Homework #1 Solutions
(Farhan Rana)

1.1

a) Consider $[\hat{A},\hat{B}^2] = [\hat{A},\hat{B}]\hat{B} + \hat{B}[\hat{A},\hat{B}] = i\hbar 2\hat{B}$. Suppose $[\hat{A},\hat{B}^n] = ni\hbar\hat{B}^{n-1}$. Then $[\hat{A},\hat{B}^{n+1}] = [\hat{A},\hat{B}^n]\hat{B} + \hat{B}^n[\hat{A},\hat{B}] = ni\hbar\hat{B}^n + i\hbar\hat{B}^n$
$= (n+1)i\hbar\hat{B}^n$ $\Rightarrow$ holds for $n=1$ and the assumption for being true for n implies truth for $n+1$ $\Rightarrow$ holds for all n (by induction).

b) Suppose $F(\hat{B}) = \sum_{n=0}^{N} c_n \hat{B}^n \Rightarrow [\hat{A},\hat{F}(B)] = \sum_{n=1}^{N} i\hbar n c_n \hat{B}^{n-1}$
$= \hat{F}'(B)$

c) First notice that for any complex number a:

$$e^{a\hat{A}}\hat{B} = \hat{B}e^{a\hat{A}} + i\hbar a\, e^{a\hat{A}}$$

by (b) above. Then consider $e^{a[\hat{A}+\hat{B}]}$ and suppose it equals $f(a)\,e^{a\hat{A}}e^{a\hat{B}}$. We need to find $f(a)$.

$$e^{a[\hat{A}+\hat{B}]} = f(a)\,e^{a\hat{A}}e^{a\hat{B}}.$$

differentiating w.r.t. a we get

$$(\hat{A}+\hat{B})\,e^{a[\hat{A}+\hat{B}]} = f'(a)\,e^{a\hat{A}}e^{a\hat{B}} + f(a)\hat{A}\,e^{a\hat{A}}e^{a\hat{B}} + f(a)\,e^{a\hat{A}}e^{a\hat{B}}\hat{B}$$
$$= f'(a)\,e^{a\hat{A}}e^{a\hat{B}} + f(a)(\hat{A}+\hat{B})\,e^{a\hat{A}}e^{a\hat{B}} + i\hbar a f(a)\,e^{a\hat{A}}e^{a\hat{B}}$$

$$\Rightarrow f'(a) + i\hbar a f(a) = 0 \xrightarrow{\text{Solution}} f(a) = f(a=0)\,e^{-i\hbar\frac{a^2}{2}}$$

But $f(a=0)=1$ if $e^{a(\hat{A}+\hat{B})}$ is to equal $f(a)e^{a\hat{A}}e^{a\hat{B}}$ for $a=0$. So $f(a) = e^{-i\hbar\frac{a^2}{2}}$. Now let $a=1$ and get the result: $e^{(\hat{A}+\hat{B})} = e^{-\frac{i\hbar}{2}} e^{\hat{A}} e^{\hat{B}}$.

d) Notice that, as shown earlier, $e^{a\hat{A}}\hat{B} = (\hat{B}+i\hbar a)e^{a\hat{A}}$

$\Rightarrow e^{\hat{A}}\hat{B}^n = (\hat{B}+i\hbar)^n e^{\hat{A}} \Rightarrow e^{\hat{A}}e^{\hat{B}} = e^{(\hat{B}+i\hbar)}e^{\hat{A}}$

$\Rightarrow e^{-i\frac{\hbar}{2}} e^{\hat{A}} e^{\hat{B}} = e^{i\frac{\hbar}{2}} e^{\hat{B}} e^{\hat{A}} \Rightarrow e^{\hat{A}+\hat{B}} = e^{i\frac{\hbar}{2}} e^{\hat{B}} e^{\hat{A}}$

e) $e^{-\frac{i}{\hbar}\alpha\hat{A}}\,\hat{B}\,e^{\frac{i}{\hbar}\alpha\hat{A}} = \left(\hat{B} + i\hbar \times \frac{-i}{\hbar}\alpha\right) e^{\frac{-i\alpha}{\hbar}\hat{A}} e^{i/\hbar\,\alpha\hat{A}} = \hat{B}+\alpha$

1.2

a) Prob. of obtaining $\lambda_j = |\langle a_j|\psi\rangle|^2 = |\alpha_j|^2$. State vector post measurement $= |a_j\rangle$

b) Prob. of obtaining $\eta_j = |\langle b_j|\psi\rangle|^2 = \left|\sum_k \alpha_k X^*_{kj}\right|^2$

State vector post measurement $= |b_j\rangle$

c) Prob. of obtaining $\lambda_j = |\langle a_j|b_j\rangle|^2 = |X_{jj}|^2$

d) $|\psi\rangle = \sum_k \alpha_k |a_k\rangle \Rightarrow$ prob. of obtaining λ is $|\alpha_1|^2+|\alpha_2|^2+|\alpha_3|^2$

(i.e. the prob. of obtaining λ is the prob. that $|\psi\rangle$ is in the eigen-subspace associated with λ).

State vector post measurement $= \dfrac{\alpha_1|a_1\rangle+\alpha_2|a_2\rangle+\alpha_3|a_3\rangle}{\sqrt{\alpha_1^2+\alpha_2^2+\alpha_3^2}}$

This is the projection of $|\psi\rangle$ onto the eigen-subspace associated with λ.

e) $|\psi\rangle = \sum_k \alpha_k |a_k\rangle = \sum_k \alpha_k \mathbb{1} |a_k\rangle = \sum_{k,j} \alpha_k |b_j\rangle\langle b_j|a_k\rangle$

$= \sum_j \left(\sum_k \alpha_k X^*_{kj}\right)|b_j\rangle = \sum_j \beta_j |b_j\rangle$

i.e first expand $|\psi\rangle$ in eigenfunctions of $\hat{B}$

So the prob of measuring η is $|\beta_1|^2 + |\beta_2|^2 + |\beta_3|^2$

$$= \left|\sum_k \alpha_k x_{k1}^*\right|^2 + \left|\sum_k \alpha_k x_{k2}^*\right|^2 + \left|\sum_k \alpha_k x_{k3}^*\right|^2$$

wave function post measurement $= \dfrac{\beta_1 |b_1\rangle + \beta_2 |b_2\rangle + \beta_3 |b_3\rangle}{\sqrt{|\beta_1|^2 + |\beta_2|^2 + |\beta_3|^2}}$

1.3 $\boxed{\text{Let } U = \dfrac{\hbar\Omega_R}{2}}$

a) $\exp(-i\omega|e_1\rangle\langle e_1|t) = \sum_{n=0}^{\infty} \dfrac{(-i\omega|e_1\rangle\langle e_1|t)^n}{n!}$

Note that $(|e_1\rangle\langle e_1|)^2 = |e_1\rangle\langle e_1|e_1\rangle\langle e_1| = |e_1\rangle\langle e_1|$

$(|e_1\rangle\langle e_1|)^3 = |e_1\rangle\langle e_1|e_1\rangle\langle e_1|e_1\rangle\langle e_1| = |e_1\rangle\langle e_1|$

$(|e_1\rangle\langle e_1|)^n = |e_1\rangle\langle e_1|$ but $(|e_1\rangle\langle e_1|)^0 = 1$

$$\exp(-i\omega|e_1\rangle\langle e_1|t) = 1 + \sum_{n=1}^{\infty} \frac{(-i\omega|e_1\rangle\langle e_1|t)^n}{n!} = 1 + |e_1\rangle\langle e_1| \sum_{n=1}^{\infty} \frac{(-i\omega t)^n}{n!}$$

$$= 1 + |e_1\rangle\langle e_1|\left(e^{-i\omega t} - 1\right)$$

b) $|\phi(t)\rangle = \exp[-i\omega|e_1\rangle\langle e_1|t]\,|\psi(t)\rangle$

$$i\hbar\frac{\partial}{\partial t}|\phi(t)\rangle = \hbar\omega|e_1\rangle\langle e_1| \exp[-i\omega|e_1\rangle\langle e_1|t]\,|\psi(t)\rangle$$

$$+ \exp[-i\omega|e_1\rangle\langle e_1|t]\, i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle$$

$$= \hbar\omega|e_1\rangle\langle e_1|\,|\phi(t)\rangle$$

$$+ \exp[-i\omega|e_1\rangle\langle e_1|t]\,\hat{H}(t)\,|\psi(t)\rangle$$

$$= \hbar\omega|e_1\rangle\langle e_1|\,|\phi(t)\rangle$$

$$+ \exp[-i\omega|e_1\rangle\langle e_1|t]\,\hat{H}\exp[i\omega|e_1\rangle\langle e_1|t]\,|\phi(t)\rangle$$

Note that

$$\exp(-i\omega|e_1\rangle\langle e_1|t)\,\hat{H}(t)\,\exp[i\omega|e_1\rangle\langle e_1|t]$$

$$= \left(1 + |e_1\rangle\langle e_1|\left(e^{-i\omega t}-1\right)\right)\Big\{ \varepsilon_1 |e_1\rangle\langle e_1| + \varepsilon_2 |e_2\rangle\langle e_1|$$

$$- U\left[e^{i\omega t} |e_1\rangle\langle e_2| + e^{-i\omega t} |e_2\rangle\langle e_1|\right]\Big\} \left(1 + |e_1\rangle\langle e_1|\left(e^{i\omega t}-1\right)\right)$$

$$= \varepsilon_1 |e_1\rangle\langle e_1| + \varepsilon_2 |e_2\rangle\langle e_2|$$

$$- U\left[|e_1\rangle\langle e_2| + |e_2\rangle\langle e_1|\right]$$

Therefore:

$$i\hbar \frac{\partial}{\partial t} |\phi(t)\rangle = \hbar\omega |e_1\rangle\langle e_1| \, |\phi(t)\rangle$$

$$+ \Big(\varepsilon_1 |e_1\rangle\langle e_1| + \varepsilon_2 |e_2\rangle\langle e_2|$$

$$- U |e_1\rangle\langle e_2| - U|e_2\rangle\langle e_1| \Big) |\phi(t)\rangle$$

$$= (\varepsilon_1 + \hbar\omega) |e_1\rangle\langle e_1| + \varepsilon_2 |e_2\rangle\langle e_2|$$

$$- U\left[|e_1\rangle\langle e_2| + |e_2\rangle\langle e_1| \right]$$

$$= \hat{H}_R \, |\phi(t)\rangle.$$

c) $|\phi(t)\rangle = c_1(t)\,|e_1\rangle + c_2(t)\,|e_2\rangle$

From the lecture notes the values of $c_1(t)$ and $c_2(t)$ are:

$$|\phi(t)\rangle = e^{-i\frac{\varepsilon t}{\hbar}} \cos\left(\frac{Ut}{\hbar}\right) |e_1\rangle + i e^{-i\frac{\varepsilon t}{\hbar}} \sin\left(\frac{Ut}{\hbar}\right) |e_1\rangle$$

where $\varepsilon = \varepsilon_2 = \varepsilon_1 + \hbar\omega$.

d) $|\psi(t)\rangle = \exp\left(i\omega\, |e_1\rangle\langle e_1| t\right) |\phi(t)\rangle$

$$= \left\{ 1 + |e_1\rangle\langle e_1| \left(e^{i\omega t} - 1 \right) \right\} \left\{ e^{-\frac{i\varepsilon t}{\hbar}} \cos\left(\frac{Ut}{\hbar}\right) |e_1\rangle + i e^{-\frac{i\varepsilon t}{\hbar}} \sin\left(\frac{Ut}{\hbar}\right) |e_2\rangle \right\}$$

$$= e^{-i\frac{(\varepsilon - \hbar\omega)t}{\hbar}} \cos\left(\frac{Ut}{\hbar}\right) |e_1\rangle + i e^{\frac{i\varepsilon t}{\hbar}} \sin\left(\frac{Ut}{\hbar}\right) |e_2\rangle$$

$$|\psi(t)\rangle = e^{-i\frac{\varepsilon_1 t}{\hbar}} \cos\left(\frac{Ut}{\hbar}\right) |e_1\rangle + i e^{-i\frac{\varepsilon_2 t}{\hbar}} \sin\left(\frac{Ut}{\hbar}\right) |e_2\rangle$$

e) $|\langle e_1|\psi(t)\rangle|^2 = \cos^2\left(\frac{Ut}{\hbar}\right)$.

f) $\langle \psi(t)|\hat{x}|\psi(t)\rangle$

$$= \frac{2U}{qE_0} \left\{ - e^{i\omega t} \sin\left(\frac{Ut}{\hbar}\right) \cos\left(\frac{Ut}{\hbar}\right) + e^{-i\omega t} \sin\left(\frac{Ut}{\hbar}\right) \cos\left(\frac{Ut}{\hbar}\right) \right\} i$$

$$= \frac{2U}{qE_0} \cdot \sin\omega t \cdot \sin\left(\frac{2Ut}{\hbar}\right).$$

$$= \frac{U}{qE_0} \left\{ \cos\left[\left(\omega - \frac{2U}{\hbar}\right) t \right] - \cos\left[\left(\omega + \frac{2U}{\hbar}\right) t \right] \right\}$$

Interesting point to note: Under the action of the electromagnetic wave the mean position of the electron oscillates with frequencies $\omega \pm \frac{2U}{\hbar}t$.

Ω_R is $\frac{2U}{\hbar}$. So the frequencies charge oscillation are $\omega \pm \Omega_R$.

Question: Does this electron, driven at frequency ω by an electromagnetic wave, radiate at frequencies $\omega \pm \Omega_R$??

1.4

a) $\hat{H} = \varepsilon_1 |e_1\rangle\langle e_1| + \varepsilon_2 |e_2\rangle\langle e_2| - U\left[|e_1\rangle\langle e_2| + |e_2\rangle\langle e_1|\right]$

b)

$$\hat{H} = \begin{bmatrix} \varepsilon_1 & -U \\ -U & \varepsilon_2 \end{bmatrix} \qquad \begin{cases} \lambda_2 \\ \lambda_1 \end{cases} \quad \frac{\varepsilon_1+\varepsilon_2}{2} \pm \frac{1}{2}\sqrt{(2U)^2 + (\varepsilon_2-\varepsilon_1)^2}$$

$$= \frac{\varepsilon_1+\varepsilon_2}{2} \pm \frac{1}{2}\sqrt{(\hbar\Omega_R)^2 + \Delta^2}$$

$\Delta = \varepsilon_2 - \varepsilon_1$

$\boxed{\Omega_R = \frac{2U}{\hbar}}$

$\boxed{\text{Let } \hbar\Omega = \sqrt{(\hbar\Omega_R)^2 + \Delta^2}}$

eigenvectors

$$|v_1\rangle = \frac{\begin{bmatrix} \hbar\Omega + \Delta \\ \hbar\Omega_R \end{bmatrix}}{\sqrt{(\hbar\Omega+\Delta)^2 + (\hbar\Omega_R)^2}} \qquad |v_2\rangle = \frac{\begin{bmatrix} \hbar\Omega - \Delta \\ -\hbar\Omega_R \end{bmatrix}}{\sqrt{(\hbar\Omega-\Delta)^2 + (\hbar\Omega_R)^2}}$$

$$|v_1\rangle = \frac{(\hbar\Omega+\Delta)|e_1\rangle + \hbar\Omega_R |e_2\rangle}{\sqrt{(\hbar\Omega+\Delta)^2 + (\hbar\Omega_R)^2}} \qquad |v_2\rangle = \frac{(\hbar\Omega-\Delta)|e_1\rangle - \hbar\Omega_R|e_2\rangle}{\sqrt{(\hbar\Omega-\Delta)^2 + (\hbar\Omega_R)^2}}$$

c) $|\psi(t=0)\rangle = |e_1\rangle$

$$= c_1|v_1\rangle + c_2|v_2\rangle$$

$$c_1 = \frac{\sqrt{(\hbar\Omega+\Delta)^2 + (\hbar\Omega_R)^2}}{2\hbar\Omega} = \frac{1}{\sqrt{2}}\left(1 + \frac{\Delta}{\hbar\Omega}\right)^{1/2}$$

$$c_2 = \frac{1}{\sqrt{2}}\left(1 - \frac{\Delta}{\hbar\Omega}\right)^{1/2}$$

$$\Rightarrow \quad |\psi(t)\rangle = c_1 e^{-i\lambda_1 t} |v_1\rangle + c_2 e^{-i\lambda_2 t} |v_2\rangle$$

and $|\langle e_1|\psi(t)\rangle|^2$

$$= \left| c_1 e^{-i\lambda_1 t} \langle e_1|v_1\rangle + c_2 e^{-i\lambda_2 t} \langle e_1|v_2\rangle \right|^2$$

$$= \left| \frac{1}{2}\left(1+\frac{\Delta}{\hbar\Omega}\right) e^{-i\lambda_1 t} + \frac{1}{2}\left(1-\frac{\Delta}{\hbar\Omega}\right) e^{-i\lambda_2 t} \right|^2$$

$$= \frac{1}{4}\left(1+\frac{\Delta}{\hbar\Omega}\right)^2 + \frac{1}{4}\left(1-\frac{\Delta}{\hbar\Omega}\right)^2 + \frac{1}{2}\left(1-\frac{\Delta^2}{\hbar^2\Omega^2}\right) \cos(\lambda_2-\lambda_1)t$$

$$= \frac{1}{2}\left\{1 + \frac{\Delta^2}{\hbar^2\Omega^2} + \frac{\Omega_R^2}{\Omega^2} \cos(\Omega t)\right\}$$

e) when detuning $\Delta \gg \hbar\Omega_R$ then $\frac{\Omega_R}{\Omega} \to 0$ and $\frac{\Delta}{\hbar\Omega} \to 1$ and $|\langle e_1|\psi(t)\rangle|^2 \longrightarrow 1$ (i.e the amplitude of oscillations keep decreasing with detuning until there are no oscillations). At the same time the frequency of oscillations (given by Ω) keeps increasing.

f) See at the end for part (f)

1.5

a) $$\mathrm{Tr}\{\hat{A}\} = \sum_k \langle v_k|\hat{A}|v_k\rangle = \sum_k \langle v_k|\hat{A}\,\hat{\mathbb{1}}|v_k\rangle$$

$$= \sum_k \langle v_k|\hat{A} \sum_j |e_j\rangle\langle e_j| \, |v_k\rangle = \sum_{kj} \langle e_j|v_k\rangle\langle v_k|\hat{A}|e_j\rangle$$

$$= \sum_j \langle e_j|\,\mathbb{1}\,\hat{A}|e_j\rangle = \sum_j \langle e_j|\hat{A}|e_j\rangle.$$

b) First show that

$$\mathrm{Tr}[\hat{A}\hat{B}] = \mathrm{Tr}\{\hat{B}\hat{A}\}$$

$$\sum_K \langle V_K|\hat{A}\hat{B}|V_K\rangle = \sum_K \langle V_K|\hat{A}\,\hat{\mathbb{1}}\,\hat{B}|V_K\rangle = \sum_{Kj} \langle V_K|\hat{A}|e_j\rangle\langle e_j|\hat{B}|V_K\rangle$$

$$= \sum_{Kj} \langle e_j|\hat{B}|V_K\rangle\langle V_K|\hat{A}|e_j\rangle = \sum_j \langle e_j|\hat{B}\hat{A}|e_j\rangle$$

$$= \mathrm{Tr}\{\hat{B} A\}.$$

Then it follows that:

$$\mathrm{Tr}\{\underline{\hat{A}\hat{B}}\hat{C}\} = \mathrm{Tr}\{\hat{C}\underline{\hat{A}\hat{B}}\}$$

and $\mathrm{Tr}\{\underline{\hat{C}}\hat{A}\hat{B}\} = \mathrm{Tr}\{\hat{B}\underline{\hat{C}\hat{A}}\}$

1.6

a) Set A:

$$\hat{\rho}_A = |e_+\rangle\langle e_+| = \frac{1}{2}(|e_1\rangle + |e_2\rangle)(\langle e_1| + \langle e_2|) = \frac{1}{2}|e_1\rangle\langle e_1| + \frac{1}{2}|e_2\rangle\langle e_2|$$

$$+ \frac{1}{2}|e_2\rangle\langle e_1| + \frac{1}{2}|e_1\rangle\langle e_2| = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Set B:

$$\hat{\rho}_B = \frac{1}{2}|e_1\rangle\langle e_1| + \frac{1}{2}|e_2\rangle\langle e_2| = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

b) Note that $\hat{\rho}_A(t) = e^{-i\frac{\hat{H}t}{\hbar}}\,\hat{\rho}_A\,e^{i\frac{\hat{H}t}{\hbar}}$

$$= \frac{1}{2}|e_1\rangle\langle e_1| + \frac{1}{2}|e_2\rangle\langle e_2| + \frac{e^{-i\frac{(\varepsilon_2-\varepsilon_1)t}{\hbar}}}{2}|e_2\rangle\langle e_1| + \frac{e^{i\frac{(\varepsilon_2-\varepsilon_1)t}{\hbar}}}{2}|e_1\rangle\langle e_2|$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2}e^{+i\frac{(\varepsilon_2-\varepsilon_1)t}{\hbar}} \\ \frac{1}{2}e^{-i\frac{(\varepsilon_2-\varepsilon_1)t}{\hbar}} & \frac{1}{2} \end{bmatrix}$$

and $\hat{\rho}_B(t) = e^{-i\frac{\hat{H}t}{\hbar}} \hat{\rho}_B e^{i\frac{\hat{H}t}{\hbar}} = \frac{1}{2}|e_1\rangle\langle e_1| + \frac{1}{2}|e_2\rangle\langle e_2|$

$$= \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$\langle K\rangle_A(t) = \mathrm{Tr}\{\hat{\rho}_A(t)\,|e_-\rangle\langle e_-|\}$$

$$= \langle e_-|\hat{\rho}_A(t)|e_-\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2}e^{i\frac{(\varepsilon_2-\varepsilon_1)t}{\hbar}} \\ \frac{1}{2}e^{-i\frac{(\varepsilon_2-\varepsilon_1)t}{\hbar}} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\frac{1}{\sqrt{2}}$$

$$= \frac{1}{4}\left(1 - e^{+i\frac{(\varepsilon_2-\varepsilon_1)t}{\hbar}} - e^{-i\frac{(\varepsilon_2-\varepsilon_1)t}{\hbar}} + 1\right)$$

$$= \frac{1}{2}\left(1 - \cos\left(\frac{(\varepsilon_2-\varepsilon_1)t}{\hbar}\right)\right) = \sin^2\left(\frac{\varepsilon_2-\varepsilon_1}{2\hbar}\right)t$$

And

$$\langle K\rangle_B(t) = \mathrm{Tr}\{\hat{\rho}_B(t)\,|e_-\rangle\langle e_-|\} = \langle e_-|\hat{\rho}_B(t)|e_-\rangle$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\frac{1}{\sqrt{2}} = \frac{1}{4}(2) = \frac{1}{2}.$$

Since $\langle K\rangle_A(t) \neq \langle K\rangle_B(t)$ a series of measurements on set A and set B to determine if the state is $|e_-\rangle$ will give different results

c) $\hat{\rho}_C = \frac{1}{2}|e_+\rangle\langle e_+| + \frac{1}{2}|e_-\rangle\langle e_-|$

$$= \frac{1}{2}\left\{\frac{1}{2}(|e_1\rangle + |e_2\rangle)(\langle e_1| + \langle e_2|)\right\} + \frac{1}{2}\left\{\frac{1}{2}(|e_1\rangle - |e_2\rangle)(\langle e_1| - \langle e_2|)\right\}$$

$$= \frac{1}{2}|e_1\rangle\langle e_1| + \frac{1}{2}|e_2\rangle\langle e_2|$$

$$= \hat{\rho}_B$$

Since $\hat{\rho}_C = \hat{\rho}_B$, no set of measurements can distinguish between set B and set C.

1.4

f) $$\frac{d\hat{N}_1(t)}{dt} = -\frac{d\hat{N}_2(t)}{dt} = \frac{-i}{\hbar} U \left[\hat{\sigma}_+(t) - \hat{\sigma}_-(t) \right] \qquad \left\{ U = \frac{\hbar\Omega_R}{2} \right.$$

$$\frac{d\hat{\sigma}_\pm(t)}{dt} = \pm \frac{i}{\hbar} \Delta\, \hat{\sigma}_\pm(t) \pm \frac{i}{\hbar} U \left[\hat{N}_2(t) - \hat{N}_1(t) \right] \qquad \left\{ \Delta = \varepsilon_2 - \varepsilon_1 \right.$$

we can write these as:

$$\frac{d\hat{N}_d(t)}{dt} = i\frac{2U}{\hbar} \left[\hat{\sigma}_+(t) - \hat{\sigma}_-(t) \right] \qquad \left\{ \hat{N}_d(t) = \hat{N}_2(t) - \hat{N}_1(t) \right.$$

$$\frac{d}{dt}\left[\hat{\sigma}_+(t) - \hat{\sigma}_-(t) \right] = \frac{i\Delta}{\hbar}\left[\hat{\sigma}_+(t) + \hat{\sigma}_-(t) \right] + i\frac{2U}{\hbar} \hat{N}_d(t)$$

$$\frac{d}{dt}\left[\hat{\sigma}_+(t) + \hat{\sigma}_-(t) \right] = \frac{i\Delta}{\hbar}\left[\hat{\sigma}_+(t) - \hat{\sigma}_-(t) \right]$$

These can be combined to give:

$$\frac{d^3\hat{N}_d(t)}{dt^3} - (\Omega)^2 \frac{d\hat{N}_d(t)}{dt} = 0 \qquad \left\{ \Omega = \sqrt{\left(\frac{2U}{\hbar}\right)^2 + \left(\frac{\Delta}{\hbar}\right)^2} \right.$$

Solution is:

$$\hat{N}_d(t) = \hat{A}\cos\Omega t + \hat{B}\sin\Omega t + \hat{C}$$

$\left\{ \hat{A}, \hat{B}, \hat{C} \right.$ are time-independent Schrodinger operators

Boundary conditions: $\hat{N}_d(t=0) = \hat{N}_d = \hat{N}_2 - \hat{N}_1$

$$\left.\frac{d\hat{N}_d(t)}{dt}\right|_{t=0} = i\frac{2U}{\hbar}\left[\hat{\sigma}_+ - \hat{\sigma}_- \right]$$

$$\left.\frac{d^2\hat{N}_d(t)}{dt^2}\right|_{t=0} = -\frac{2U}{\hbar}\frac{\Delta}{\hbar}\left[\hat{\sigma}_+ + \hat{\sigma}_- \right] - \left(\frac{2U}{\hbar}\right)^2 \hat{N}_d$$

These give:

$$\hat{A} = \frac{\left(\frac{2U}{\hbar}\right)^2}{\Omega^2}\hat{N}_d + \frac{\left(\frac{2U}{\hbar}\frac{\Delta}{\hbar}\right)}{\Omega}\left[\hat{\sigma}_+ - \hat{\sigma}_- \right] \qquad \hat{B} = \frac{i\frac{2U}{\hbar}\left[\hat{\sigma}_+ - \hat{\sigma}_- \right]}{\Omega}$$

$$\hat{c} = \frac{\left(\frac{\Delta}{\hbar}\right)^2}{\Omega^2}\hat{N}_d - \frac{1}{\Omega}\left(\frac{2U}{\hbar}\right)\left(\frac{\Delta}{\hbar}\right)\left[\hat{\sigma}_+ - \hat{\sigma}_-\right].$$

Finally :

$$|\langle e_1|\Psi(t)\rangle|^2 = \langle\Psi(t)|e_1\rangle\langle e_1|\Psi(t)\rangle = \langle\Psi(t=0)|\hat{N}_1(t)|\Psi(t=0)\rangle$$

$$= \langle e_1|\hat{N}_1(t)|e_1\rangle = \langle e_1|\frac{\hat{N}_2+\hat{N}_1-\hat{N}_d(t)}{2}|e_1\rangle$$

$$= \frac{1 + \frac{\left(\frac{2U}{\hbar}\right)^2}{\Omega^2}\cos\Omega t + \frac{\left(\frac{\Delta}{\hbar}\right)^2}{\Omega^2}}{2}. \qquad \left\{ U = \frac{\hbar\Omega_R}{2} \right\}.$$