ECE 531 Homework 4 Solutions

3.1

a) $$|e_3\rangle = \frac{e^{i\phi_{23}/2}}{\sqrt{2}}\left[|V_a\rangle - |V_b\rangle\right] \Rightarrow \hat{\rho}_{31} = \langle e_3|\hat{\rho}|1\rangle = \frac{e^{-i\phi_{23}/2}}{\sqrt{2}}\left\{\hat{\rho}_{a1} - \hat{\rho}_{b1}\right\}.$$

b)

$$\frac{d\hat{\rho}_{a1}(t)}{dt} = -i\left(\frac{\Delta_{13}}{\hbar} - \frac{\Omega_{R23}}{2}\right)\hat{\rho}_{a1}(t) - i\frac{\Omega_{R13}}{\sqrt{8}}\left[\hat{\rho}_{aa}(t) - \hat{\rho}_{11}(t)\right]e^{-i\phi_{13}+i\phi_{23}/2}$$

$$\frac{d\hat{\rho}_{b1}(t)}{dt} = -i\left(\frac{\Delta_{13}}{\hbar} + \frac{\Omega_{R23}}{2}\right)\hat{\rho}_{b1}(t) + i\frac{\Omega_{R13}}{\sqrt{8}}\left[\hat{\rho}_{bb}(t) - \hat{\rho}_{11}(t)\right]e^{-i\phi_{13}+i\phi_{23}/2}$$

c) Lets just consider decoherence.. Then,

$$\frac{d\hat{\rho}_{31}(t)}{dt} = -\gamma_{13}\hat{\rho}_{31}(t) \quad \text{and} \quad \frac{d\hat{\rho}_{21}(t)}{dt} = 0$$

But from part (a): $$\hat{\rho}_{31}(t) = \frac{e^{-i\phi_{23}/2}}{\sqrt{2}}\left\{\hat{\rho}_{a1}(t) - \hat{\rho}_{b1}(t)\right\}$$

Similarly: $$\hat{\rho}_{21}(t) = \frac{e^{i\phi_{23}/2}}{\sqrt{2}}\left\{\hat{\rho}_{a1}(t) + \hat{\rho}_{b1}(t)\right\}.$$

Therefore the above differential equations give:

$$\frac{d\hat{\rho}_{a1}(t)}{dt} - \frac{d\hat{\rho}_{b1}(t)}{dt} = -\gamma_{13}\left(\hat{\rho}_{a1}(t) - \hat{\rho}_{b1}(t)\right)$$

$$\frac{d\hat{\rho}_{a1}(t)}{dt} + \frac{d\hat{\rho}_{b1}(t)}{dt} = 0$$

Adding and subtracting gives:

$$\left.\begin{aligned}\frac{d}{dt}\hat{\rho}_{a1}(t) &= -\frac{\gamma_{13}}{2}\left(\hat{\rho}_{a1}(t) - \hat{\rho}_{b1}(t)\right)\\ \frac{d}{dt}\hat{\rho}_{b1}(t) &= +\frac{\gamma_{13}}{2}\left(\hat{\rho}_{a1}(t) - \hat{\rho}_{b1}(t)\right)\end{aligned}\right\}$$

This is how decoherence enters the equations for $\hat{\rho}_{a1}(t)$ and $\hat{\rho}_{b1}(t)$

So we get:

$$\frac{d}{dt}\hat{\rho}_{a1}(t) = -\frac{\gamma_{13}}{2}\left(\hat{\rho}_{a1}(t) - \hat{\rho}_{b1}(t)\right) - i\left(\frac{\Delta_{13}}{\hbar} - \frac{\Omega_{R23}}{2}\right)\hat{\rho}_{a1}(t) - i\frac{\Omega_{R13}}{\sqrt{8}}\left[\hat{\rho}_{aa} - \hat{\rho}_{11}\right]e^{-i\phi_{13}+i\frac{\phi_{23}}{2}}$$

$$\frac{d}{dt}\hat{\rho}_{b1}(t) = +\frac{\gamma_{13}}{2}\left(\hat{\rho}_{a1}(t) - \hat{\rho}_{b1}(t)\right) - i\left(\frac{\Delta_{13}}{\hbar} + \frac{\Omega_{R23}}{2}\right)\hat{\rho}_{b1}(t) + i\frac{\Omega_{R13}}{\sqrt{8}}\left[\hat{\rho}_{aa} - \hat{\rho}_{11}\right]e^{-i\phi_{13}+i\frac{\phi_{23}}{2}}$$

d) Subtract the equations obtained in part (c):

$$\frac{d}{dt}\hat{\rho}_a(t) = -\frac{\gamma_{13}}{2}\hat{\rho}_a(t) - i\frac{\Delta_{13}}{\hbar}\hat{\rho}_a(t) + i\frac{\Omega_{R23}}{2}\hat{\rho}_s(t) - i\frac{\Omega_{R13}}{\sqrt{2}}\left[\hat{\rho}_{99} - \hat{\rho}_{11}\right]e^{-i\phi_{13}+i\frac{\phi_{23}}{2}}$$

Now add:

$$\frac{d}{dt}\hat{\rho}_s(t) = -i\frac{\Delta_{13}}{\hbar}\hat{\rho}_s(t) + i\frac{\Omega_{R23}}{2}\hat{\rho}_a(t)$$

Differentiate w.r.t. time the equation for $\hat{\rho}_a(t)$ and then use the equation for $\hat{\rho}_s(t)$ to get:

$$\frac{d^2\hat{\rho}_a}{dt^2} + \left(\gamma_{13} + 2i\frac{\Delta_{13}}{\hbar}\right)\frac{d\hat{\rho}_a}{dt} + \hat{\rho}_a\left\{\left(\frac{\Omega_{R23}}{2}\right)^2 - \left(\frac{\Delta_{13}}{\hbar}\right)^2 + i\frac{\Delta_{13}}{\hbar}\gamma_{13}\right\}$$

$$= -\frac{\Delta_{13}}{\hbar}\frac{\Omega_{R13}}{\sqrt{2}}e^{-i\phi_{13}+i\phi_{23}/2}$$

I will let you guys figure out the equation for $\hat{\rho}_s(t)$.

e) Because of damping in the above equation (due to γ_{13}) $\hat{\rho}_a(t)$ will have a well defined steady state.

f) Steady state value of $\hat{\rho}_a$ is,

$$\hat{\rho}_a = \frac{-\frac{\Delta_{13}}{\hbar}\frac{\Omega_{R13}}{\sqrt{2}}e^{-i\phi_{13}+i\phi_{23}/2}}{\left[i\frac{\Delta_{13}}{\hbar}\left(\gamma_{13} + i\frac{\Delta_{13}}{\hbar}\right) + \left(\frac{\Omega_{R23}}{2}\right)^2\right]}.$$

$\left\{ \Omega_{R13} = \frac{qE_{13}(\vec{d}_{13}\cdot\hat{n})}{\hbar} \right.$

$$\chi(\omega_{13}) = \frac{qN(\vec{d}_{13}\cdot\hat{n})}{\varepsilon_0}\frac{2\hat{\rho}_{31}}{E_{13}} = \frac{qN(\vec{d}_{13}\cdot\hat{n})}{\varepsilon_0}2\frac{e^{-i\phi_{23}/2}}{\sqrt{2}}\cdot\frac{\hat{\rho}_a}{E_{13}}$$

$$= \frac{q^2N(\vec{d}_{13}\cdot\hat{n})^2}{\varepsilon_0\hbar}\frac{-\Delta_{13}/\hbar}{i\gamma_{13}\frac{\Delta_{13}}{\hbar} - \left(\frac{\Delta_{13}}{\hbar}\right)^2 + \left(\frac{\Omega_{R23}}{2}\right)^2}$$

$\left\{ \right.$ Same as in the lecture notes other than a factor of $\hbar$ missing in the lecture notes

3.2

a) Try $\hat{B}(t) = e^{i\omega_{23}\hat{N}_3 t}$ { This "reduces" the energy of the state $|e_3\rangle$ by $\hbar\omega_{23}$.

$\Rightarrow |\phi_R(t)\rangle = \hat{B}(t)|\psi(t)\rangle$.

Note: $\hat{B}(t)\left[-\hbar\omega_{23}\hat{N}_3 + \hat{H}(t)\right]\hat{B}^\dagger(t) = \hat{H}_R$

where
$$\hat{H}_R = \varepsilon_1|e_1\rangle\langle e_1| + \varepsilon_2|e_2\rangle\langle e_2| + (\varepsilon_3 - \hbar\omega_{23})|e_3\rangle\langle e_3|$$
$$- U\left[|e_1\rangle\langle e_2| + |e_2\rangle\langle e_1|\right] - \frac{\hbar\Omega_{R23}}{2}\left[e^{i\phi_{23}}|e_2\rangle\langle e_3| + e^{-i\phi_{23}}|e_3\rangle\langle e_2|\right]$$
= time independent!

Note: Had you tried to "boost" the energy of $|e_2\rangle$ by $\hbar\omega_{23}$ then the term in the Hamiltonian containing U would end up becoming time dependent.

In matrix form:
{ $\varepsilon_3 - \hbar\omega_{23} = \varepsilon_2$ since detuning $= 0$

$$\hat{H}_R = \begin{bmatrix} \varepsilon_1 & -U & 0 \\ -U & \varepsilon_2 & -\frac{\hbar\Omega_{R23}}{2}e^{i\phi_{23}} \\ 0 & -\frac{\hbar\Omega_{R23}}{2}e^{-i\phi_{23}} & \varepsilon_3 - \hbar\omega_{23} \end{bmatrix}$$
in basis $|e_1\rangle, |e_2\rangle, |e_3\rangle$.

Now diagonalize the lower 2X2 Hamiltonian by rotating basis to get

$$\hat{H}_R = \begin{bmatrix} \varepsilon_1 & -\frac{U}{\sqrt{2}} & -\frac{U}{\sqrt{2}} \\ -\frac{U}{\sqrt{2}} & \lambda_a & 0 \\ -\frac{U}{\sqrt{2}} & 0 & \lambda_b \end{bmatrix}$$
in basis $|e_1\rangle\ |v_a\rangle\ |v_b\rangle$

$$|v_a\rangle = \frac{1}{\sqrt{2}}\left[|e_2\rangle + e^{-i\phi_{23}}|e_3\rangle\right]$$
$$|v_b\rangle = \frac{1}{\sqrt{2}}\left[|e_2\rangle - e^{-i\phi_{23}}|e_3\rangle\right].$$

For tunneling to happen, ε_1 should either be $\varepsilon_2 - \frac{\hbar\Omega_{R23}}{2}$ or $\varepsilon_2 + \frac{\hbar\Omega_{R23}}{2}$

{ $\lambda_a = \varepsilon_2 - \frac{\hbar\Omega_{R23}}{2}$, $\lambda_b = \varepsilon_2 + \frac{\hbar\Omega_{R23}}{2}$

$\Rightarrow$ it is as if the lower state is split!

3.3

a) $\frac{\partial E_0(z)}{\partial z} = ik \frac{\chi(\omega)}{2} E_0(z)$ and $\frac{\partial E_0^*(z)}{\partial z} = -ik \chi^*(\omega) E_0^*(z)$.

b) Real part of $\chi(\omega)$ changes the effective propagation vector from k to $k\left(1+\frac{\chi'(\omega)}{2}\right)$

c) Imaginary part of $\chi(\omega)$ causes the wave amplitude to decay as $e^{-k\frac{\chi''(\omega)}{2}z}$ $\{$or grow instead of decay if $\chi''(\omega)<0\}$.

3.4

a) $\frac{\partial E_m(t)}{\partial t} = i \frac{\omega_m \chi(\omega_m)}{2} E_m(t)$ and $\frac{\partial E_m^*(t)}{\partial t} = -i\omega_m \frac{\chi^*}{2}(\omega_m) E_m^*(t)$

b) Suppose $\chi(\omega_m) = \chi'(\omega_m) + i\chi''(\omega_m)$

$\Rightarrow \frac{\partial E_m(t)}{\partial t} = \frac{i\,\omega_m \chi'(\omega_m)}{2} E_m(t) - \frac{\omega_m \chi''(\omega_m)}{2} E_m(t)$.

$\Rightarrow E_m(t) = E_m(t=0)\, e^{\frac{i\,\omega_m \chi'(\omega_m)t}{2}}\, e^{-\frac{\omega_m \chi''(\omega_m)t}{2}}$

Note that the actual field goes as $E_m(t)e^{-i\omega_m t}$ which means that field goes as $E_m(t=0)\, e^{-i\left[\omega_m\left(1 - \frac{\chi'(\omega_m)}{2}\right)\right]t}\, e^{-\omega_m \frac{\chi''(\omega_m)}{2}t}$

So the real part of $\chi(\omega_m)$ modifies the frequency ω_m.

Recall that for a two-level system medium:

$$\chi(\omega_m) \approx \frac{q^2 N(\vec{d}\cdot\hat{n})^2}{\varepsilon_0}\left[\rho_{22}-\rho_{11}\right] \frac{\hbar\omega - (\varepsilon_2-\varepsilon_1) - i\frac{\hbar}{T_2}}{\left[\hbar\omega-(\varepsilon_2-\varepsilon_1)\right]^2 + \left(\frac{\hbar}{T_2}\right)^2}$$

So if $\rho_{22}<\rho_{11}$, the modified frequency will be pushed away from the two-level system resonance. If $\rho_{22}>\rho_{11}$, the modified frequency is pulled towards the resonance.

c) As seen above the imaginary part of $\chi(\omega_m)$ makes the wave amplitude increase or decrease in time depending on the sign of $\chi''(\omega_m)$.