Homework # 10 Solutions (Farhan Rana, Cornell U.)

Part I :

a)
$$\langle \hat{N}_{out} \rangle = \int_{d_a}^{d_a+V_g\Delta\tau} dz \left\langle \hat{b}^{+}\left(z, \frac{d_a+V_g\Delta\tau}{V_g}\right) \hat{b}\left(z, \frac{d_a+V_g\Delta\tau}{V_g}\right) \right\rangle$$

$$= \int_{d_a}^{d_a+V_g\Delta\tau} dz \left\langle \hat{b}^{+}(z-d_a-V_g\Delta\tau, 0)\, \hat{b}(z-d_a-V_g\Delta\tau, 0) \right\rangle L$$

$$= \int_{-V_g\Delta\tau}^{0} dz \left\langle \hat{b}^{+}(z,0)\, \hat{b}(z,0) \right\rangle L = \langle \hat{N}_{in} \rangle L = P_0 L$$

$$\langle \hat{N}_{out}^2 \rangle = \int_{d_a}^{d_a+V_g\Delta\tau} dz_1 \int_{d_a}^{d_a+V_g\Delta\tau} dz_2 \left\langle \hat{b}^{+}\left(z_1, \frac{d_a+V_g\Delta\tau}{V_g}\right) \hat{b}\left(z_1, \frac{d_a+V_g\Delta\tau}{V_g}\right) \hat{b}^{+}\left(z_2, \frac{d_a+V_g\Delta\tau}{V_g}\right) \hat{b}\left(z_2, \frac{d_a+V_g\Delta\tau}{V_g}\right) \right\rangle$$

$$= \int_{d_a}^{d_a+V_g\Delta\tau} dz_1 \int_{d_a}^{d_a+V_g\Delta\tau} dz_2 \left\langle \hat{b}^{+}(z_1-d_a-V_g\Delta\tau,0)\, \hat{b}(z_1-d_a-V_g\Delta\tau,0)\, \hat{b}^{+}(z_2-d_a-V_g\Delta\tau,0)\, \hat{b}(z_2-d_a-V_g\Delta\tau,0) \right\rangle L^2$$

$$+ \int_{d_a}^{d_a+V_g\Delta\tau} dz_1 \int_{d_a}^{d_a+V_g\Delta\tau} dz_2 \left\{ \begin{array}{l} \left\langle \hat{b}^{+}(z_1-d_a-V_g\Delta\tau,0)\, \hat{b}(z_2-d_a-V_g\Delta\tau,0) \right\rangle L \\ \frac{2\gamma}{v_g} \int_0^{d_a} dz' \int_0^{d_a} dz'' \; \delta(z'-z'')\, \delta\left(\frac{z_1-z_2}{V_g}\right) e^{-\gamma(d_a-z')}\, e^{-\gamma(d_a-z'')} \end{array} \right\}$$

$$= \langle \hat{N}_{in}^2 \rangle L^2 + \langle \hat{N}_{in} \rangle L(1-L)$$

$$\Rightarrow \langle \Delta\hat{N}_{out}^2 \rangle = \langle \hat{N}_{out}^2 \rangle - \langle \hat{N}_{out} \rangle^2 = \langle \Delta\hat{N}_{in}^2 \rangle L^2 + \langle \hat{N}_{in} \rangle L(1-L)$$

$$= P_0 L^2 + P_0 L(1-L) = P_0 L$$

Part II :

b)
$$\langle \hat{N}_{out} \rangle = \int_{d_b}^{d_b+V_g\Delta\tau} dz \left\langle \hat{b}^{+}\left(z, \frac{d_b+V_g\Delta\tau}{V_g}\right) \hat{b}\left(z, \frac{d_b+V_g\Delta\tau}{V_g}\right) \right\rangle$$

$$= \int_{d_b}^{d_b+V_g\Delta\tau} dz \left\langle \hat{b}^{+}(z-d_b-V_g\Delta\tau,0)\, \hat{b}(z-d_b-V_g\Delta\tau,0) \right\rangle G$$

$$+ \frac{1}{V_g} \int_{d_b}^{d_b+V_g\Delta\tau} dz \int_0^{d_b} dz_1 \int_0^{d_b} dz_2 \; 2g\, n_{sp}\, \delta(z_1-z_2)\, \delta\left(\frac{z_1-z_2}{V_g}\right) e^{2g(d_b-z_1)}$$

$$\langle \hat{N}_{out} \rangle = \langle \hat{N}_{in} \rangle G + \frac{\Delta\omega\Delta\tau}{2\pi} n_{sp} (G-1).$$

Also,

$$\begin{aligned}
\langle \hat{N}_{out}^2 \rangle = & \int_{d_b}^{d_b+v_g\Delta\tau} dz_1 \int_{d_b}^{d_b+v_g\Delta\tau} dz_2 \Big\{ \langle \hat{b}^+(z_1-d_b-v_g\Delta\tau,0)\, \hat{b}(z_1-d_b-v_g\Delta\tau,0) \\
& \qquad \hat{b}^+(z_2-d_b-v_g\Delta\tau,0)\, \hat{b}(z_2-d_b-v_g\Delta\tau,0) \rangle G^2 \\
& + \langle \hat{b}^+(z_1-d_b-v_g\Delta\tau,0)\, \hat{b}(z_2-d_b-v_g\Delta\tau,0) \rangle G \\
& \quad \times \frac{2g(n_{sp}-1)}{v_g} \int_0^{d_b} dz' \int_0^{d_b} dz'' \; \delta(z'-z'')\, \delta\!\left(\frac{z_1-z_2}{v_g}\right) e^{g(d_b-z')} e^{g(d_b-z'')} \\
& + \langle \hat{b}(z_1-d_b-v_g\Delta\tau,0)\, \hat{b}^+(z_2-d_b-v_g\Delta\tau,0) \rangle G \\
& \quad \times \frac{2g\, n_{sp}}{v_g} \int_0^{d_b} dz' \int_0^{d_b} dz'' \; \delta(z'-z'')\, \delta\!\left(\frac{z_1-z_2}{v_g}\right) e^{g(d_b-z')} e^{g(d_b-z'')} \\
& + \int_0^{d_b} dz' \int_0^{d_b} dz'' \int_0^{d_b} dz''' \int_0^{d_b} dz'''' \; \frac{(2g n_{sp})^2}{v_g v_g} \delta(z'-z'')\, \delta(z'''-z'''') \, \delta\!\left(\frac{z'-z''}{v_g}\right) \delta\!\left(\frac{z'''-z''''}{v_g}\right) e^{2g(d_b-z')} e^{2g(d_b-z''')} \\
& + \int_0^{d_b} dz' \int_0^{d_b} dz'' \int_0^{d_b} dz''' \int_0^{d_b} dz'''' \; \frac{(2g)^2 n_{sp}(n_{sp}-1)}{v_g v_g} \delta(z'-z'''')\, \delta(z''-z''') \, e^{2g(d_b-z')} e^{2g(d_b-z'')} \\
& \qquad \delta\!\left(\frac{z_1-z_2}{v_g}\right) \delta\!\left(\frac{z_1-z_2}{v_g}\right) \\
& + \frac{\Delta\omega\Delta\tau}{2\pi} \langle \hat{N}_{in} \rangle 2n_{sp}\, G(G-1) \Big\}
\end{aligned}$$

$$\begin{aligned}
= & \langle \hat{N}_{in}^2 \rangle G^2 + \langle \hat{N}_{in} \rangle (2n_{sp}-1) G(G-1) + \frac{\Delta\omega\Delta\tau}{2\pi} \langle \hat{N}_{in} \rangle 2n_{sp}\, G(G-1) \\
& + \left[\frac{\Delta\omega\Delta\tau}{2\pi} n_{sp} (G-1) \right]^2 + \frac{\Delta\omega\Delta\tau}{2\pi} n_{sp}\, G(G-1) \\
& + \frac{\Delta\omega\Delta\tau}{2\pi} n_{sp}(n_{sp}-1) (G-1)^2
\end{aligned}$$

$$\begin{aligned}
= & \langle \hat{N}_{in}^2 \rangle G^2 + \langle \hat{N}_{in} \rangle (2n_{sp}-1) G(G-1) + \frac{\Delta\omega\Delta\tau}{2\pi} n_{sp} (G-1) \left[n_{sp}(G-1) + 1 \right] \\
& + \left[\frac{\Delta\omega\Delta\tau}{2\pi} n_{sp} (G-1) \right]^2 + \frac{\Delta\omega\Delta\tau}{2\pi} \langle \hat{N}_{in} \rangle 2n_{sp}\, G(G-1)
\end{aligned}$$

$$\Rightarrow \langle \Delta \hat{N}_{out}^2 \rangle = \langle \Delta \hat{N}_{in}^2 \rangle G^2 + \langle \hat{N}_{in} \rangle (2n_{sp}-1) G(G-1) + \frac{\Delta\omega \Delta\tau}{2\pi} n_{sp}(G-1)\left[n_{sp}(G-1)+1\right]$$

c)

$$\langle \hat{N}_{in} \rangle \big|_{amp} = \langle \hat{N}_{out} \rangle \big|_{fiber} = P_o L$$

$$\langle \Delta \hat{N}_{in}^2 \rangle \big|_{amp} = \langle \Delta \hat{N}_{out}^2 \rangle \big|_{fiber} = P_o L$$

$$\Rightarrow \langle \hat{N}_d \rangle = \langle \hat{N}_{out} \rangle \big|_{amp} = P_o L G + \frac{\Delta\omega \Delta\tau}{2\pi} n_{sp} (G-1)$$

$$\langle \Delta \hat{N}_d^2 \rangle = \langle \Delta \hat{N}_{out}^2 \rangle \big|_{amp} = P_o L G^2 + P_o L (2n_{sp}-1) G(G-1) + \frac{\Delta\omega \Delta\tau}{2\pi} n_{sp} (G-1)\left[n_{sp}(G-1)+1\right]$$

Part III :

d) See attached plot.

$P_o L \big|_{min} \approx 100$

e) See attached plot

$P_o L \big|_{min} \approx 300$

f) Need $P_o L > 100$.

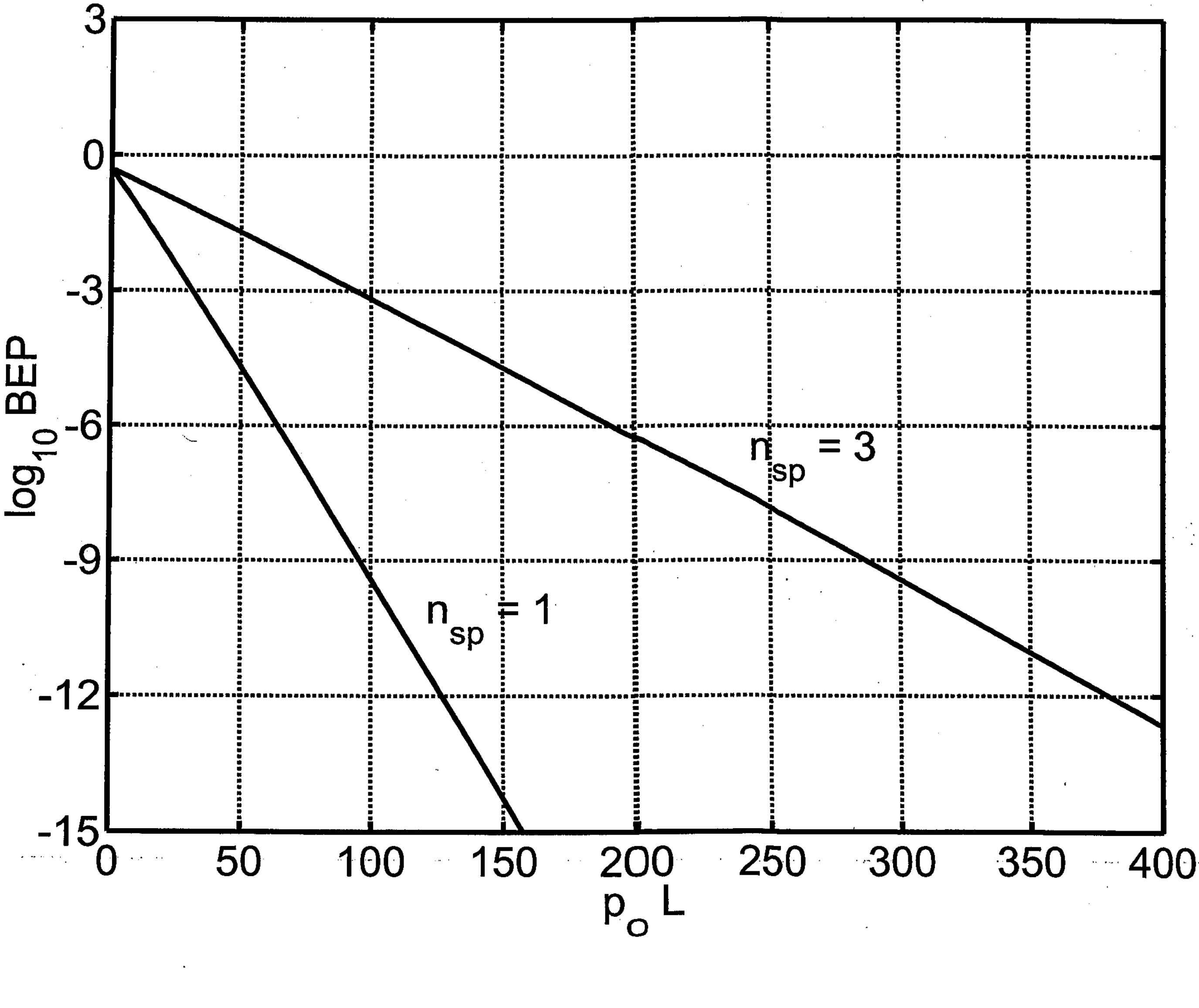
$$\Rightarrow L > \frac{100}{10^6} = 10^{-4}.$$

$$\Rightarrow e^{-2\gamma d_a} > 10^{-4}$$

$$\Rightarrow d_a < -\frac{\ln(10^{-4})}{2\gamma} = 200 \text{ km}.$$

$\Rightarrow d_a < 200$ km.

when $d_a \approx 200$ km $L \approx 10^{-4} = 0.0001$



3
0
-3
-6
-9
-12
-15
log10 BEP
0
50
100
150
200
250
300
350
400
po L
nsp = 3
nsp = 1

9) Note that at the output of the fiber

$$\langle \hat{N}_{out} \rangle = \langle \hat{N}_{in} \rangle L$$
$$\langle \Delta \hat{N}_{out}^2 \rangle = \langle \Delta \hat{N}_{in}^2 \rangle L^2 + \langle \hat{N}_{in} \rangle L(1-L)$$

For a number-state input; $\langle \hat{N}_{in} \rangle = P_0$
$\langle \Delta \hat{N}_{in}^2 \rangle = 0$.

$$\Rightarrow \quad \langle \hat{N}_{out} \rangle = P_0 L$$
$$\langle \Delta \hat{N}_{out}^2 \rangle = P_0 L (1-L)$$

Now since fiber attenuation L is 10^{-4} $\Rightarrow$ $1-L \approx 1$

$$\Rightarrow \quad \langle \Delta \hat{N}_{out}^L \rangle \approx P_0 L$$

Therefore, for a number state input the output number fluctuations are the same as for a coherent state input when attenuation is large (i.e $L \ll 1$). For a number state input, the output state is approximately a statistical mixture of number states with Poisson distribution. In an earlier homework, you showed that the output distribution for a beam splitter is a binomial distribution for a number state input. When the attenuation is Large, a binomial distribution Looks like a Poisson distribution. Therefore, the BEP for a number state input will be no better than that for a Coherent state input with the same average

$$\begin{cases} \langle \hat{N}_d(t) \rangle = N_d(t) = N_d = -1. \\ \langle \hat{a}(t) \rangle = b e^{-i\omega t} \end{cases}$$

10.1.

a) From chapter 11: $\langle \hat{\sigma}_-(t) \rangle = -K \dfrac{N_d(t)\langle \hat{a}(t)\rangle}{(\hbar\omega - \Delta\varepsilon) + i\frac{\hbar}{T_2}}$

$$\Rightarrow \quad \frac{d\langle \hat{a}(t)\rangle}{dt} = \left(-i\omega_0 - \frac{1}{\tau_p}\right)\langle \hat{a}(t)\rangle - \frac{i}{\hbar}K^* \langle \hat{\sigma}_-(t)\rangle + \sqrt{\frac{v_g}{\tau_p}}\,\alpha_{in} e^{-i\omega t}$$

$$\Rightarrow \quad -i\,b\left(\omega - \omega_0 + \frac{i}{\tau_p}\right) = + \frac{i|K|^2}{\hbar}\frac{N_d\, b}{\left(\hbar\omega - \Delta\varepsilon + i\frac{\hbar}{T_2}\right)} + \sqrt{\frac{v_g}{\tau_p}}\,\alpha_{in}.$$

$$b = \frac{i\left(\omega - \frac{\Delta\varepsilon}{\hbar} + \frac{i}{T_2}\right)\sqrt{\frac{v_g}{\tau_p}}\,\alpha_{in}}{\left(\omega - \omega_0 + \frac{i}{\tau_p}\right)\left(\omega - \frac{\Delta\varepsilon}{\hbar} + \frac{i}{T_2}\right) + \frac{|K|^2}{\hbar^2}N_d.} \qquad \{N_d = -1\}.$$

Also:

$$\alpha_{out} = \langle \hat{d}_R(z,t)\rangle = \sqrt{\frac{1}{v_g\tau_p}}\; b$$

$$\Rightarrow \quad \frac{\alpha_{out}}{\alpha_{in}} = \frac{i\left(\omega - \frac{\Delta\varepsilon}{\hbar} + \frac{i}{T_2}\right)\frac{1}{\tau_p}}{\left(\omega - \omega_0 + \frac{i}{\tau_p}\right)\left(\omega - \frac{\Delta\varepsilon}{\hbar} + \frac{i}{T_2}\right) - \frac{|K|^2}{\hbar^2}.}$$

= Transmission amplitude for light to go through the cavity.

b) The resonant frequencies are given by the poles of the denominator. For $\omega_0 = \frac{\Delta\varepsilon}{\hbar}$, the poles are at the complex frequencies:

$$\omega \cong \omega_0 - \frac{i}{2}\left[\frac{1}{\tau_p} + \frac{1}{T_2}\right] \pm \frac{|K|}{\hbar} \longrightarrow$$ The real part is split by the vacuum Rabi frequency $\frac{2|K|}{\hbar}$.

c) For $T_2 \to \infty$, null occurs when $\omega = \frac{\Delta\varepsilon}{\hbar} (= \omega_0)$. For $T_2 < \infty$, the transmission for $\omega = \frac{\Delta\varepsilon}{\hbar} (= \omega_0)$ equals:

$$\frac{\alpha_{out}}{\alpha_{in}} = \frac{1}{1 + \frac{|K|^2}{\hbar^2}T_2\tau_p}$$

d) An empty cavity has ~~two~~ one resonant mode at ω_0. An interacting system of a cavity plus a two-level system has two resonant modes: one at $\omega_0 + \frac{|K|}{\hbar}$ and at $\omega_0 - \frac{|K|}{\hbar}$. Both are excited by an incident wave at frequency ω_0. The transmission amplitudes through these two modes interfere destructively in the output to give a null in the transmission.

e) one may neglect the term $\frac{|K|^2}{\hbar^2}$ in this case to get:

$$\frac{\alpha_{out}}{\alpha_{in}} = \frac{i}{i+(\omega-\omega_0)\tau_p} = \frac{1}{1-i(\omega-\omega_0)\tau_p} \Rightarrow$$ only a single resonance at ω_0 (cavity behaves like an empty cavity).

f) From handouts:

$$N_d = \frac{-1}{1+2g_d T_1|b|^2} = \frac{-1}{1+g_d T_2|b|^2} \qquad g_d = \frac{|K|^2}{\hbar^2}\frac{1/T_2}{\left(\omega-\frac{\Delta\epsilon}{\hbar}\right)^2+\left(\frac{1}{T_2}\right)^2}.$$

g) Suppose $\omega \simeq \frac{\Delta\epsilon}{\hbar}$, then $g_d \simeq \frac{|K|^2}{\hbar^2}T_2$. and $N_d \simeq \frac{-1}{1+\frac{|K|^2}{\hbar^2}T_2^2|b|^2}$.

And if $\frac{|K|}{\hbar} \gg \frac{1}{T_2}$ then for even small values of $|b|^2$, N_d will be a very small (negative) number.

h) The transmission at $\omega=\omega_0=\frac{\Delta\epsilon}{\hbar}$ is: $\frac{\alpha_{out}}{\alpha_{in}} = \frac{1}{1-\frac{|K|^2}{\hbar^2}T_2\tau_p N_d}$.

and $N_d \cong \frac{-1}{1+\frac{|K|^2}{\hbar^2}T_2^2|b|^2}$. If $|b|^2 \gg 1$ then $N_d \simeq 0$ and $\frac{\alpha_{out}}{\alpha_{in}} \cong 1$.

To keep $\frac{\alpha_{out}}{\alpha_{in}} \ll 1$ at $\omega=\omega_0=\frac{\Delta\epsilon}{\hbar}$, one must have

$T_2|b|^2 \ll \tau_p$