Department of Electrical and Computer Engineering, Cornell University

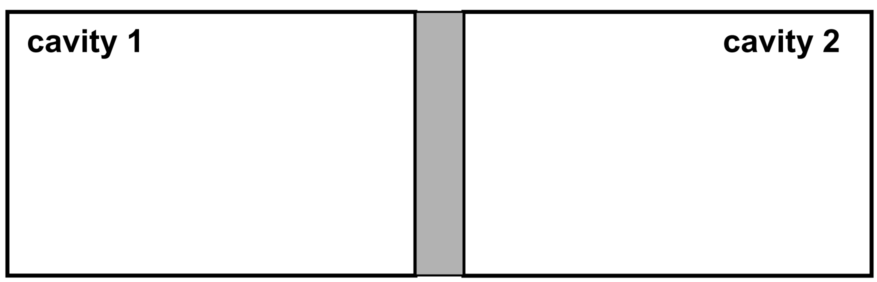
ECE 5310: Quantum Optics for Photonics and Optoelectronics

Fall 2013

Homework 6 **Due on Oct. 18, 2013 (self-grade)**

Problem 6.1 (Coupled photonic microcavities)

Consider a system consisting of two identical microcavities, labeled as 1 and 2, as shown in the **cartoon** figure below. The cavities are coupled in the sense that photons initially present in one cavity can "tunnel" into the other cavity, and vice versa. It is the photonic version of the coupled quantum well problem. Each cavity supports only a single mode of the electromagnetic field of frequency ω_o . In integrated micro-photonics such coupled cavities are obtained by evanescent coupling of fields between the cavities.



Two coupled photonic micro-cavities

The Hamiltonian describing the system is (assuming one relevant mode in each cavity),

$$\hat{H} = \hbar\omega_o \left(\hat{a}_1^+ \hat{a}_1 + \frac{1}{2}\right) + \hbar\omega_o \left(\hat{a}_2^+ \hat{a}_2 + \frac{1}{2}\right) - U \left(\hat{a}_1^+ a_2 + \hat{a}_2^+ a_1\right)$$

where $\hat{a}_1$ and $\hat{a}_2$ are destruction operators for the photons in field modes $\vec{U}_1(r)$ and $\vec{U}_2(r)$ that are localized in cavity 1 and in cavity 2, respectively. The number states of the coupled system are written as $|m\rangle_1 \otimes |p\rangle_2$ which means "m" photons in the mode of cavity 1, and "p" photons in the mode of cavity 2. The set of all such states, for all values of "m" and "p" from zero to infinity, constitute the Hilbert space of the problem. The term in the Hamiltonian proportional to $-U\left(\hat{a}_1^+ a_2 + \hat{a}_2^+ a_1\right)$ models the coupling between the two cavities. The term $\hat{a}_2^+ a_1$ acts on a quantum state by destroying a photon in mode 1 (if at least one photon is present in mode 1) and creating a photon in mode 2. Thus, the action of this term is to transfer a photon from cavity 1 to cavity 2. Similarly, the term $\hat{a}_1^+ a_2$ transfers a photon from cavity 2 to cavity 1. In this sense, the term $-U\left(\hat{a}_1^+ a_2 + \hat{a}_2^+ a_1\right)$ is somewhat similar to the term $-U\left(|e_1\rangle\langle e_2| + |e_2\rangle\langle e_1|\right)$ in the electronic coupled quantum well problem discussed in the lecture notes. **The big difference between electrons and photons is that while only a single electron may occupy a**

quantum state, many photons can be in the same mode. The commutation relations among the creation and destruction operators can be written in compact form as $\left[\hat{a}_j , \hat{a}_k^+\right] = \delta_{jk}$.

a) Suppose the initial state of the system is $|\psi(t=0)\rangle = |1\rangle_1 \otimes |0\rangle_2$, i.e. one photon in cavity 1 and no photons in cavity 2. Using the Shrodinger picture of time evolution, find the state $|\psi(t)\rangle$ of the system at time t . You will need to expand $|\psi(t)\rangle$ in terms of some suitable states, plug it into the Shrodinger equation, and then solve for the time dependent expansion coefficients with appropriate initial conditions.

Hint: If you are prudent, you will realize that the expansion may only include the following two states and not all the states in the Hilbert space of the problem (this is because of photon number conservation),

$$|\psi(t)\rangle = c_0(t)|1\rangle_1 \otimes |0\rangle_2 + c_1(t)|0\rangle_1 \otimes |1\rangle_2$$

b) Define the following Shrodinger operators,

$$\hat{n}_1 = \hat{a}_1^+ \hat{a}_1$$
$$\hat{n}_2 = \hat{a}_2^+ \hat{a}_2$$
$$\hat{s}_+ = \hat{a}_2^+ \hat{a}_1$$
$$\hat{s}_- = \hat{a}_1^+ \hat{a}_2$$

Write the Hamiltonian in terms of the operators $\hat{n}_1, \hat{n}_2, \hat{s}_+, \hat{s}_-$ and derive the Heisenberg equations for the time evolution of each of the following Heisenberg operators: $\hat{n}_1(t), \hat{n}_2(t), \hat{s}_+(t), \hat{s}_-(t)$.

Hint: the commutation relations among the operators $\hat{n}_1(t), \hat{n}_2(t), \hat{s}_+(t), \hat{s}_-(t)$ are rather similar to those obtained in the electron problem discussed in the lecture notes.

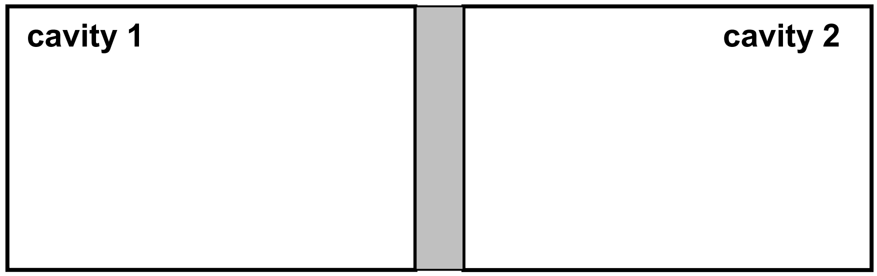
c) Solve the Heisenberg equations derived in part (b) with appropriate initial conditions, and obtain the operators $\hat{n}_1(t)$ and $\hat{n}_2(t)$ as a function of time.

d) Now we make the problem more interesting, and you will see the differences between the photonic coupled cavity and the electronic coupled quantum well system discussed in the lecture notes. Suppose the initial state of the system is $|\psi(t=0)\rangle = |n\rangle_1 \otimes |0\rangle_2$, i.e. "n" photons in cavity 1 and no photons in cavity 2. Using the Shrodinger picture of time evolution, expand the state $|\psi(t)\rangle$ of the system at time t in terms of some suitable states with time dependent expansion coefficients. If you are prudent, you will realize that the expansion may not include all the states in the Hilbert space of the problem. Derive the time differential equations for the coefficients of your expansion.

e) As in part (d) suppose that the initial state of the system is $|\psi(t=0)\rangle = |n\rangle_1 \otimes |0\rangle_2$, i.e. "n" photons in cavity 1 and no photons in cavity 2. Using any method of your own choice (Shrodinger or Heisenberg) find the average number of photons in cavity 1 and also the average number of photons in cavity 2 as a function of time for $t \geq 0$. Hint: one of the methods (Shrodinger or Heisenberg) will give you the result with orders of magnitude less work.

Problem 6.2 (The mystery of the coupled photonic cavities)

Consider a system consisting of two identical and coupled cavities (as in the previous problem),



Two coupled photonic micro-cavities

The Hamiltonian describing the system is,

$$\hat{H} = \hbar\omega_o \left(\hat{a}_1^+ \hat{a}_1 + \frac{1}{2}\right) + \hbar\omega_o \left(\hat{a}_2^+ \hat{a}_2 + \frac{1}{2}\right) - U\left(\hat{a}_1^+ a_2 + \hat{a}_2^+ a_1\right)$$

where $\hat{a}_1$ and $\hat{a}_2$ are destruction operators for the field modes $\vec{U}_1(r)$ and $\vec{U}_2(r)$ that are localized in cavity 1 and in cavity 2, respectively. You solved some aspects of this problem in the previous question.

Two ECE531 students, Alice and Bob, are having a debate about this problem. Alice says, "We defined photons earlier in the course to be the energy eigenstates of free electromagnetic fields. So how can a state of the form $|n\rangle_1 \otimes |p\rangle_2$, obtained by operating the vacuum state $|0\rangle_1 \otimes |0\rangle_2$ with creation operators, $\hat{a}_1^+$ and $\hat{a}_2^+$, be called a state with "n" photons in cavity 1 and "p" photons in cavity 2? Clearly, states like $|n\rangle_1 \otimes |p\rangle_2$ are not the energy eigenstates of the Hamiltonian, and the corresponding field modes $\vec{U}_1(r)$ and $\vec{U}_2(r)$ that are localized in cavity 1 and cavity 2, respectively, are not the eigenmodes of Maxwell's equations for the entire two-cavity system taken as a big single cavity". To further make her point, Alice argues that, in principle, one can find the actual eigenmodes, $\vec{U}_+(r)$ and $\vec{U}_-(r)$, of the entire two-cavity system by solving Maxwell's equation for the two-cavity system. Alice claims that if this is done, and if one defines operators $\hat{d}_-$ and $\hat{d}_+$ for the actual eigenmodes, then the Hamiltonian above could also be written simply as,

$$\hat{H} = \hbar\omega_- \left(\hat{d}_-^+ \hat{d}_- + \frac{1}{2}\right) + \hbar\omega_+ \left(\hat{d}_+^+ \hat{d}_+ + \frac{1}{2}\right)$$

After a lot of hard work Alice is able to find the actual eigenmodes $\vec{U}_+(r)$ and $\vec{U}_-(r)$ and the corresponding eigenenergies, $\hbar\omega_+$ and $\hbar\omega_-$, and therefore writes the Hamiltonian in the desired form above. Of course, $\hbar\omega_+$ and $\hbar\omega_-$ come out to be different from $\hbar\omega_o$.

Bob disagrees with Alice. Alice and Bob decide that the "proof of the pudding is in the eating." So they get three photodetectors. Photodetector A can detect photons only if the photon energy is $\hbar\omega_o$. Photodetector B can detect photons of energy $\hbar\omega_+$ only, and photodetector C can detect photons of energy $\hbar\omega_-$ only. Alice and Bob then stick all three photodetectors in cavity 1.

a) Suppose the initial state of the system is, $|\psi(t=0)\rangle = \hat{a}_1^+ \, |0\rangle_1 \otimes |0\rangle_2$ (and somehow Alice and Bob are able to prepare this state by some means), which of the three photodetectors, if any, in your opinion will detect a photon? Just give an opinion if you don't know the answer.

Due to some technical problems in their photodetectors, Alice and Bob could not get convincing results from their experiment. You have been hired by Alice and Bob to resolve their paradox.

Clearly, the states $|m\rangle_1 \otimes |p\rangle_2$ are not the eigenstates of the Hamiltonian. To see this note that,

$$\hat{H}\,|m\rangle_1 \otimes |p\rangle_2 = \hbar\omega_o\,(m+p+1)|m\rangle_1 \otimes |p\rangle_2 - U\left[\sqrt{m(p+1)}\,|m-1\rangle_1 \otimes |p+1\rangle_2 + \sqrt{p(m+1)}\,|m+1\rangle_1 \otimes |p-1\rangle_2\right]$$

So next you will find all the eigenstates and the eigenvalues of the Hamiltonian by a method called "operator diagonalization". You have already done something like this when you looked at the photon spin operator.

b) Write the Heisenberg equations for the operators $\hat{a}_1(t), \hat{a}_2(t)$ in the matrix form,

$$i\hbar \frac{d}{dt}\begin{bmatrix} \hat{a}_1(t) \\ \hat{a}_2(t) \end{bmatrix} = \begin{bmatrix} & \mathrm{M} & \end{bmatrix}\begin{bmatrix} \hat{a}_1(t) \\ \hat{a}_2(t) \end{bmatrix}$$

Find the matrix "M", and the eigenvalues $\lambda_-, \lambda_+ \; (\lambda_+ > \lambda_-)$ of the matrix "M".

c) Define two new destruction operators, $\hat{d}_-$ and $\hat{d}_+$, each as a linear combination of the two destruction operators, $\hat{a}_1$ and $\hat{a}_2$, such that the time development of the corresponding Heisenberg operators, $\hat{d}_-(t)$ and $\hat{d}_+(t)$, is given by,

$$i\hbar \frac{d}{dt}\begin{bmatrix} \hat{d}_-(t) \\ \hat{d}_+(t) \end{bmatrix} = \begin{bmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{bmatrix}\begin{bmatrix} \hat{d}_-(t) \\ \hat{d}_+(t) \end{bmatrix}$$

The above criteria determines $\hat{d}_-, \hat{d}_+$ up to a multiplicative constant. You must choose the normalization of the operators $\hat{d}_-, \hat{d}_+$ such they satisfy the commutation relations $\left[\hat{d}_-, \hat{d}_-^+\right] = 1$ and $\left[\hat{d}_+, \hat{d}_+^+\right] = 1$. If you have managed to obtain the correct results you may verify that $\left[\hat{d}_-, \hat{d}_+^+\right] = \left[\hat{d}_+, \hat{d}_-^+\right] = 0$.

d) Using your results in part (c), write each of $\hat{a}_1$ and $\hat{a}_2$ as a linear combination of $\hat{d}_-$ and $\hat{d}_+$.

e) Using your results in part (d), write the Hamiltonian operator in terms of the operators $\hat{d}_-, \hat{d}_-^+$, $\hat{d}_+, \hat{d}_+^+$. Does it agree in form with what Alice proposed? If so, what are the energies $\hbar\omega_+$ and $\hbar\omega_-$?

What you just did is called **operator diagonalization of a Hamiltonian**. You defined new operators such that there are no coupling terms in the Hamiltonian when it is written in terms of the new operators. Of course, the problem of physical interpretation remains (i.e. what do the new operators mean? What do they create and what do they destroy?). This you will tackle next.

The state $|0\rangle_- \otimes |0\rangle_+$ is defined as the ground state of the Hamiltonian. It has the property that it is the ground state of the "+" mode and the ground state of the "-" mode in the sense that,

$$\hat{d}_- \, |0\rangle_- \otimes |0\rangle_+ = 0$$
$$\hat{d}_+ \, |0\rangle_- \otimes |0\rangle_+ = 0$$

(recall that the ground state of a simple harmonic oscillator is defined as the state $|0\rangle$ such that $\hat{a}\,|0\rangle = 0$)

f) Prove, using the operator relations obtained in part (c), that the state $|0\rangle_- \otimes |0\rangle_+$, as defined by the two relations above, is the same physical quantum state as $|0\rangle_1 \otimes |0\rangle_2$. Recall that $|0\rangle_1 \otimes |0\rangle_2$ is defined as the state with the following properties,

$$\hat{a}_1 \, |0\rangle_1 \otimes |0\rangle_2 = 0$$
$$\hat{a}_2 \, |0\rangle_1 \otimes |0\rangle_2 = 0$$

g) Argue that the state $|n\rangle_- \otimes |q\rangle_+$, obtained as $\frac{\left(d_-^+\right)^n}{\sqrt{n!}} \frac{\left(d_+^+\right)^q}{\sqrt{q!}} |0\rangle_- \otimes |0\rangle_+$, is an eigenstate of the Hamiltonian. What is the corresponding eigenvalue? Hint. Start from the fact that the commutation relations for the new operators are $\left[\hat{d}_- \, , \hat{d}_-^+\right] = 1$, $\left[\hat{d}_+ \, , \hat{d}_+^+\right] = 1$, and $\left[\hat{d}_- \, , \hat{d}_+^+\right] = \left[\hat{d}_+ \, , \hat{d}_-^+\right] = 0$. (Recall that we were able to get the energy eigenstates of a simple harmonic oscillator just from the commutation relations of the creation and destruction operators.)

h) Write the state $|n\rangle_- \otimes |q\rangle_+$ as a linear superposition of the states $|m\rangle_1 \otimes |p\rangle_2$. Hint: use your solution of part (c). You may also find the following operator expansions helpful,

$$\left(\hat{A} + \hat{B}\right)^n = \sum_{m=0}^{n} \frac{n!}{(n-m)!\, m!} \hat{A}^{n-m} \, \hat{B}^m$$
$$\left(\hat{A} - \hat{B}\right)^n = \sum_{m=0}^{n} \frac{n!}{(n-m)!\, m!} \hat{A}^{n-m} \left(-\hat{B}\right)^m$$

provided $\left[\hat{A} \, , \hat{B}\right] = 0$

Since the states $|n\rangle_- \otimes |q\rangle_+$ are eigenstates of the Hamiltonian, you have been able to construct the eigenstates from the states $|m\rangle_1 \otimes |p\rangle_2$.

i) Now write the state $|n\rangle_1 \otimes |q\rangle_2$ as a linear superposition of the states $|m\rangle_- \otimes |p\rangle_+$.

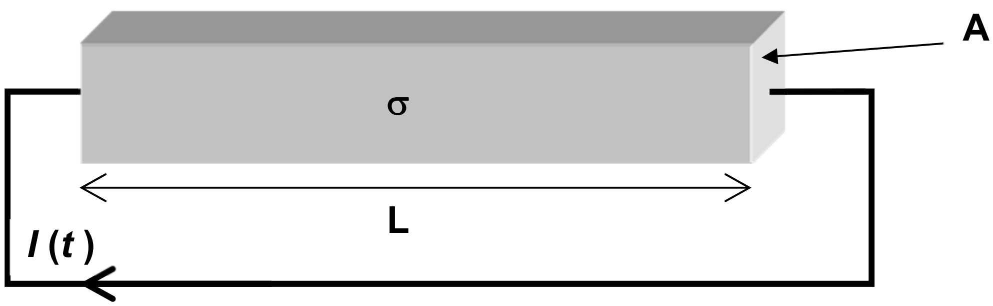
j) Can we say the state $|n\rangle_1 \otimes |q\rangle_2$ has "n" photons in cavity 1 and "q" photons in cavity 2? If yes, explain why? If not, explain why not?

k) Can we say the state $|n\rangle_- \otimes |q\rangle_+$ has "n" photons in mode "-" and "q" photons in cavity "+"? If yes, explain why? If not, explain why not?

l) Now that you are done the math, you can go back and try to help Alice and Bob. Suppose the initial state of the system is $|\psi(t=0)\rangle = \hat{a}_1^+ \, |0\rangle_1 \otimes |0\rangle_2$. Explain which photodetector (A, B, or C) will detect a photon?

Problem 6.3 (Thermal noise in resistors: Johnson-Nyquist noise)

In this problem you will find the thermal noise (or the Johnson-Nyquist noise) in electrical resistors using the methods developed in the course. Consider a slab of material of length L, cross-sectional area A, and conductivity σ, as shown in the figure below,



The resistance of the slab is given as $R = \dfrac{L}{\sigma A}$. The conductivity is related to the electron density n, and electron scattering rate $1/\tau$, by the relation $\sigma = \dfrac{n q^2 \tau}{m}$, where q is the electron charge and m is the electron mass.

The total current can be written as,

$$I(t) = \frac{q}{L} \sum_{j=1}^{N} v_j(t)$$

where the sum is over the velocities of all the $N\,(= nAL)$ electrons in the slab. We assume (as in the lecture notes) that in the **absence of any applied electric field**, the velocity $v_j(t)$ of the *j*-th electron obeys a Langevin equation,

$$\frac{d\, v_j(t)}{dt} = -\frac{1}{\tau} v_j(t) + \frac{F_j(t)}{m}$$

where the Langevin forces $F_j(t)$ have the following properties,

$$\langle F(t) \rangle = 0$$
$$\langle F_j(t_1) F_k(t_2) \rangle = A\, \delta_{jk}\, \delta(t_1 - t_2)$$

The second relation implies that velocity kicks imparted to different electrons are uncorrelated.

a) What must be the value of A so that $\langle v_j^2(t) \rangle = \frac{K_B T}{m}$ for large values of time t ?

b) Because the Langevin forces "kick" the electron velocities, the resulting velocity fluctuations of electrons result in current fluctuations in the circuit and these current fluctuations can be measured experimentally. Find the current correlation function $R_{II}(t_1, t_2) = \langle I(t_1) I(t_2) \rangle$ when the times t_1 and t_2 are both very large so that all initial conditions are irrelevant?

c) Using the result in part (b), calculate the spectral density $S_{II}(\omega)$ of the current.

d) Find the spectral density $S_{II}(\omega)$ of the current by solving the Langevin equation for the velocity of each electron entirely in the frequency domain.

e) What is the low frequency value of the current spectral density? Write $S_{II}(\omega = 0)$ in terms of the resistance R of the slab. Does it seem familiar?

If you did everything right, you obtained the expression for the spectral density of thermal noise in resistors (also called the Johnson noise).

Problem 6.4 (Field Heisenberg equations and some polarization algebra)

In the lecture handouts we found the following equal-time commutation relations between the fields:

$$\left[\hat{E}_a(\vec{r},t), \hat{E}_b(\vec{r}\,',t)\right] = 0$$
$$\left[\hat{H}_a(\vec{r},t), \hat{H}_b(\vec{r}\,',t)\right] = 0$$
$$\left[\hat{E}_a(\vec{r},t), \hat{H}_b(\vec{r}\,',t)\right] = -i\hbar c^2 \sum_c \varepsilon_{abc} \partial_c \delta^3(\vec{r} - \vec{r}\,')$$

a) Find the Heisenberg equations for the time evolution of the fields:

$$i\hbar \frac{\partial \hat{\vec{E}}(\vec{r},t)}{\partial t} = \left[\hat{\vec{E}}(\vec{r},t), \hat{H}\right] = ?$$
$$i\hbar \frac{\partial \hat{\vec{H}}(\vec{r},t)}{\partial t} = \left[\hat{\vec{H}}(\vec{r},t), \hat{H}\right] = ?$$

Hint: Might be easier to find the equation for one component first and then assemble the result in vector notation.

b) Prove the following 3 relations:

$$\sum_{j=1}^{2} \hat{\varepsilon}_j(\hat{k}).\hat{e}_a \;\; \hat{\varepsilon}_j(\hat{k}).\hat{e}_b = \delta_{ab} - \frac{k_a k_b}{k^2}$$
$$\sum_{j=1}^{2} \left(\hat{k} \times \hat{\varepsilon}_j(\hat{k})\right).\hat{e}_a \;\; \left(\hat{k} \times \hat{\varepsilon}_j(\hat{k})\right).\hat{e}_b = \delta_{ab} - \frac{k_a k_b}{k^2}$$
$$\sum_{j=1}^{2} \hat{\varepsilon}_j(\hat{k}).\hat{e}_a \;\; \left(\hat{k} \times \hat{\varepsilon}_j(\hat{k})\right).\hat{e}_b = \sum_c \varepsilon_{abc} \frac{k_c}{k}$$