ECE5310 Homework 3 Solutions

3.1

a) $\dot{V}_y = -\frac{1}{T_2} V_y + \Omega_R V_z \Rightarrow \ddot{V}_y = -\frac{1}{T_2}\dot{V}_y + \Omega_R\left[-\frac{V_z+1}{T_1} - \Omega_R V_y\right]$

$\Rightarrow \quad \ddot{V}_y + \left(\frac{1}{T_1}+\frac{1}{T_2}\right)\dot{V}_y + \left(\Omega_R^2 + \frac{1}{T_1T_2}\right)V_y = -\frac{\Omega_R}{T_1}$

Similarly, $\ddot{V}_z + \left(\frac{1}{T_1}+\frac{1}{T_2}\right)\dot{V}_z + \left(\Omega_R^2 + \frac{1}{T_1T_2}\right)V_z = -\frac{1}{T_1T_2}$

b)

Let $\gamma = \frac{1}{T_1}+\frac{1}{T_2}$ $\qquad \Omega = \sqrt{\Omega_R^2 - \frac{1}{4}\left(\frac{1}{T_1}-\frac{1}{T_2}\right)^2}$

Since $|\psi(t=0)\rangle = |e_1\rangle \Rightarrow V_z(t=0) = -1 \qquad V_y(t=0) = 0$

$\Rightarrow \dot{V}_z\Big|_{t=0} = 0 \qquad \dot{V}_y\Big|_{t=0} = -\Omega_R$

Solution: $V_z(t) = e^{-\gamma/2\, t}\left\{A\cos\Omega t + B\sin\Omega t\right\} + C$

Boundary conditions $\Rightarrow A+C = -1 \qquad B = \frac{\gamma A}{2\Omega}$

For the particular solution choose $C = \dfrac{-1/T_1T_2}{\Omega_R^2 + \frac{1}{T_1T_2}}$

$$\Rightarrow \quad A = -1 - C = \frac{-\Omega_R^2}{\Omega_R^2 + \frac{1}{T_1 T_2}} \qquad \Rightarrow \quad B = -\frac{\gamma}{2\Omega} \cdot \frac{\Omega_R^2}{\Omega_R^2 + \frac{1}{T_1 T_2}}$$

Similarly, $V_y(t) = e^{-\gamma/2\, t}\left\{ D \cos\Omega t + E \sin \Omega t \right\} + F$

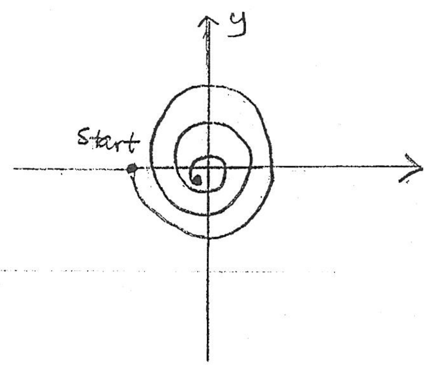
$F = -\dfrac{\Omega_R/T_1}{\Omega_R^2 + \frac{1}{T_1 T_2}}$. boundary conditions $\Rightarrow$ $D + F = 0$. $\Rightarrow$ $D = -F$.

$$\Rightarrow D = \frac{\Omega_R/T_1}{\Omega_R^2 + \frac{1}{T_1T_2}} \quad \text{and} \quad E = \frac{\gamma}{2\Omega} D - \frac{\Omega_R}{\Omega}$$

$$= \frac{\gamma}{2\Omega}\,\frac{\Omega_R/T_1}{\Omega_R^2 + \frac{1}{T_1T_2}} - \frac{\Omega_R}{\Omega}$$

Time period $= \dfrac{2\pi}{\Omega}$

c) $$\vec{V}(t \to \infty) = \frac{-\frac{\Omega_R}{T_1}\hat{y} - \frac{1}{T_1T_2}\hat{z}}{\Omega_R^2 + \frac{1}{T_1T_2}}.$$



d) $V_x(t) = 0 \Rightarrow V_y(t) = i\, 2\rho_{21}(t)\, e^{i\omega t}$

$\Rightarrow \quad \rho_{21}(t) = -\frac{i}{2} V_y(t) e^{-i\omega t} \quad \left\{ \text{and } \rho_{12}(t) = \rho_{21}^*(t) \right\}$

Also $\rho_{22}(t) = \dfrac{1 + V_z(t)}{2}$ $\quad \rho_{11}(t) = \dfrac{1 - V_z(t)}{2}$. A necessary condition for the quantum state to be described by a state vector (i.e for the quantum state to be a pure state) is that the determinant of the 2x2 matrix $\hat{\rho}(t)$ be zero, i.e

$\rho_{11}(t)\rho_{22}(t) - \rho_{21}(t)\rho_{12}(t) = 0$. This is not the case as $t \to \infty$.

So the quantum state as $t \to \infty$ is not a pure state and cannot be described by a state vector $|\psi(t)\rangle$ as $t \to \infty$.

e) Yes. The individual components of $\hat{\rho}(t)$ as $t \to \infty$ are given by the relations in part (d) above.

3.2

a) After the first $\frac{\pi}{2}$ pulse, the equations for V_y and V_x are:

$$\frac{d^2 V_y}{dt^2} + \left(\frac{\Delta}{\hbar}\right)^2 V_y = 0 \qquad \frac{d^2 V_x}{dt^2} + \left(\frac{\Delta}{\hbar}\right)^2 V_x = 0$$

Solution subject to the initial condition $V_y(t=0) = -1$

and $\left.\frac{dV_y}{dt}\right|_{t=0} = 0$ is: $V_y(t) = -\cos\left(\frac{\Delta t}{\hbar}\right)$ and $V_x(t) = \sin\left(\frac{\Delta}{\hbar}t\right)$

So after time T, $V_y(T) = -\cos\left(\frac{\Delta T}{\hbar}\right)$ and $V_x(T) = \sin\left(\frac{\Delta}{\hbar}T\right)$

Now we use these values as initial conditions when the second $\frac{\pi}{2}$ pulse comes. During the second $\frac{\pi}{2}$ pulse, $V_z(t)$ satisfies: $\frac{d^3 V_z}{dt^3} + \Omega^2 \frac{dV_z}{dt} = 0$. Boundary conditions for this phase of evolution are:

$$V_z(t=0) = 0 \qquad \left.\frac{dV_z}{dt}\right|_{t=0} = -\Omega_R V_y(t=0) = \Omega_R \cos\left(\frac{\Delta T}{\hbar}\right)$$

and $$\left.\frac{d^2 V_z}{dt^2}\right|_{t=0} = -\Omega_R \left.\frac{dV_y}{dt}\right|_{t=0} = -\Omega_R\left[\frac{\Delta}{\hbar} V_x(t=0) + \Omega_R V_z(t=0)\right]$$

$$= -\Omega_R \frac{\Delta}{\hbar} \sin\left(\frac{\Delta}{\hbar}T\right).$$

Solution is:

$$V_z(t) = \frac{\Omega_R}{\Omega}\cos\left(\frac{\Delta}{\hbar}T\right)\sin(\Omega t) + \frac{\Omega_R \frac{\Delta}{\hbar}\sin\left(\frac{\Delta}{\hbar}T\right)}{\Omega^2}\left[\cos(\Omega t) - 1\right].$$

$$\Rightarrow V_z\left(t = \frac{\pi}{2\Omega}\right) = \frac{\Omega_R}{\Omega}\cos\left(\frac{\Delta T}{\hbar}\right) - \frac{\Omega_R \Delta/\hbar}{\Omega^2}\sin\left(\frac{\Delta T}{\hbar}\right) \approx \cos\left(\frac{\Delta T}{\hbar}\right) \left\{ \text{for } \frac{\Delta}{\hbar} \ll \Omega_R \right.$$

$$\Rightarrow P_{22} = \frac{1}{2}\left[1 + V_z\right] = \frac{1}{2}\left[1 + \cos\left(\frac{\Delta T}{\hbar}\right)\right].$$

b) $|c_1|^2 = \frac{1}{2}\left[1 - \cos\left(\frac{\Delta T}{\hbar}\right)\right]$ $\quad |c_2|^2 = \frac{1}{2}\left[1 + \cos\left(\frac{\Delta T}{\hbar}\right)\right]$

c) $\langle \hat{N}_1 \rangle = |c_1|^2$ $\quad \langle \hat{N}_1^2 \rangle = \langle \hat{N}_1 \rangle = |c_1|^2$ since $\hat{N}_1^2 = \hat{N}_1$

$$\Rightarrow \langle \Delta \hat{N}_1^2 \rangle = \langle \hat{N}_1^2 \rangle - \langle \hat{N}_1 \rangle^2 = |c_1|^2 (1 - |c_1|^2).$$

$$\Rightarrow \langle \Delta \hat{N}_2^2 \rangle = |c_2|^2 (1 - |c_2|^2)$$

d) $\langle \hat{N}_2^{all} \rangle = \frac{1}{N_a} \sum_{k=1}^{N_a} \langle \hat{N}_2^k \rangle = |c_2|^2$

e) $\langle (\hat{N}_2^{all})^2 \rangle = \frac{1}{N_a^2} \sum_{k=1}^{N_a} \sum_{j=1}^{N_a} \langle \hat{N}_2^k \hat{N}_2^j \rangle = \frac{1}{N_a^2} \sum_{k=j=1}^{N_a} \langle \hat{N}_2^k \rangle + \frac{1}{N_a^2} \sum_{k \neq j}^{N_a(N_a-1)} \langle \hat{N}_2^k \hat{N}_2^j \rangle$

$$= \frac{N_a |c_2|^2}{N_a^2} + \frac{N_a(N_a-1)|c_2|^4}{N_a^2}$$

$$= \frac{|c_2|^2(1-|c_2|^2)}{N_a} + |c_2|^4$$

$$\Rightarrow \langle (\Delta \hat{N}_2^{all})^2 \rangle = \langle (\hat{N}_2^{all})^2 \rangle - \langle \hat{N}_2^{all} \rangle^2 = \frac{|c_2|^2 (1 - |c_2|^2)}{N_a}.$$

f) Joint measurements result in reduced uncertainty. This is essentially saying that averaging will reduce error.

h) $\langle \Delta \hat{\omega}^2 \rangle = \frac{4}{T^2} \langle (\Delta \hat{N}_2^{all})^2 \rangle = \frac{4}{T^2 N_a} |c_2|^2 (1 - |c_2|^2)$. But since $|c_2|^2 = \frac{1}{2}$

$$\langle \Delta \hat{\omega}^2 \rangle = \frac{1}{T^2 N_a}.$$

g) we have $\rho_{22} = \frac{1}{2}\left[1 + \cos\left(\frac{\Delta}{\hbar} T\right)\right]$ $\quad \left\{ \Delta = \Delta\varepsilon - \hbar\omega \right.$

$$\Rightarrow \frac{d\rho_{22}}{d\Delta} = -\frac{\sin\left(\frac{\Delta}{\hbar} T\right)}{2} \left(\frac{T}{\hbar}\right).$$ But $d\Delta = -\hbar d\omega$.

$$\Rightarrow d\rho_{22} = \frac{T}{2} \sin\left(\frac{\Delta}{\hbar} T\right) d\omega$$ $\left\{ \begin{array}{l} \text{At the operating point} \\ \text{we have } \sin\left(\frac{\Delta T}{\hbar}\right) = 1 \end{array} \right.$

$$d\rho_{22} = \frac{T}{2} d\omega.$$ $\longrightarrow$ (This relates the change in frequency to the change in the upper state occupancy.

$$\Rightarrow \Delta\omega = \frac{2}{T}\left[N_2^{all} - 0.5\right].$$

i) we have: $\Delta\omega[n+1] - (1-r)\Delta\omega[n] = -r F[n]$ — ①

letting $n \to n+p$: $\Delta\omega[n+1+p] - (1-r)\Delta\omega[n+p] = -r F[n+p]$ — ②

Multiply ① by ② and sum over 'p' from $-\infty$ to $+\infty$: since

$\sum_{p=-\infty}^{+\infty} \langle \Delta\omega[n]\Delta\omega[n+p]\rangle$ does not depend on 'n', the

above procedure gives (after some relabeling of indices):

$$\sum_{p=-\infty}^{\infty}\langle \Delta\omega[n]\Delta\omega[n+p]\rangle\left\{1 - 2(1-r) + (1-r)^2\right\} = r^2 \sum_{p=-\infty}^{\infty} \langle F[n]F[n+p]\rangle$$

$$\Rightarrow \left\langle \sum_{p=-\infty}^{\infty} \Delta\omega[n]\Delta\omega[n+p]\right\rangle = \frac{4}{T^2}\langle (\Delta\hat{N}_2^{all})^2\rangle = \sum_{p=-\infty}^{\infty}\langle \Delta\omega[n]\Delta\omega[p]\rangle$$

j) The thing to note is that $\langle \Delta\omega[n]\Delta\omega[n+p]\rangle$ would quickly go to zero as $\sim (1-r)^{|p|}$ as p increases. This can be seen directly from the equation $\Delta\omega[n+1] = (1-r)\Delta\omega[n] - rF[n]$ assuming $F[n]=0$ for all 'n'. Therefore,

$$\langle \Delta\Omega^2\rangle = \left\langle \frac{1}{N^2}\sum_{n=-N/2}^{N/2} \Delta\omega[n] \sum_{p=-N/2}^{N/2} \Delta\omega[p]\right\rangle$$

$$\simeq \frac{N}{N^2}\sum_{p=-N/2}^{N/2}\langle \Delta\omega[n]\Delta\omega[p]\rangle = \frac{4}{T^2 N}\langle (\Delta\hat{N}_2^{all})^2\rangle = \frac{4}{T^2 N N_a 4}$$

$$= \frac{1}{T^2 N N_a}.$$

k) $$\sigma = \sqrt{\frac{\langle \Delta\Omega^2\rangle}{\omega_0^2}} = \sqrt{\frac{1}{\omega_0^2 T^2 N N_a}} = \frac{1}{\omega_0 T}\sqrt{\frac{T_c}{\tau}}\sqrt{\frac{1}{N_a}}.$$

l) $\sigma \simeq 10^{-15}$