5.1

a) $$[\hat{q}, \hat{p}] = i\hbar \Rightarrow \langle p | [\hat{q}, \hat{p}] | q \rangle = i\hbar \langle p | q \rangle.$$

$$\Rightarrow \quad \langle p | \hat{q}\hat{p} | q \rangle - \langle p | \hat{p}\hat{q} | q \rangle = i\hbar \langle p | q \rangle$$

$$\Rightarrow \quad (i\hbar + pq)\langle p | q \rangle = \langle p | \hat{q} \hat{p} | q \rangle = \langle p | \hat{q} \, \mathbb{1} \, \mathbb{1} \, \hat{p} | q \rangle$$

$$= \int dp' \int dq' \, \langle p | \hat{q} | q' \rangle \langle q' | p' \rangle \langle p' | \hat{p} | q \rangle$$

$$= \int_{-\infty}^{\infty} dp' \int_{-\infty}^{\infty} dq' \, q' p' \langle p | q' \rangle \langle q' | p' \rangle \langle p' | q \rangle.$$

Solution:

$$\langle p | q \rangle = \frac{e^{-i\frac{pq}{\hbar}}}{\sqrt{2\pi\hbar}}$$

b) $$i\hbar \frac{\partial}{\partial t} | \psi(t) \rangle = \hat{H} | \psi(t) \rangle = \left(\frac{\hat{p}^2}{2} + \frac{1}{2}\omega_m^2 \hat{q}^2 \right) | \psi(t) \rangle$$

$$\Rightarrow \quad i\hbar \frac{\partial}{\partial t} \langle q | \psi(t) \rangle = \langle q | \frac{\hat{p}^2}{2} | \psi(t) \rangle + \langle q | \frac{1}{2}\omega_m^2 \hat{q}^2 | \psi(t) \rangle$$

$$i\hbar \frac{\partial}{\partial t} \psi(q,t) = -\frac{\hbar^2}{2} \frac{\partial^2}{\partial q^2} \psi(q,t) + \frac{1}{2}\omega_m^2 q^2 \psi(q,t)$$

$$\left\{ \text{since} \quad \langle p | q \rangle = \frac{e^{-i\frac{pq}{\hbar}}}{\sqrt{2\pi\hbar}} \Rightarrow \langle q | \hat{p} | \psi(t) \rangle = \frac{\hbar}{i}\frac{\partial}{\partial q} \psi(q,t) \right\}$$

c) $$|0\rangle = \int dq \, \langle q | 0 \rangle \, | q \rangle$$

$$= \int dq \, \phi_0(q) \, | q \rangle \quad \text{where} \quad \phi_0(q) = \left(\frac{\omega_m}{\pi\hbar} \right)^{1/4} \exp\left(-\frac{\omega_m}{2\hbar} q^2 \right)$$

d) $$|2\rangle = \int dq \, \langle q | 2 \rangle \, | q \rangle$$

$$= \int dq \, \phi_2(q) \, | q \rangle \quad \phi_2(q) = \left(\frac{\omega_m}{4\pi\hbar} \right)^{1/4} \left[2\frac{\omega_m q^2}{\hbar} - 1 \right] \exp\left(-\frac{\omega_m q^2}{2\hbar} \right)$$

e) $$P(W) = |\phi_1(W)|^2 = \left[\frac{4}{\pi}\left(\frac{\omega_m}{\hbar}\right)^3\right]^{1/2} W^2 \exp\left(-\frac{\omega_m W^2}{\hbar}\right)$$

f) since $$P(W) = |\phi_2(W)|^2 = \sqrt{\frac{\omega_m}{4\pi\hbar}}\left[\frac{2\omega_m W^2}{\hbar} - 1\right]^2 \exp\left(-\frac{\omega_m W^2}{\hbar}\right)$$

Other than the field strengths $\pm\infty$, $W = \pm\sqrt{\frac{\hbar}{2\omega_m}}$ are never going to be experimentally measured since they correspond to the nodes of $\phi_2(W)$

g) write $|0\rangle$ as

$$|0\rangle = \int dp \, \langle p|0\rangle \, |p\rangle.$$

so $$P(W) = |\langle p|0\rangle|^2_{p\to W}$$

$$\langle p|0\rangle = \int dq \, \langle p|q\rangle \langle q|0\rangle$$

$$= \int dq \, \frac{e^{-\frac{ipq}{\hbar}}}{\sqrt{2\pi\hbar}} \left(\frac{\omega_m}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{\omega_m q^2}{2\hbar}\right)$$

$$\langle p|0\rangle = \left(\frac{1}{\pi\hbar\omega_m}\right)^{1/4} \exp\left(-\frac{p^2}{2\hbar\omega_m}\right).$$

$$\Rightarrow P(W) = \frac{1}{\sqrt{\pi\hbar\omega_m}} \exp\left(-\frac{p^2}{\hbar\omega_m}\right)\Big|_{p=W} = \frac{1}{\sqrt{\pi\hbar\omega_m}} \exp\left(-\frac{W^2}{\hbar\omega_m}\right).$$

h) After the measurement $|\psi\rangle = |q\rangle$

$$|q\rangle = \left(\sum_{n=0}^{\infty} |n\rangle\langle n|\right)|q\rangle = \sum_{n=0}^{\infty} \phi_n^*(q)\,|n\rangle$$

The photons came about as a result of the measurement process. You cannot measure the field strengths so accurately without disturbing the system. This is same as what happens in text book QM — If one tries to measure the position of an electron very accurately, one disturbs its momentum (and vice versa).

5.2

a) $$\langle\psi|\hat{n}_m|\psi\rangle = \frac{3}{4}n + \frac{1}{4}(n+2) = n + \frac{1}{2}$$

$$\langle\psi|\hat{n}_m^2|\psi\rangle = \frac{3}{4}n^2 + \frac{1}{4}(n+2)^2 = n^2 + n + 1.$$

$$\langle\psi|\Delta\hat{n}_m^2|\psi\rangle = 3/4 \qquad \Delta\hat{n}_m = \hat{n}_m - \langle\hat{n}_m\rangle$$

$\Rightarrow$ std. deviation $= \sqrt{\langle\Delta\hat{n}_m^2\rangle} = \sqrt{3}/2$

b) $$\langle\hat{n}_m\rangle = Tr\{\hat{\rho}\,\hat{n}_m\} = \frac{1}{4}\{n + (n+2)\} = \frac{n+1}{2}$$

$$\langle\hat{n}_m^2\rangle = \frac{1}{4}\{n^2 + (n+2)^2\} \Rightarrow \sqrt{\langle\Delta\hat{n}_m^2\rangle} = \sqrt{\frac{n^2}{4} + \frac{n}{2} + \frac{3}{4}}$$

c) $$\langle\hat{n}_p\rangle = Tr\{\hat{\rho}\,\hat{n}_p\} = \frac{1}{4}\{(n-1) + (n+2)\} = \frac{2n+1}{4}.$$

$$\langle\hat{n}_p^2\rangle = Tr\{\hat{\rho}\,\hat{n}_p^2\} = \frac{1}{4}\{(n-1)^2 + (n+2)^2\} = \frac{2n^2+2n+5}{4}.$$

$$\Rightarrow \sqrt{\langle\Delta\hat{n}_p^2\rangle} = \sqrt{\frac{n^2+n+19/4}{4}}.$$

d) Same as in part (a). $\left\{ \text{i.e } \langle \hat{N} \rangle = \langle \hat{n}_m \rangle \ \& \ \langle \Delta \hat{N}^2 \rangle = \langle \Delta \hat{n}_m^2 \rangle \right\}$

e).

$$\langle \hat{N} \rangle = Tr\{ \hat{\rho}\, \hat{N} \} = Tr\{ \hat{\rho} [\hat{n}_m + \hat{n}_p] \} = Tr\{ \hat{\rho}\, \hat{n}_m \} + Tr\{ \hat{\rho}\, \hat{n}_p \}$$

$$= \frac{1}{4} \left\{ n + (n+2) + (n-1) + (n+2) \right\} = n + \frac{3}{4}.$$

$$\langle \hat{N}^2 \rangle = Tr\{ \hat{\rho}\, \hat{N}^2 \} = Tr\{ \hat{\rho} [\hat{n}_m^2 + \hat{n}_p^2 + \hat{n}_p \hat{n}_m + \hat{n}_m \hat{n}_p] \}$$

$$= Tr\{ \hat{\rho}\, \hat{n}_m^2 \} + Tr\{ \hat{\rho}\, \hat{n}_p^2 \} = n^2 + \frac{3}{2} n + 9/4$$

$$\Rightarrow \sqrt{\langle \Delta \hat{N}^2 \rangle} = \left\{ n^2 + \frac{3}{2} n + 9/4 - \left(n + \frac{3}{4}\right)^2 \right\}^{1/2}$$

$$= \sqrt{9/4 - \frac{9}{16}} = \sqrt{\frac{27}{16}} \quad \underline{\text{Ans}}$$

f) $\langle \hat{N} \rangle = Tr\{ \hat{\rho}\, \hat{N} \}$

$$= Tr\{ \hat{\rho}\, \hat{n}_m \} + Tr\{ \hat{\rho}\, \hat{n}_p \} = \frac{1}{3} [n + n + 2] + \frac{1}{3} [n - 1 + n + 2]$$

$$= \frac{4}{3} n + 1$$

$$\langle \hat{N}^2 \rangle = Tr\{ \hat{\rho}\, \hat{N}^2 \} = Tr\{ \hat{\rho} [\hat{n}_m^2 + \hat{n}_p^2 + 2 \hat{n}_p \hat{n}_m] \}$$

$$= \frac{1}{3} \left(n^2 + (n+2)^2 \right) + \frac{1}{3} \left((n-1)^2 + (n+2)^2 \right)$$

$$+ \frac{2}{3} \left[n (n-1) \right]$$

$$= 2n^2 + 4/3\, n + 3. \quad \Rightarrow \sqrt{\langle \Delta \hat{N}^2 \rangle} = \sqrt{\frac{2}{9} n^2 - 4/3\, n + 2}.$$

5.3

a) Only the last term in the expression for the density operator will contribute a non-zero value (equal to 1) under the trace operation.

b)

1) $\langle \hat{n}_m \rangle = n$

2) $\langle \hat{n}_p \rangle = n$

3) $\langle \hat{a}_m \rangle = 0$

4) $\langle \hat{a}_p \rangle = \sqrt{n+1}$

5) $\langle \hat{a}_m^+ \rangle = 0$

6) $\langle \hat{a}_p^+ \rangle = \sqrt{n+1}$

7) $\langle \hat{n}_m \hat{n}_p \rangle = n^2$

8) $\langle \hat{a}_m^+ \hat{a}_p \rangle = \sqrt{n}\sqrt{n+1}$

9) $\langle \hat{a}_p^+ \hat{a}_m \rangle = \sqrt{n}\sqrt{n+1}$

10) $\langle \hat{a}_m^+ \hat{a}_p^+ \rangle = 0$

11) $\langle \hat{a}_m \hat{a}_p \rangle = 0$