## ECE 5310: Applied Quantum Optics for Photonics and Optoelectronics

**Fall 2013**

**Homework 5** **Due on Oct. 08, 2013 (self-grade)**

**Midterm:** Midterm will take place in mid October immediately after the Fall break.

### Problem 5.1: (Explorations in quantum optics)

This problem will explore the concept of a "photon" in more detail. You will learn how the probability distribution of the field strengths can be calculated for different photonic states. This problem will tie together much of what you learned about basic quantum mechanics earlier in the course and bring together many fundamental concepts. Most of the calculations are simple and can be done in one line provided you understand the concepts.

The operator equations for a simple harmonic oscillator (particle in a quadratic potential well) are,

$$\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega_o^2\,\hat{x}^2 = \hbar\omega_o\left(\hat{a}^+\hat{a} + \frac{1}{2}\right)$$

where,

$$\hat{a} = \frac{1}{\sqrt{2\hbar\omega_o}}\left[\omega_o\,\hat{x} + i\,\hat{p}\right] \qquad \hat{a}^+ = \frac{1}{\sqrt{2\hbar\omega_o}}\left[\omega_o\,\hat{x} - i\,\hat{p}\right]$$

$$\left[\hat{x},\hat{p}\right] = i\hbar \qquad \left|\hat{a},\hat{a}^+\right| = 1$$

Just from the commutation relation $\left[\hat{x},\hat{p}\right] = i\hbar$, and the fact that operators $\hat{x}$ and $\hat{p}$ have eigenkets $|x\rangle$ and $|p\rangle$ that satisfy the completeness relations, we were able to transform the Shrodinger equation in the operator form,

$$i\hbar\frac{\partial|\psi(t)\rangle}{\partial t} = \hat{H}\,|\psi(t)\rangle$$

into a differential equation of the form ,

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2}\frac{\partial^2\psi(x,t)}{\partial x^2} + \frac{1}{2}\omega_o^2\,x^2\psi(x,t)$$

The time-independent form of the above equation yielded the eigenenergies and eigenfunctions of a simple harmonic oscillator.

$$-\frac{\hbar^2}{2}\frac{\partial^2\phi_n(x)}{\partial x^2} + \frac{1}{2}\omega_o^2\,x^2\phi_n(x) = E_n\,\phi_n(x)$$

The eigenfunctions $\phi_n(x)$ (for n=0,1,2,3,…..) are Hermite Gaussians, and are given below,

$$\langle x|0\rangle = \phi_0(x) = \left(\frac{\omega_o}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{\omega_o x^2}{2\hbar}\right) \qquad \text{energy: } \frac{\hbar\omega_o}{2}$$

$$\langle x|1\rangle = \phi_1(x) = \left(\frac{4}{\pi}\left(\frac{\omega_o}{\hbar}\right)^3\right)^{1/4} x \exp\left(-\frac{\omega_o x^2}{2\hbar}\right) \qquad \text{energy: } \hbar\omega_o + \frac{\hbar\omega_o}{2}$$

$$\langle x|2\rangle = \phi_2(x) = \left(\frac{\omega_o}{4\pi\hbar}\right)^{1/4}\left[2\frac{\omega_o x^2}{\hbar} - 1\right]\exp\left(-\frac{\omega_o x^2}{2\hbar}\right) \qquad \text{energy: } 2\hbar\omega_o + \frac{\hbar\omega_o}{2}$$

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$$\langle x|n\rangle = \phi_n(x) \qquad \text{energy: } n\hbar\omega_o + \frac{\hbar\omega_o}{2}$$

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Now consider a single mode of electromagnetic field inside a cavity. **Since we are only considering a single mode of the field, I will drop the mode index "m" from all the operators and the states (but keep it for the mode frequency).** The Hamiltonian is,

$$\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega_m^2\,\hat{q}^2 = \hbar\omega_m\left(\hat{a}^+\hat{a} + \frac{1}{2}\right)$$

where,

$$\hat{a} = \frac{1}{\sqrt{2\hbar\omega_m}}\left[\omega_m\,\hat{q} + i\,\hat{p}\right] \qquad \hat{a}^+ = \frac{1}{\sqrt{2\hbar\omega_m}}\left[\omega_m\,\hat{q} - i\,\hat{p}\right]$$

$$[\hat{q},\hat{p}] = i\hbar \qquad [\hat{a},\hat{a}^+] = 1$$

The electric and magnetic fields (in the Schrodinger picture) are related directly to the operators $\hat{p}$ and $\hat{q}$ as follows,

$$\hat{\vec{E}}(\vec{r}) = -\frac{\hat{p}}{\sqrt{\varepsilon_o\varepsilon_m}}\vec{U}_m(\vec{r}) \qquad \hat{\vec{H}}(\vec{r}) = \frac{\hat{q}}{\mu_o\sqrt{\varepsilon_o\varepsilon_m}}\nabla\times\vec{U}_m(\vec{r})$$

Since the electric and magnetic fields are physical observables, the operators $\hat{p}$ and $\hat{q}$ are also physical observables and are Hermitian. These operators represent the field amplitudes. They will have eigenstates $|q\rangle$ and $|p\rangle$ that will satisfy the completeness relations,

$$\int_{-\infty}^{\infty} dq\,|q\rangle\langle q| = \hat{1} \text{ and } \int_{-\infty}^{\infty} dp\,|p\rangle\langle p| = \hat{1}$$

a) Starting from the commutator for the field variables $[\hat{q},\hat{p}] = i\hbar$, and using the completeness relations for the eigenkets $|q\rangle$ and $|p\rangle$, show that $\langle p|q\rangle = \dfrac{\exp\left(-i\frac{p}{\hbar}q\right)}{\sqrt{2\pi\,\hbar}}$.

b) We can expand any quantum state of a **single mode electromagnetic field** in the $|q\rangle$ eigenkets as follows: $|\psi(t)\rangle = \int_{-\infty}^{\infty} dq\;\; \psi(q,t)\,|q\rangle$. Show that $\psi(q,t)$ satisfies the differential equation,

$$i\hbar\frac{\partial\psi(q,t)}{\partial t} = -\frac{\hbar^2}{2}\frac{\partial^2\psi(q,t)}{\partial q^2} + \frac{1}{2}\omega_o^2\,q^2\psi(q,t)$$

c) The vacuum state $|0\rangle$ of the electromagnetic field (associated with the mode under consideration) can be expanded in the $|q\rangle$ basis set. Find the coefficients of this expansion.

d) The number state $|2\rangle$ of the electromagnetic field can be expanded in the $|q\rangle$ basis set. Find the coefficients of this expansion.

e) Suppose we prepare the single photon state $|1\rangle$ inside the cavity. If a measurement is made at the location $\vec{r}$ to determine the magnetic field strength, given by the operator $\hat{\vec{H}}(r)$, what is the probability $P(w)$ of obtaining the result $\frac{w}{\mu_o\sqrt{\varepsilon_o\varepsilon_m}}\nabla\times\vec{U}_m(r)$ ? Your answer $P(w)$ must be normalized such that,

$$\int_{-\infty}^{\infty} P(w)\,dw = 1.$$

f) Suppose we prepare the number state $|2\rangle$ inside the cavity. If a measurement is made at the location $\vec{r}$ to determine the magnetic field strength, given by the operator $\hat{\vec{H}}(r)$, then what field strengths $w$, if any, can never be obtained as a result of this measurement.

g) Suppose we have a vacuum state $|0\rangle$ inside the cavity. If a measurement is made at the location $\vec{r}$ to determine the electric field strength, given by the operator $\hat{\vec{E}}(r)$, what is the probability $P(w)$ of obtaining the result $\left(-\frac{w}{\sqrt{\varepsilon_o\varepsilon_m}}\vec{U}_m(r)\right)$ ? Note that $P(w)$ must be normalized such that $\int_{-\infty}^{\infty} P(w)\,dw = 1$.

h) Suppose the field is in the state $|2\rangle$ (i.e. a state with a definite number of photons). A magnetic field measurement is made at the location $\vec{r}$ and the result $\frac{q}{\mu_o\sqrt{\varepsilon_o\varepsilon_m}}\nabla\times\vec{U}_m(r)$ is obtained. What is the state of the field immediately after the measurement? Write your answer as a linear superposition of photon number states $|n\rangle$. Try to explain where the other photons came from?

## Problem 5.2: (Working with photon number states)

In this problem you will get some practice with different photon states. The Hamiltonian of the field in a cavity is,

$$\hat{H} = \sum_{m=1}^{\infty} \hbar\omega_m \left(\hat{a}_m^+\hat{a}_m + \frac{1}{2}\right)$$

where $\left[\hat{a}_m, \hat{a}_n^+\right] = \delta_{nm}$. The photon number operator for the mode "m" is $\hat{n}_m = \hat{a}_m^+\hat{a}_m$. And the photon number operator for all the modes is $\hat{N} = \sum_{m=1}^{\infty}\hat{n}_m$.

a) Suppose the state of the field is described by a state vector $|\psi\rangle = \frac{\sqrt{3}}{2}|n\rangle_m + \frac{1}{2}|n+2\rangle_m$ (*** recall that $|n\rangle_m$ means "$n$" photons in mode "$m$" AND zero photons in every other mode).

If a measurement is made to determine the number of photons in mode "$m$", what is going to be the mean result? What is going to be the standard deviation?

b) Suppose the state of the field is described by a density operator,

$$\hat{\rho} = \frac{1}{4}\Big[\, |n\rangle_{m\,m}\langle n| + |n+2\rangle_{m\,m}\langle n+2| + |n-1\rangle_{p\,p}\langle n-1| + |n+2\rangle_{p\,p}\langle n+2| \Big]$$

If a measurement is made to determine the number of photons in mode "$m$", what is going to be the mean result? What is going to be the standard deviation?

Hint: Here you will encounter the first case of complexity creeping in. When you take averages w.r.t. a density operator you do a trace operation and for that purpose you use any complete basis set. Here you have a problem that deals with two modes of the field so the appropriate complete basis set to use is not the following,

$$\sum_{n=0}^{\infty} |n\rangle_{m\;\,m}\langle n| = \hat{1}$$

Since this only works in the Hilbert space of mode "$m$". You need to use a complete basis for the Hilbert space of the two modes,

$$\sum_{n=0}^{\infty}\sum_{q=0}^{\infty} \Big(|n\rangle_m \otimes |q\rangle_p\Big)\Big({}_m\langle n| \otimes {}_p\langle q|\Big) = \hat{1}$$

If I ever gave you a density operator that dealt with three modes, you should use a complete basis set in the Hilbert space of those three modes. Make sure you understand all the confusing indexing and labeling procedures before attempting the problem.

c) For the state of the field given in part (b), if a measurement is made to determine the number of photons in mode "$p$", what is going to be the mean result? What is going to be the standard deviation?

d) For the state given in part (a), if a measurement is made to determine the TOTAL number of photons in all the modes what is going to be the mean result? What is going to be the standard deviation?

e) For the state given in part (b), if a measurement is made to determine the TOTAL number of photons in all the modes what is going to be the mean result? What is going to be the standard deviation?

f) Suppose the state of the field is described by a density operator,

$$\hat{\rho} = \frac{1}{3}\Big[\, \Big(|n\rangle_m \otimes |n-1\rangle_p\Big)\Big({}_m\langle n| \otimes {}_p\langle n-1|\Big) + |n+2\rangle_{m\,m}\langle n+2| + |n+2\rangle_{p\,p}\langle n+2| \Big]$$

If a measurement is made to determine the TOTAL number of photons in all the modes what is going to be the mean result? What is going to be the standard deviation?

**Points to ponder:**
In real life, the way you will do a measurement of the number of photons in mode "$m$" (or "$p$") is by sticking a photodetector inside the cavity that will respond to only photons at frequency of mode "$m$" (or "$p$"). You can do a measurement of the TOTAL photon number by using a photodetector that responds to frequencies of both the modes, but then you may not get information about how many photons were in each mode since you made a total photon number measurement. Notice that the process of measurement destroys the photon (i.e. destroys the state we want to measure). Can you think of a photon number measurement scheme that would NOT destroy the photons it was trying to count?

## Problem 5.3: (More averages)

In this problem you will get some more practice with different photon number states. Calculating these averages quickly can only be learnt through practice. The Hamiltonian of the field in a cavity is,

$$\hat{H} = \sum_{m=1}^{\infty} \hbar\omega_m \left( \hat{a}_m^{+}\hat{a}_m + \frac{1}{2} \right)$$

where $\left[\hat{a}_m, \hat{a}_n^{+}\right] = \delta_{nm}$. The photon number operator for the mode "m" is $\hat{n}_m = \hat{a}_m^{+}\hat{a}_m$ .

a) The quantum state of the field is (notice the "off-diagonal" elements):

$$\hat{\rho} = \left[ \begin{array}{l} \left(|n\rangle_m \otimes |n\rangle_p\right)\left({}_m\langle n+1| \otimes {}_p\langle n-1|\right) + \left(|n+1\rangle_m \otimes |n-1\rangle_p\right)\left({}_m\langle n| \otimes {}_p\langle n|\right) + |n\rangle_{p\,p}\langle n+1| \\ + |n+1\rangle_{p\,p}\langle n| + |n\rangle_{m\,m}\langle n+2| + |n+2\rangle_{m\,m}\langle n| + \left(|n\rangle_m \otimes |n\rangle_p\right)\left({}_m\langle n| \otimes {}_p\langle n|\right) \end{array} \right]$$

confirm that $\mathrm{Tr}[\hat{\rho}]=1$.

b) For the state in part (a) calculate the following:

(1) $\langle \hat{n}_m \rangle$

(2) $\langle \hat{n}_p \rangle$

(3) $\langle \hat{a}_m \rangle$

(4) $\langle \hat{a}_p \rangle$

(5) $\langle \hat{a}_m^{+} \rangle$

(6) $\langle \hat{a}_p^{+} \rangle$

(7) $\langle \hat{n}_m\ \hat{n}_p \rangle$

(8) $\langle \hat{a}_m^{+}\ \hat{a}_p \rangle$

(9) $\langle \hat{a}_p^{+}\ \hat{a}_m \rangle$

(10) $\langle \hat{a}_m^{+}\ \hat{a}_p^{+} \rangle$

(11) $\langle \hat{a}_m\ \hat{a}_p \rangle$