(Home Work #2 Solutions) Farhan Rana

2.1 $\boxed{U_K = \frac{\hbar\Omega_{RK}}{2}}$

a) $|\psi(t)\rangle = C_0(t)|e_0\rangle + \sum_k C_k(t)\, |e_k\rangle\langle e_k|$

Plug in $i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$ to get

$$i\hbar \frac{dC_0(t)}{dt}|e_0\rangle + \sum_k i\hbar \frac{d}{dt}C_k(t)|e_k\rangle = \left\{C_0(t)\,\varepsilon_0 - \sum_k U_k e^{i\omega t} C_k(t)\right\}|e_0\rangle$$
$$+ \sum_k \left\{\varepsilon_k - U_k e^{-i\omega t} C_0(t)\right\}|e_k\rangle$$

Take the inner product with first $\langle e_0|$ and then $\langle e_k|$ to get the desired results.

b) $C_0(t) = b_0(t)\, e^{-i\frac{\varepsilon_0 t}{\hbar}}$ $\quad C_k(t) = b_k(t)\, e^{-i\frac{\varepsilon_k t}{\hbar}}$

The idea behind this substitution is to define $b_0(t)$ and $b_k(t)$ such that all the fast time dependence of $C_0(t)$ and $C_k(t)$ is explicitly factored out. The procedure is simple and so will not be repeated here.

c) We have:

$$\frac{db_0(t)}{dt} = \sum_k i\frac{U_k}{\hbar}\, e^{-i\left(\frac{\varepsilon_k - \varepsilon_0 - \hbar\omega}{\hbar}\right)t}\, b_k(t) \quad \text{—— (1)}$$

$$\frac{db_k(t)}{dt} = i\frac{U_k}{\hbar}\, e^{i\left(\frac{\varepsilon_k - \varepsilon_0 - \hbar\omega}{\hbar}\right)t}\, b_0(t) \quad \text{—— (2)}$$

Solve (2) first $\{$initial condition $b_k(t=0) = 0\}$

$$b_k(t) = i\frac{U_k}{\hbar}\int_0^t dt' \left\{ e^{i\left(\frac{\varepsilon_k - \varepsilon_0 - \hbar\omega}{\hbar}\right)t'}\, b_0(t') \right\}$$

and plug in (1)

$$\frac{db_0(t)}{dt} = -\sum_k \frac{U_k^2}{\hbar^2}\int_0^t dt' \left\{ e^{-i\frac{(\varepsilon_k - \varepsilon_0 - \hbar\omega)}{\hbar}(t-t')}\, b_0(t') \right\} \quad \text{—— (3)}$$

Note that if the k dependence of U_k^2 is not strong, the summation over k will give something close to a delta function in $(t-t')$ $\{$i.e $\delta(t-t')\}$. What this means is that the final result depends on values of $b_0(t')$ close to $b_0(t)$. Therefore, we can let $b_0(t') \approx b_0(t)$ and remove it from the integral.

$$\frac{db_0(t)}{dt} = -\sum_k \frac{U_k^2}{\hbar^2} \int_0^t dt' \left\{ e^{-i\left(\frac{\varepsilon_k-\varepsilon_0-\hbar\omega}{\hbar}\right)(t-t')} \right\} b_0(t).$$

$$\Rightarrow \frac{d|b_0(t)|^2}{dt} = -2\,\text{Real}\left\{ \sum_k \frac{U_k^2}{\hbar^2} \int_0^t dt' \left\{ e^{-i\left(\frac{\varepsilon_k-\varepsilon_0-\hbar\omega}{\hbar}\right)(t-t')} \right\} \right\} |b_0(t)|^2$$

$$2\,\text{Real}\left\{ \sum_k \frac{U_k^2}{\hbar^2} \int_0^t dt' \; e^{-i\left(\frac{\varepsilon_k-\varepsilon_0-\hbar\omega}{\hbar}\right)(t-t')} \right\}$$

$$= 2 \sum_k \frac{U_k^2}{\hbar^2} \frac{\sin\left(\frac{\varepsilon_k-\varepsilon_0-\hbar\omega}{\hbar}\right)t}{\left(\frac{\varepsilon_k-\varepsilon_0-\hbar\omega}{\hbar}\right)}.$$

Consider the function $\frac{\sin xt}{x}$. Its width is $\approx \frac{2\pi}{t}$ in x-space.

Note that $\int_{-\infty}^{\infty} dx \frac{\sin xt}{x} = \pi$ and $\left.\frac{\sin xt}{x}\right|_{x=0} = t$

As t becomes large (large compared to the relative energy level spacings $\Delta\varepsilon_k$ of the upper states i.e $\frac{\Delta\varepsilon_k t}{\hbar} \gg 2\pi$
This is because the width of the $\frac{\sin\left(\frac{\varepsilon_k-\varepsilon_0-\hbar\omega}{\hbar}\right)t}{\left(\frac{\varepsilon_k-\varepsilon_0-\hbar\omega}{\hbar}\right)}$ function in the $(\varepsilon_k-\varepsilon_0-\hbar\omega)$ space is $\frac{2\pi\hbar}{t}$) then we can replace

$$\frac{\sin\left(\frac{\varepsilon_k-\varepsilon_0-\hbar\omega}{\hbar}\right)t}{\left(\frac{\varepsilon_k-\varepsilon_0-\hbar\omega}{\hbar}\right)} \text{ by } \pi\,\delta\left(\frac{\varepsilon_k-\varepsilon_0-\hbar\omega}{\hbar}\right) = \pi\hbar\,\delta(\varepsilon_k-\varepsilon_0-\hbar\omega).$$

So then we get:

$$\frac{d|b_0(t)|^2}{dt} = -\gamma\, |b_0(t)|^2$$

where $\gamma = \frac{2\pi}{\hbar} \sum_k U_k^2\, \delta(\varepsilon_k - \varepsilon_0 - \hbar\omega)$

2.2 $\boxed{U = \frac{\hbar\Omega_R}{2}}$

a) $\hat{H}(t) = \varepsilon_1 \hat{N}_1 + \varepsilon_2 \hat{N}_2 - U\left[e^{-i\omega t}\hat{\sigma}_+ + e^{i\omega t}\hat{\sigma}_-\right]$

In the Hiesenberg picture:

$$\hat{H}(t) = \varepsilon_1 \hat{N}_1(t) + \varepsilon_2 \hat{N}_2(t) - U\left[e^{-i\omega t}\hat{\sigma}_+(t) + e^{i\omega t}\hat{\sigma}_-(t)\right]$$

$$i\hbar \frac{d\hat{N}_1(t)}{dt} = \left[\hat{N}_1(t), \hat{H}(t)\right]$$

$$= -U e^{-i\omega t}\left[\hat{N}_1(t), \hat{\sigma}_+(t)\right] - U e^{i\omega t}\left[\hat{N}_1(t), \hat{\sigma}_-(t)\right]$$

$$= U e^{-i\omega t}\hat{\sigma}_+(t) - U e^{i\omega t}\hat{\sigma}_-(t)$$

$$i\hbar \frac{d\hat{\sigma}_+(t)}{dt} = \left[\hat{\sigma}_+(t), \hat{H}(t)\right]$$

$$= \varepsilon_1\left[\hat{\sigma}_+(t), \hat{N}_1(t)\right] + \varepsilon_2\left[\hat{\sigma}_+(t), \hat{N}_2(t)\right] - U e^{i\omega t}\left[\hat{\sigma}_+(t), \hat{\sigma}_-(t)\right]$$

$$= -(\varepsilon_2 - \varepsilon_1)\hat{\sigma}_+(t) - U e^{i\omega t}\left[\hat{N}_2(t) - \hat{N}_1(t)\right]$$

Also, $\left(i\hbar \frac{d\hat{\sigma}_+(t)}{dt}\right)^\dagger = i\hbar \frac{d\hat{\sigma}_-(t)}{dt}$ since $\left(\hat{\sigma}_+(t)\right)^\dagger = \hat{\sigma}_-(t)$.

and $\frac{d\hat{N}_2(t)}{dt} = -\frac{d\hat{N}_1(t)}{dt}$.

b) This is pretty straight forward so I will skip it.

c) We have:

$$\frac{d\hat{N}_1(t)}{dt} = -i\frac{U}{\hbar}\left[\hat{\sigma}_{++}(t) - \hat{\sigma}_{--}(t)\right]$$

$$\frac{d\hat{N}_2(t)}{dt} = i\frac{U}{\hbar}\left[\hat{\sigma}_{++}(t) - \hat{\sigma}_{--}(t)\right]$$

If $\Delta\hat{N}(t) = \hat{N}_2(t) - \hat{N}_1(t)$ then

$$\left.\begin{aligned} \frac{d\Delta\hat{N}(t)}{dt} &= i\frac{2U}{\hbar}\Delta\hat{\sigma}(t) \\ \text{Also } \frac{d\Delta\hat{\sigma}(t)}{dt} &= i\frac{2U}{\hbar}\Delta\hat{N}(t) \end{aligned}\right\} \quad \text{where } \Delta\hat{\sigma}(t) = \hat{\sigma}_{++}(t) - \hat{\sigma}_{--}(t)$$

$\Delta\hat{N}(t)$ and $\Delta\hat{\sigma}(t)$ are the coupled quantities. Diff. the equation for $\Delta\hat{N}(t)$ and use the equation for $\Delta\hat{\sigma}(t)$ to get:

$$\frac{d^2}{dt^2}\Delta\hat{N}(t) = -\left(\frac{2U}{\hbar}\right)^2\Delta\hat{N}(t) = -\Omega_R^2\,\Delta\hat{N}(t).$$

$$\Rightarrow \Delta\hat{N}(t) = \hat{A}\cos(\Omega_R t) + \hat{B}\sin\Omega_R t$$

Boundary Conditions:

$$\Delta\hat{N}(t=0) = \hat{A} = \hat{N}_2 - \hat{N}_1 = \Delta\hat{N}$$

$$\left.\frac{d\Delta\hat{N}(t)}{dt}\right|_{t=0} = \Omega_R\hat{B} = i\frac{2U}{\hbar}\Delta\hat{\sigma}(t=0) = i\Omega_R(\hat{\sigma}_+ - \hat{\sigma}_-)$$

$$\Rightarrow \Delta\hat{N}(t) = \hat{N}_2(t) - \hat{N}_1(t) = \Delta\hat{N}\cos(\Omega_R t) + i(\hat{\sigma}_+ - \hat{\sigma}_-)$$

Also. $\hat{N}_2(t) + \hat{N}_1(t) = \hat{N}_2 + \hat{N}_1$ since $\frac{d}{dt}\left(\hat{N}_2(t) + \hat{N}_1(t)\right) = 0$

$$\hat{N}_1(t) = \frac{(\hat{N}_2(t) + \hat{N}_1(t)) - \Delta\hat{N}(t)}{2} = \frac{(\hat{N}_2 + \hat{N}_1) - \Delta\hat{N}(t)}{2}$$

$$= \hat{N}_1 \cos^2\left(\frac{\Omega_R t}{2}\right) + \hat{N}_2 \sin^2\left(\frac{\Omega_R t}{2}\right) - \frac{i}{2}(\hat{\sigma}_+ - \hat{\sigma}_-) \sin(\Omega_R t).$$

d). $\hat{\rho}(t=0) = |e_2\rangle\langle e_2| = \hat{N}_2$.

Note that $\hat{N}_1 \cdot \hat{N}_2 = 0 \qquad \hat{N}_2 \cdot \hat{N}_2 = \hat{N}_2 \qquad \hat{N}_2 \cdot \hat{\sigma}_- = 0$

and $\hat{N}_2 \cdot \hat{\sigma}_+ = \hat{\sigma}_+$

$$\Rightarrow \mathrm{Tr}\{\hat{\rho}(t=0)\,\hat{N}_1(t)\} = \mathrm{Tr}\{\hat{N}_2 \cdot \hat{N}_1(t)\} = \mathrm{Tr}\left\{\hat{N}_2 \sin^2\left(\frac{\Omega_R t}{2}\right) - i\frac{\hat{\sigma}_+}{2}\sin(\Omega_R t)\right\}$$

$$= \sin^2\left(\frac{\Omega_R t}{2}\right).$$

2.3 $\boxed{U = \frac{\hbar\Omega_R}{2}}$

a) Choose a vector perpendicular to $\vec{\Omega}$ (and therefore in the plane of rotation) that is in the x-z plane. This vector should have unit magnitude. I choose it to be $\dfrac{\frac{\Delta}{\hbar}\hat{x} + \Omega_R\hat{z}}{\sqrt{\Omega_R^2 + (\frac{\Delta}{\hbar})^2}} = \hat{n}$

then $\cos\theta = \hat{z}\cdot\hat{n} = \dfrac{\Omega_R}{\sqrt{\Omega_R^2 + (\frac{\Delta}{\hbar})^2}} = \dfrac{\Omega_R}{|\vec{\Omega}|}$

b) The population difference is the projection of $\vec{V}(t)$ on the z-axis. The max. value of this projection is obrious from the figure to be $\cos(2\theta)$.

$$\cos 2\theta = 2\cos^2\theta - 1 = 2\frac{\Omega_R^2}{|\vec{\Omega}|^2} - 1 = \frac{\Omega_R^2 - (\Delta/\hbar)^2}{|\vec{\Omega}|^2}$$

From Lecture notes:

$$|\langle e_1|\psi(t)\rangle|^2 = \rho_{11}(t) = \frac{1}{2}\left[1 + \frac{(\Delta/\hbar)^2}{|\vec{\Omega}|^2} + \frac{(2U/\hbar)^2}{|\vec{\Omega}|^2}\cos|\vec{\Omega}|t\right]$$

and $\rho_{22}(t) = 1 - \rho_{11}(t)$

So $\rho_{22}(t) - \rho_{11}(t) = 1 - 2\rho_{11}(t)$

$$= -\frac{(\Delta/\hbar)^2}{|\vec{\Omega}|^2} - \frac{(2U/\hbar)^2}{|\vec{\Omega}|^2}\cos|\vec{\Omega}|t$$

$$\rho_{22}(t) - \rho_{11}(t)\Big|_{max} = \frac{(2U/\hbar)^2 - (\Delta/\hbar)^2}{|\vec{\Omega}|^2} = \frac{\Omega_R^2 - (\Delta/\hbar)^2}{|\vec{\Omega}|^2}$$

So they agree.

c) First take

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}\left[|e_1\rangle + i|e_2\rangle\right]$$

$$\hat{\rho}(t=0) = |\psi(t=0)\rangle\langle\psi(t=0)| = \frac{1}{2}\left\{|e_1\rangle\langle e_1| + |e_2\rangle\langle e_2| - i|e_1\rangle\langle e_2| + i|e_2\rangle\langle e_1|\right\}$$

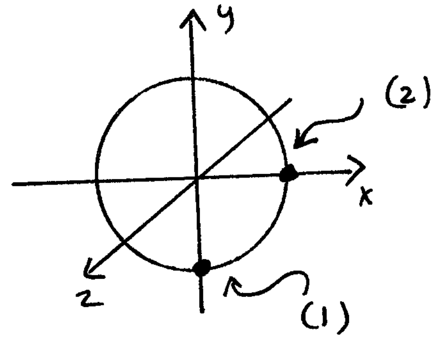
$$= \begin{bmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{bmatrix}$$

$\Rightarrow$

$V_x(t=0) = 0$

$V_y(t=0) = -1$

$V_z(t=0) = 0$



Then take:

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}\left[\, |e_1\rangle + |e_2\rangle \,\right]$$

$$\hat{\rho}(t=0) = |\psi(t=0)\rangle\langle\psi(t=0)| = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$V_x(t=0) = 1$

$V_y(t=0) = 0$

$V_z(t=0) = 0$

d) For initial state (1):

axis of rotation: the x-axis

plane of rotatation: the y-z plane at $x=0$

direction of rotation: Clockwise when looking at the y-z plane from positive x-axis.

For initial state (2):

The state vector $\vec{V}(t)$ does not change with time!