Department of Electrical and Computer Engineering, Cornell University

ECE 5310: Quantum Optics for Photonics and Optoelectronics

Fall 2013

Homework 2 **Due on Sep. 17, 2013 (self-grade)**

OPTIONAL READING: A good (short but nice) description of density operators can be found in compliment E_{III} of chapter 3 of "Quantum Mechanics: Volume I" by Claude Cohen-Tanoudji.

Problem 2.1: (Fermi's Golden Rule for Time-Dependent Problems)

In the class we saw that in systems time dependent behavior can take the form of periodic oscillations or just a one-shot decay (or a one-shot transition) described by Fermi's Golden Rule. And which of the above happens depends upon whether the final possible state(s) is a single state or a large number of closely spaced states (i.e. a continuum of states). In this problem you will derive the Fermi's Golden Rule for transitions caused by electromagnetic radiation.

In the last homework set you saw that the time *dependent* Hamiltonian that described the interaction between a two level system and a time dependent electric filed is,

$$\hat{H}(t) = \varepsilon_1 |e_1\rangle\langle e_1| + \varepsilon_2 |e_2\rangle\langle e_2| - \frac{\hbar\Omega_R}{2}\left[\exp(i\omega t)|e_1\rangle\langle e_2| + \exp(-i\omega t)|e_2\rangle\langle e_1|\right]$$

where the parameter $\hbar\Omega_R$ is related to the electric field strength.

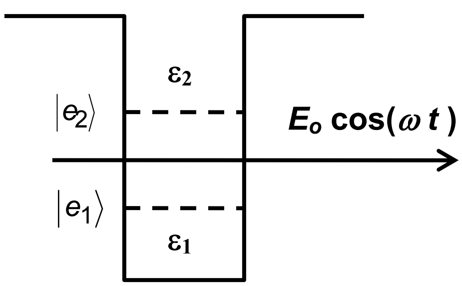


Fig. p2.1a: A driven two level system

Here the problem is changed a little. The upper state $|e_2\rangle$ is replaced by a large collection of states $|e_k\rangle$, with energies ε_k, that form a continuum characterized with a density of states $D(\varepsilon)$ (i.e. the number of states within an energy interval $d\varepsilon$ is $D(\varepsilon)\,d\varepsilon$) as shown in the figure below.

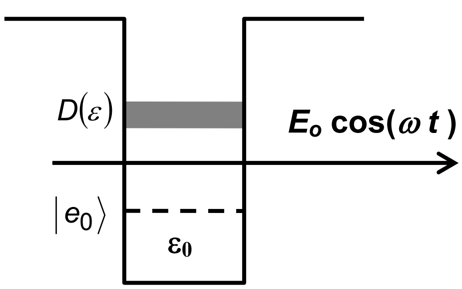


Fig. p2.1b: The upper level is replaced with a continuum of states.

The Hamiltonian of the problem becomes,

$$\hat{H}(t) = \varepsilon_0 |e_0\rangle\langle e_0| \; + \; \sum_k \varepsilon_k |e_k\rangle\langle e_k| \; - \; \sum_k \frac{\hbar\Omega_{Rk}}{2}\left[\exp(i\omega t)|e_0\rangle\langle e_k| \; + \; \exp(-i\omega t)|e_k\rangle\langle e_0|\right]$$

Suppose the initial quantum state of the particle at $t = 0$ is,

$$|\psi(t=0)\rangle = |e_0\rangle$$

The state of the particle at time t can be written as,

$$|\psi(t)\rangle = c_0(t)|e_0\rangle + \sum_k c_k(t)|e_k\rangle$$

a) Following the method used in the problem of particle decay from a well given in the lecture notes show that the probability $P(t)$ that the particle is in the initial state decays approximately as,

$$\frac{d\,P(t)}{dt} \approx -\gamma\, P(t)$$

where the "transition rate" γ from the lower state to the upper states is given by the Fermi's Golden Rule,

$$\gamma = \frac{2\pi}{\hbar}\sum_k \left(\frac{\hbar\Omega_{Rk}}{2}\right)^2 \delta(\varepsilon_k - \varepsilon_0 - \hbar\omega) = \frac{2\pi}{\hbar}\int_{-\infty}^{\infty} d\varepsilon \left(\frac{\hbar\Omega_R(\varepsilon)}{2}\right)^2 D(\varepsilon)\,\delta(\varepsilon - \varepsilon_0 - \hbar\omega)$$

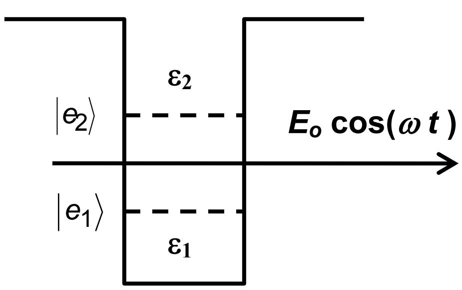
$$= \frac{2\pi}{\hbar}\left[\frac{\hbar\Omega_R(\varepsilon_0 + \hbar\omega)}{2}\right]^2 D(\varepsilon_0 + \hbar\omega)$$

Problem 2.2: (Driven Two-Level System in the Heisenberg Picture)

The object of this problem is to demonstrate the link between time *dependent* and time *independent* problems in the Heisenberg formalism. You did this in the Shrodinger formalism in the last homework.

Consider again the time *dependent* Hamiltonian of a driven two level system,

$$\hat{H}(t) = \varepsilon_1 |e_1\rangle\langle e_1| \; + \; \varepsilon_2 |e_2\rangle\langle e_2| \; - \; \frac{\hbar\Omega_R}{2}\left[\exp(i\omega t)|e_1\rangle\langle e_2| \; + \; \exp(-i\omega t)|e_2\rangle\langle e_1|\right]$$



Define the following Shrodinger operators as in the lecture notes,

$$\hat{N}_1 = |e_1\rangle\langle e_1| \qquad \hat{N}_2 = |e_2\rangle\langle e_2| \qquad \hat{\sigma}_+ = |e_2\rangle\langle e_1| \qquad \hat{\sigma}_- = |e_1\rangle\langle e_2|$$

So the Hamiltonian becomes,

$$\hat{H}(t) = \varepsilon_1\hat{N}_1 + \varepsilon_2\hat{N}_2 - \frac{\hbar\Omega_R}{2}\left[\hat{\sigma}_+ \exp(-i\omega t) + \hat{\sigma}_- \exp(i\omega t)\right]$$

Carefully study all the commutation relations given in the lecture notes among these operators.

a) Derive the Heisenberg equations for the Heisenberg operators $\hat{N}_1(t),\quad \hat{N}_2(t),\quad \hat{\sigma}_+(t),\quad \hat{\sigma}_-(t)$. Hint: you need to derive just the equation for $\hat{N}_1(t)$ and $\hat{\sigma}_+(t)$, and recognize that $\frac{d\,\hat{N}_2(t)}{dt} = -\frac{d\,\hat{N}_1(t)}{dt}$, and that the equation for $\hat{\sigma}_-(t)$ is just the adjoint of the equation for $\hat{\sigma}_+(t)$. Your answer should be:

$$\frac{d\,\hat{N}_1(t)}{dt} = -i\,\frac{\Omega_R}{2}\left[\hat{\sigma}_+(t)\exp(-i\omega\,t) - \hat{\sigma}_-(t)\exp(i\omega\,t)\right]$$

$$\frac{d\,\hat{N}_2(t)}{dt} = i\,\frac{\Omega_R}{2}\left[\hat{\sigma}_+(t)\exp(-i\omega\,t) - \hat{\sigma}_-(t)\exp(i\omega\,t)\right]$$

$$\frac{d\,\hat{\sigma}_+(t)}{dt} = i\,\frac{(\varepsilon_2 - \varepsilon_1)}{\hbar}\,\hat{\sigma}_+(t) + i\,\frac{\Omega_R}{2}\exp(i\omega\,t)\left[\hat{N}_2(t) - \hat{N}_1(t)\right]$$

$$\frac{d\,\hat{\sigma}_-(t)}{dt} = -i\,\frac{(\varepsilon_2 - \varepsilon_1)}{\hbar}\,\hat{\sigma}_-(t) - i\,\frac{\Omega_R}{2}\exp(-i\omega\,t)\left[\hat{N}_2(t) - \hat{N}_1(t)\right]$$

b) Assume **no detuning**, i.e. $\Delta = \varepsilon_2 - (\varepsilon_1 + \hbar\omega) = 0$ (the frequency of the electromagnetic field is exactly the same as the energy level separation). Define new quantities $\hat{\sigma}_{++}(t),\ \hat{\sigma}_{--}(t)$ as follows,

$$\hat{\sigma}_{++}(t) = \hat{\sigma}_+(t)\exp(-i\omega t) \qquad\qquad \hat{\sigma}_{--}(t) = \hat{\sigma}_-(t)\exp(i\omega t)$$

and show that the Heisenberg equations of part (a) then look like those of a time *independent* problem that we discussed in the lecture notes. Your answer should be:

$$\frac{d\,\hat{N}_1(t)}{dt} = -i\,\frac{\Omega_R}{2}\left[\hat{\sigma}_{++}(t) - \hat{\sigma}_{--}(t)\right]$$

$$\frac{d\,\hat{N}_2(t)}{dt} = i\,\frac{\Omega_R}{2}\left[\hat{\sigma}_{++}(t) - \hat{\sigma}_{--}(t)\right]$$

$$\frac{d\,\hat{\sigma}_{++}(t)}{dt} = i\,\frac{\Omega_R}{2}\left[\hat{N}_2(t) - \hat{N}_1(t)\right]$$

$$\frac{d\,\hat{\sigma}_{--}(t)}{dt} = -i\,\frac{\Omega_R}{2}\left[\hat{N}_2(t) - \hat{N}_1(t)\right]$$

c) From the lecture notes, confirm that the solution of the above system results in the following expression for $\hat{N}_1(t)$,

$$\hat{N}_1(t) = \hat{N}_1\cos^2\left(\frac{\Omega_R}{2}t\right) + \hat{N}_2\sin^2\left(\frac{\Omega_R}{2}t\right) - \frac{i}{2}(\hat{\sigma}_+ - \hat{\sigma}_-)\sin(\Omega_R t)$$

Hint: note the following when imposing the boundary conditions:

$$\hat{\sigma}_{++}(t=0) = \hat{\sigma}_+(t=0) = \hat{\sigma}_+ \qquad\qquad \hat{\sigma}_{--}(t=0) = \hat{\sigma}_-(t=0) = \hat{\sigma}_-$$

d) Calculate the average value of $\hat{N}_1(t)$ given that the initial state of the system is described by the density matrix $\hat{\rho}(t=0) = |e_2\rangle\langle e_2| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Hint: Use the formula: $\langle \hat{N}_1 \rangle(t) = \mathrm{Tr}\{\hat{\rho}(t=0)\,\hat{N}_1(t)\}$.

Problem 2.3: (A Bloch Sphere Problem)

In parts (a)-(b) you have to graphically calculate the maximum population difference when the detuning Δ is non-zero using the Bloch sphere picture. If you find yourself doing too much computation, you are off track.

a) The plane of rotation of the vector $\vec{V}(t)$ is perpendicular to the direction given by $\vec{\Omega} = -\Omega_R\,\hat{x} + \frac{\Delta}{\hbar}\hat{z}$. Find the angle θ between the plane of rotation and the z-axis for positive values of detuning Δ (see Fig. p2.4d below).

b) Show that the maximum population difference is given by $\cos(2\theta)$. Compute its value and compare with the answer in the lecture notes.

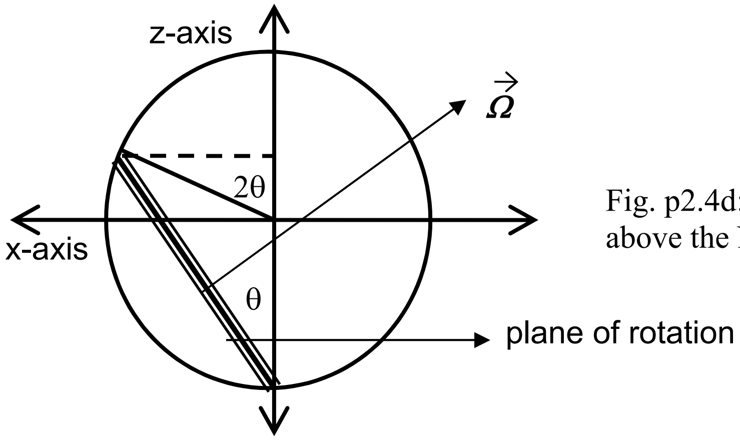


Fig. p2.4d: Bloch sphere seen from above the North pole.

Parts (c)-(d) are together

Consider two possible initial quantum states:

$$(1)\ |\psi(t=0)\rangle = \frac{1}{\sqrt{2}}\left[|e_1\rangle + i|e_2\rangle\right]$$

$$(2)\ |\psi(t=0)\rangle = \frac{1}{\sqrt{2}}\left[|e_1\rangle + |e_2\rangle\right]$$

For each quantum state shown above do the following:

c) Write the density operator at time $t=0$ in matrix form. Find the three components of the vector $\vec{V}(t=0)$? Indicate the initial state (given by $\vec{V}(t=0)$) as a black dot on a neatly labeled 3-dimensional Bloch Sphere.

d) At time $t=0$, a time dependent electric field with **zero detuning** is turned on that interacts with the system. Describe the motion of the state vector $\vec{V}(t)$ for $t \geq 0$ (i.e. indicate the axis of rotation and the plane of rotation and the direction of rotation). Does something strange happen when the initial state is (2)?

SOMETHING TO PONDER UPON: Recall that in lecture notes you showed that the time *dependent* problem given by the Hamiltonian,

$$\hat{H}(t) = \varepsilon_1|e_1\rangle\langle e_1| + \varepsilon_2|e_2\rangle\langle e_2| - \frac{\hbar\Omega_R}{2}\left[\exp(i\omega t)|e_1\rangle\langle e_2| + \exp(-i\omega t)|e_2\rangle\langle e_1|\right]$$

can be transformed into a time *independent* problem described by the Hamiltonian,

$$\hat{H}_R = (\varepsilon_1 + \hbar\omega)|e_1\rangle\langle e_1| \; + \; \varepsilon_2|e_2\rangle\langle e_2| \; - \; \frac{\hbar\Omega_R}{2}\left[|e_1\rangle\langle e_2| \; + \; |e_2\rangle\langle e_1|\right]$$

In case of no detuning (i.e. when $\Delta = \varepsilon_2 - (\varepsilon_1 + \hbar\omega) = 0$) one eigenstate of the Hamiltonian $\hat{H}_R$ is given by the state (2) above. Do you think this has anything to do with the dynamics you observed in part (d) when the initial state is (2)? What if the initial state was the other eigenstate of $\hat{H}_R$?