sqrt{2}}$

e) $\hat{H} = \hbar\omega_0\left(\hat{a}_1^+\hat{a}_1 + \hat{a}_2^+\hat{a}_2 + 1\right) - U\left(\hat{a}_2^+\hat{a}_1 + \hat{a}_1^+ a_1\right)$

using (d): $\hat{H} = (\hbar\omega_0 - U)\left(\hat{d}_-^+\hat{d}_- + \frac{1}{2}\right) + (\hbar\omega_0 + U)\left(\hat{d}_+^+\hat{d}_+ + \frac{1}{2}\right)$.

It agrees with Alice's proposal & $\hbar\omega_- = \lambda_- = \hbar\omega_0 - U$
$\hbar\omega_+ = \lambda_+ = \hbar\omega_0 + U$.

f) We know that $\hat{d}_-|0\rangle_-\otimes|0\rangle_+ = 0 \Rightarrow \frac{1}{\sqrt{2}}(\hat{a}_1 + \hat{a}_2)|0\rangle_-\otimes|0\rangle_+ = 0$

$\Rightarrow \hat{a}_1|0\rangle_-\otimes|0\rangle_+ = -\hat{a}_2|0\rangle_-\otimes|0\rangle_+$ — (1)

Also,

$$\hat{d}_+ |0\rangle_- \otimes |0\rangle_+ = 0 \quad \Rightarrow \quad \frac{1}{\sqrt{2}}(\hat{a}_1 - \hat{a}_2)|0\rangle_- \otimes |0\rangle_+ = 0$$

$$\Rightarrow \hat{a}_1 |0\rangle_- \otimes |0\rangle_+ = \hat{a}_2 |0\rangle_- \otimes |0\rangle_+ \quad — (2)$$

The only state for which (1) + (2) can be both true is the state $|0\rangle_1 \otimes |0\rangle_2$ since $\hat{a}_1 |0\rangle_1 \otimes |0\rangle_2 = \hat{a}_2 |0\rangle_1 \otimes |0\rangle_2 = 0$.

g) Since $[\hat{d}_-, \hat{d}_-^+] = 1$ and $\hat{d}_- |0\rangle_- \otimes |0\rangle_+ = 0$

$\Rightarrow$ the state $\frac{(\hat{d}_-^+)^n}{\sqrt{n!}} |0\rangle_- \otimes |0\rangle_+ = |n\rangle_- \otimes |0\rangle_+$ is an eigenstate of the Hamiltonian $\hat{H} = (\hbar\omega_0 - U)(\hat{d}_-^+\hat{d}_- + \frac{1}{2}) + (\hbar\omega_0 + U)(\hat{d}_+^+\hat{d}_+ + \frac{1}{2})$ with eigenvalue $n(\hbar\omega_0 - U) + \underbrace{\hbar\omega_0}_{\text{vacuum part}}$.

This just follows from the commutation relations and the form of the Hamiltonian.

Also, $|0\rangle_- \otimes |q\rangle = \frac{(\hat{d}_+^+)^q}{\sqrt{q!}} |0\rangle_- \otimes |0\rangle_+$ is an eigenstate of the Hamiltonian with eigenvalue $q(\hbar\omega_0 + U) + \hbar\omega_0$

And the state $|n\rangle_- \otimes |q\rangle_+ = \frac{(\hat{d}_-^+)^n}{\sqrt{n!}} \frac{(\hat{d}_+^+)^q}{\sqrt{q!}} |0\rangle_- \otimes |0\rangle_+$ is, therefore, also the eigenstate of the hamiltonian with eigenvalue

$$n(\hbar\omega_0 - U) + q(\hbar\omega_0 + U) + \hbar\omega_0$$

h) $$|n\rangle_- \otimes |q\rangle_+ = \frac{(d_-^+)^n}{\sqrt{n!}} \cdot \frac{(\hat{d}_+^+)^q}{\sqrt{q!}} |0\rangle_- \otimes |0\rangle_+$$

Substitute $\hat{d}_-^+ = \frac{\hat{a}_1^+ + \hat{a}_2^+}{\sqrt{2}}$ $\quad \hat{d}_+^+ = \frac{\hat{a}_1^+ - \hat{a}_2^+}{\sqrt{2}}$

$$\Rightarrow |n\rangle_- \otimes |q\rangle_+ = \frac{1}{2^{n/2}} \cdot \frac{1}{2^{q/2}} \cdot \frac{(\hat{a}_1^+ + \hat{a}_2^+)^n}{\sqrt{n!}} \frac{(\hat{a}_1^+ - \hat{a}_2^+)^q}{\sqrt{q!}} |0\rangle_1 \otimes |0\rangle_2$$

$$|n\rangle_- \otimes |q\rangle_+ = \frac{1}{2^{n/2}} \cdot \frac{1}{2^{q/2}} \sum_{m=0}^{n} \sum_{p=0}^{q} \frac{(\hat{a}_1^+)^{n-m+q-p}}{\sqrt{n!}} \frac{(\hat{a}_2^+)^{m+p}}{\sqrt{q!}} (-1)^p \frac{n!}{m!(n-m)!} \frac{q!}{p!(q-p)!} |0\rangle_1 \otimes |0\rangle_2$$

$$= \frac{1}{2^{n/2}} \cdot \frac{1}{2^{q/2}} \sum_{m=0}^{n} \sum_{p=0}^{q} \frac{\sqrt{(n-m+q-p)!}\sqrt{n!}}{m!\,(n-m)!} \frac{\sqrt{(m+p)!}\sqrt{q!}}{p!(q-p)!} (-1)^p \, |n-m+q-p\rangle_1 \otimes |m+p\rangle_2$$

(i) $$|n\rangle_1 \otimes |q\rangle_2 = \frac{(\hat{a}_1^+)^n}{\sqrt{n!}} \cdot \frac{(\hat{a}_2^+)^q}{\sqrt{q!}} \cdot |0\rangle_1 \otimes |0\rangle_2$$

$$= \frac{1}{2^{n/2}} \cdot \frac{1}{2^{q/2}} \cdot \left(\hat{d}_- + \hat{d}_+\right)^n \left(\hat{d}_- - \hat{d}_+\right)^q \, |0\rangle_- \otimes |0\rangle_+$$

$$= \frac{1}{2^{n/2}} \cdot \frac{1}{2^{q/2}} \sum_{m=0}^{n} \sum_{p=0}^{q} \frac{\sqrt{(n-m+q-p)!}\sqrt{n!}}{m!\,(n-m)!} \frac{\sqrt{(m+p)!}\sqrt{q!}}{p!\,(q-p)!} (-1)^p \, |n-m+q-p\rangle_- \otimes |m+p\rangle_+$$

(j) Not really. $|n\rangle_1 \otimes |q\rangle_2$ is a linear superpositions of the "actual" photon states $|m\rangle_- \otimes |p\rangle_+$

(k) Yes. $|n\rangle_- \otimes |q\rangle_+$ represents n photons in mode "−" and q photons in mode "+". These are honest-to-God photons that can be detected with a photodetector.

(l) $$|\psi(t=0)\rangle = \hat{a}_1^+ |0\rangle_1 \otimes |0\rangle_2 = \frac{\hat{d}_-^+ + \hat{d}_+^+}{\sqrt{2}} \, |0\rangle_- \otimes |0\rangle_+$$

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}\left[|1\rangle_- \otimes |0\rangle_+ + |0\rangle_- \otimes |1\rangle_+\right]$$

So $|\psi(t=0)\rangle$ is a linear superposition of one photon in mode "−" and zero in "+", and 0 in mode "−" and one in mode "+". So photodetectors B and C have equal probability of detecting a single photon, and as soon as either one of the detector B or C

detects a photon, the quantum state collapses to $|0\rangle_- \otimes |0\rangle_+$.
The photon is destroyed in the process of detection.

6.3

a) From the equation

$$\frac{dv_j(t)}{dt} = -\frac{1}{\tau} v_j(t) + \frac{F_j(t)}{m}$$

we know that for t_1, t_2 large

$$\langle v_j(t_1) v_j(t_2)\rangle = \frac{A}{m^2}\frac{\tau}{2} e^{-\frac{|t_1-t_2|}{\tau}}$$

$$\Rightarrow \langle v_j^2(t)\rangle = \frac{k_BT}{m} = \frac{A}{m^2}\frac{\tau}{2} \Rightarrow A = \frac{2mk_BT}{\tau}$$

b) $$I(t) = \frac{q}{L}\sum_{j=1}^{N} v_j(t)$$

$$\Rightarrow \langle I(t_1) I(t_2)\rangle = \frac{q^2}{L^2}\sum_{j=1}^{N}\sum_{k=1}^{N}\langle v_j(t_1) v_k(t_2)\rangle$$

Since $\langle F_j(t_1) F_k(t_2)\rangle \propto \delta_{jk}$. $\Rightarrow \langle v_j(t_1) v_k(t_2)\rangle$ is also $\propto \delta_{jk}$.

$$\Rightarrow \langle I(t_1) I(t_2)\rangle = \frac{q^2}{L^2}\sum_{j=1}^{N}\langle v_j(t_1) v_j(t_2)\rangle$$

$$= \frac{q^2 N}{L^2}\frac{k_BT}{m} e^{-\frac{|t_1-t_2|}{\tau}} = \frac{q^2 An}{L}\frac{k_BT}{m} e^{-\frac{|t_1-t_2|}{\tau}}.$$

c) $S_{II}(\omega) = \frac{A}{L}\frac{nq^2\tau}{m}\cdot 2k_BT. \ (1/\tau)^2 / [\omega^2 + (1/\tau)^2]$

d) $$\frac{dv_j(t)}{dt} = -\frac{1}{\tau} v_j(t) + \frac{F_j(t)}{m} \Rightarrow v_j(\omega) = \frac{F_j(\omega)/m}{-i\left(\omega + \frac{i}{\tau}\right)}$$

$$\Rightarrow \langle v_j^*(\omega') v_k(\omega)\rangle = \frac{\langle F_j^*(\omega') F_k(\omega)\rangle \frac{1}{m^2}}{i\left(\omega' - \frac{i}{\tau}\right) - i\left(\omega + \frac{i}{\tau}\right)} = \frac{\delta_{jk}\frac{2k_BT}{m\tau} 2\pi\delta(\omega'-\omega)}{\omega^2 + \left(\frac{1}{\tau}\right)^2}$$

Also $I(\omega) = \frac{q}{L} \sum_{j=1}^{N} v_j(\omega)$

$$\Rightarrow \langle I^*(\omega') I(\omega) \rangle = \frac{q^2}{L^2} \sum_{j=1}^{N} \sum_{k=1}^{N} \delta_{jk} \frac{\frac{2k_BT}{m\tau}}{\omega^2 + \frac{1}{\tau^2}} 2\pi\, \delta(\omega' - \omega).$$

$$= \frac{q^2 N}{L^2} \frac{\frac{2k_BT}{m\tau}}{\omega^2 + \frac{1}{\tau^2}} 2\pi\, \delta(\omega' - \omega).$$

$$S_{II}(\omega) = \int \frac{d\omega'}{2\pi} \langle I^*(\omega') I(\omega) \rangle = \frac{q^2 N}{L^2} \frac{\frac{2k_BT}{m\tau}}{\omega^2 + \frac{1}{\tau^2}}$$

$$= \frac{q^2 A n}{L} \frac{\frac{2k_BT}{m\tau}}{\omega^2 + \frac{1}{\tau^2}} = \frac{A}{L} \frac{n q^2 \tau}{m} 2k_BT \cdot \frac{(1/\tau)^2}{\omega^2 + (1/\tau)^2}.$$

e) $$S_{II}(\omega) = \frac{2k_BT}{R} \cdot \frac{(1/\tau)^2}{\omega^2 + (1/\tau)^2}.$$

$$\Rightarrow \left\{ S_{II}(\omega) \text{ for } \omega \ll 1/\tau \right\} = \frac{2k_BT}{R} = \text{thermal noise.}$$

$$S_{II}(\omega = 0) = \frac{2k_BT}{R}.$$

6.4

We have $\left[\hat{E}_a(\vec{r},t),\ \hat{H}_b(\vec{r}\,',t)\right] = -i\hbar c^2 \sum_c \varepsilon_{abc}\, \partial_c\, \delta^3(\vec{r}-\vec{r}\,')$

$$\hat{H}(t) = \int d^3\vec{r}\ \tfrac{1}{2}\varepsilon_0\, \hat{\vec{E}}(\vec{r},t)\cdot\vec{E}(\vec{r},t) + \tfrac{1}{2}\mu_0\, \hat{\vec{H}}(\vec{r},t)\cdot\hat{\vec{H}}(\vec{r},t)$$

$$\Rightarrow \left[\hat{E}_a(\vec{r},t),\ \hat{H}(t)\right] = \int d^3\vec{r}\,'\ \tfrac{1}{2}\mu_0 \left[\vec{E}_a(\vec{r},t),\ \hat{\vec{H}}(\vec{r}\,',t)\cdot\hat{\vec{H}}(\vec{r}\,',t)\right].$$

$$= \tfrac{1}{2}\mu_0 \int d^3\vec{r}\,'\ 2\left\{+i\hbar c^2 \sum_{bc} \varepsilon_{abc}\, \hat{H}_b(\vec{r}\,',t)\, \partial_c'(\vec{r}-\vec{r}\,')\right\}.$$

Integrate by parts:

$$= -\mu_0\, i\hbar c^2 \sum_{bc} \varepsilon_{abc}\, \partial_c\, \hat{H}_b(\vec{r},t)$$

$$= +i\hbar\, \mu_0 c^2 \sum_{bc} \varepsilon_{abc}\, \partial_b\, \hat{H}_c(\vec{r},t)$$

$$= +\frac{i\hbar}{\varepsilon_0} \sum_{bc} \varepsilon_{abc}\, \partial_b\, \hat{H}_c(\vec{r},t)$$

$$\Rightarrow\quad i\hbar \frac{\partial \hat{\vec{E}}}{\partial t}(\vec{r},t) = \left[\hat{\vec{E}}(\vec{r},t),\hat{H}(t)\right]$$

Look at it component at a time: $i\hbar \frac{\partial}{\partial t}\hat{E}_a(\vec{r},t) = +\frac{i\hbar}{\varepsilon_0}\sum_{bc}\varepsilon_{abc}\partial_b \hat{H}_c(\vec{r},t)$

$$\Rightarrow\quad \varepsilon_0 \frac{\partial \hat{E}_a(\vec{r},t)}{\partial t} = \sum_{bc} \varepsilon_{abc}\, \partial_b\, \hat{H}_c(\vec{r},t)$$

or $$\varepsilon_0 \frac{\partial \hat{\vec{E}}(\vec{r},t)}{\partial t} = \nabla \times \hat{\vec{H}}(\vec{r},t)$$

Similarly, the equation for $\hat{H}(\vec{r},t)$ will be:

$$\mu_0 \frac{\partial \hat{\vec{H}}(\vec{r},t)}{\partial t} = -\nabla \times \hat{\vec{E}}(\vec{r},t)$$

Heisenberg equations for fields are just Maxwell's equations!

b) i) $\sum_{j=1,2} \hat{\varepsilon}_j(\hat{k}) \cdot \hat{e}_a \; \hat{\varepsilon}_j(\hat{k}) \cdot \hat{e}_b = \delta_{ab} - \frac{k_a k_b}{k^2}$

Note that at any point in k-space, the 3 vectors $\hat{\varepsilon}_1(\vec{k})$, $\hat{\varepsilon}_2(\vec{k})$ and $\hat{k}$ are mutually orthogonal and therefore any vector can be expanded in terms of these unit vectors. (just like we use $\hat{x}$, $\hat{y}$ and $\hat{z}$ — in our notation $\hat{e}_a$, $\hat{e}_b$, and $\hat{e}_c$ — to expand vectors). So in particular $\hat{e}_a$ can be written as,

$$\hat{e}_a = \sum_{j=1}^{2} \left[\hat{e}_a \cdot \hat{\varepsilon}_j(\vec{k})\right] \hat{\varepsilon}_j(\vec{k}) + (\hat{e}_a \cdot \hat{k})\,\hat{k}$$

and

$$\hat{e}_b = \sum_{j=1}^{2} \left[\hat{e}_b \cdot \hat{\varepsilon}_j(\vec{k})\right] \hat{\varepsilon}_j(\vec{k}) + (\hat{e}_b \cdot \hat{k})\,\hat{k}$$

Take the dot product of the two equations above:

$$\hat{e}_a \cdot \hat{e}_b = \sum_{j=1}^{2} \sum_{j'=1}^{2} \left[\hat{e}_a \cdot \hat{\varepsilon}_j(\vec{k})\right]\left[\hat{e}_b \cdot \hat{\varepsilon}_{j'}(\vec{k})\right] \hat{\varepsilon}_j(\vec{k}) \cdot \hat{\varepsilon}_{j'}(\vec{k}) + (\hat{e}_a \cdot \hat{k})(\hat{e}_b \cdot \hat{k})$$

$$\delta_{ab} = \sum_{j=1}^{2} \left[\hat{e}_a \cdot \hat{\varepsilon}_j(\vec{k})\right]\left[\hat{e}_b \cdot \hat{\varepsilon}_j(\vec{k})\right] + \frac{k_a}{k}\frac{k_b}{k}.$$

$$\Rightarrow \sum_{j=1}^{2} \left[\hat{\varepsilon}_j(\vec{k}) \cdot \hat{e}_a\right]\left[\hat{\varepsilon}_j(\vec{k}) \cdot \hat{e}_b\right] = \delta_{ab} - \frac{k_a k_b}{k^2}$$

ii) This is pretty easy. Suppose $\hat{k} \times \hat{\varepsilon}_1(\vec{k}) = \hat{\varepsilon}_2(\vec{k}) \Rightarrow \hat{k} \times \hat{\varepsilon}_2(\vec{k}) = -\hat{\varepsilon}_1(\vec{k})$

$$\Rightarrow \sum_{j=1}^{2} \left[\hat{k} \times \hat{\varepsilon}_j(\vec{k})\right] \cdot e_a \left[\hat{k} \times \hat{\varepsilon}_j(\vec{k})\right] \cdot e_b = \sum_{j=1}^{2} \left[\hat{\varepsilon}_j(\vec{k}) \cdot e_a\right]\left[\hat{\varepsilon}_j(\vec{k}) \cdot e_b\right] = \delta_{ab} - \frac{k_a k_b}{k^2}$$

iii) $\sum_{j=1}^{2} \hat{\varepsilon}_j(\vec{k}) \cdot \hat{e}_a \quad [\hat{k} \times \hat{\varepsilon}_j(\vec{k})] \cdot \hat{e}_b$

consider $[\hat{k} \times \hat{\varepsilon}_j(\vec{k})] \cdot \hat{e}_b = [\hat{e}_b \times \hat{k}] . \hat{\varepsilon}_j(\vec{k}) = \sum_{dpr} [\hat{\varepsilon}_j(\vec{k}) \cdot \hat{e}_d] \varepsilon_{dpr} [\hat{e}_b \cdot \hat{e}_p][\hat{k} \cdot \hat{e}_r]$

$$= \sum_{dr} [\hat{\varepsilon}_j(\vec{k}) \cdot \hat{e}_d] \; \varepsilon_{dbr} \frac{k_r}{k}.$$

$$\Rightarrow \sum_{j=1}^{2} [\hat{\varepsilon}_j(\vec{k}) \cdot \hat{e}_a][\hat{k} \times \hat{\varepsilon}_j(\vec{k})] \cdot \hat{e}_b = \sum_{dr} \sum_{j=1}^{2} [\hat{\varepsilon}_j(\vec{k}) \cdot \hat{e}_a][\hat{\varepsilon}_j(\vec{k}) \cdot \hat{e}_d] \varepsilon_{dbr} \frac{k_r}{k}.$$

$$= \sum_{dr} \left[\delta_{ad} - \frac{k_a k_d}{k^2}\right] \varepsilon_{dbr} \frac{k_r}{k} = \sum_{dr} \delta_{ad} \; \varepsilon_{dbr} \frac{k_r}{k} = \sum_{r} \varepsilon_{abr} \frac{k_r}{k}.$$