Department of Electrical and Computer Engineering, Cornell University

ECE 5310: Quantum Optics for Photonics and Optoelectronics

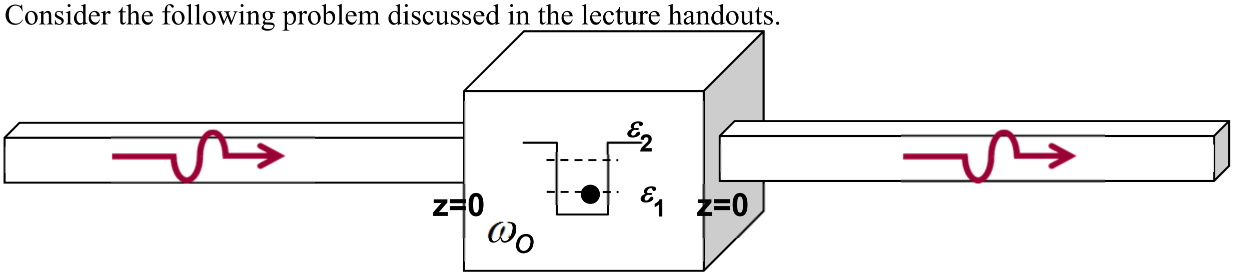
Fall 2013

Homework 10 (Last one) **Due on Dec.05 at 5:00 PM (self-grade)**

Problem 10.1 (Experimental Observation of Vacuum Rabi Splitting and Single-Photon Nonlinearities in Optical Cavities)

You might find this review article related to the problem interesting: Nature Physics, 2, 81-90 (2006).
Also, those of you who attended M. D. Lukin's LASSP talk on 11/26 will find this problem particularly relevant.

Consider the following problem discussed in the lecture handouts.



A cavity is connected to two waveguides, as shown. A weak CW coherent state (of tunable frequency) is sent from the left (input) waveguide and the transmitted signal is measured at the right (output) waveguide. For simplicity, the coordinates for both the waveguides are chosen such that the cavity is located at the origin. For the left waveguide the field operators are,

$$\hat{b}_L(z,t)e^{-i\omega t} \text{ and } \hat{b}_R(z,t)e^{-i\omega t}$$

And for the right waveguide, the field operators are,

$$\hat{d}_L(z,t)e^{-i\omega t} \text{ and } \hat{d}_R(z,t)e^{-i\omega t}$$

Note that the field operators are expanded around a frequency ω which is not the same as the cavity mode frequency ω_o. The frequency ω is the center frequency of the coherent state that will be sent into the cavity from the outside. The cavity contains a two level system in the ground state. We will assume strong coupling regime where,

$$\frac{2|k|}{\hbar} >> \frac{1}{\tau_p}, \frac{1}{T_2}$$

You may also assume that one is in the "Purcell" regime where,

$$\frac{1}{\tau_p} >> \frac{1}{T_2}$$

The goal of this problem is to find out the resonance frequencies that will be observed in this optical transmission experiment. Since the input state is weak, and we will be interested in the "linear response" of the system, it is safe to assume, *to the order we are interested in*, that the populations does not change during the experiment (and the electron is in the ground state). This also means that the average photon number inside the cavity remains very small (much less than unity) during the experiment.

The relevant equations for the polarization are,

$$\frac{d\hat{\sigma}_+(t)}{dt} = \left(i\frac{\Delta\varepsilon}{\hbar} - \frac{1}{T_2} \right)\hat{\sigma}_+(t) - \frac{i}{\hbar}k^*\,\hat{a}^+(t)\left[\hat{N}_2(t) - \hat{N}_1(t)\right] + \hat{F}_+(t)e^{i\omega t}$$

$$\frac{d\hat{\sigma}_-(t)}{dt} = \left(-i\frac{\Delta\varepsilon}{\hbar} - \frac{1}{T_2} \right)\hat{\sigma}_-(t) + \frac{i}{\hbar}k\left[\hat{N}_2(t) - \hat{N}_1(t)\right]\hat{a}(t) + \hat{F}_-(t)e^{-i\omega t}$$

where, one may assume for parts (a)-(e) of this problem,

$$\left\langle \hat{N}_2(t) - \hat{N}_1(t) \right\rangle = -1$$

The equations for the cavity field operators are,

$$\frac{d\hat{a}(t)}{dt} = \left(-i\omega_o - \frac{1}{2\tau_p} - \frac{1}{2\tau_p} \right)\hat{a}(t) - \frac{i}{\hbar}k^*\hat{\sigma}_-(t) + \sqrt{\frac{v_g}{\tau_p}}\,\hat{b}_R(0,t)e^{-i\omega t} + \sqrt{\frac{v_g}{\tau_p}}\,\hat{d}_L(0,t)e^{-i\omega t}$$

$$\frac{d\hat{a}^+(t)}{dt} = \left(i\omega_o - \frac{1}{2\tau_p} - \frac{1}{2\tau_p} \right)\hat{a}^+(t) + \frac{i}{\hbar}k\,\hat{\sigma}_+(t) + +\sqrt{\frac{v_g}{\tau_p}}\,\hat{b}_R^+(0,t)e^{i\omega t} + \sqrt{\frac{v_g}{\tau_p}}\,\hat{d}_L^+(0,t)e^{i\omega t}$$

And outside the cavity,

$$\sqrt{v_g}\,\hat{d}_R(z=0,t)e^{-i\omega t} = \sqrt{\frac{1}{\tau_p}}\,\hat{a}(t) - \sqrt{v_g}\,\hat{d}_L(z=0,t)e^{-i\omega t}$$

$$\sqrt{v_g}\,\hat{b}_L(z=0,t)e^{-i\omega t} = \sqrt{\frac{1}{\tau_p}}\,\hat{a}(t) - \sqrt{v_g}\,\hat{b}_R(z=0,t)e^{-i\omega t}$$

We may also assume that in the input waveguide,

$$\left\langle \hat{b}_R(z,t)e^{-i\omega t} \right\rangle = \alpha_{in}e^{-i\omega t}$$

and in the output waveguide,

$$\left\langle \hat{d}_R(z,t)e^{-i\omega t} \right\rangle = \alpha_{out}e^{-i\omega t}$$

The goal of the problem is to find,

$$\frac{\alpha_{out}}{\alpha_{in}}$$

as a function of the frequency ω. This is the transmission coefficient for the coupled cavity/two-level system.

In steady state, we expect that the average values will have the following form,

$$\left\langle \hat{\sigma}_-(t) \right\rangle = \chi_- e^{-i\omega t}$$

$$\left\langle \hat{a}(t) \right\rangle = b\,e^{-i\omega t}$$

And the average photon number inside the cavity is then,

$$\left\langle \hat{a}^+(t)\hat{a}(t) \right\rangle \approx |b|^2$$

Here, χ_- and b are complex numbers that are independent of time. By assumption, $|b|^2 << 1$.

You can take the average of all the equations right from the beginning, and then solve them in steady state. Assume zero detuning,

$$\Delta = \Delta\varepsilon - \hbar\omega_o = 0$$

a) Find the ratio,

$$\frac{\alpha_{out}}{\alpha_{in}}$$

as a function of the frequency ω by solving all the relevant equations.

b) Find the frequencies at which resonances would occur in the transmission spectrum.

c) If $T_2 \to \infty$, find the frequencies at which there will be a null in the transmission spectrum (and all the light is reflected). Note that if T_2 is not infinite, but large, then the minimum transmission is determined by the product,

$$\frac{|k|^2}{\hbar^2}\tau_p T_2$$

d) Come up with a **physical** reason to explain (without using math) why is there a null in the transmission spectrum (as found in part (c)).

e) Suppose the cavity is in the weak coupling regime,

$$\frac{2|k|}{\hbar} << \frac{1}{\tau_p}, \frac{1}{T_2}$$

Find the frequencies at which resonances would occur in the transmission spectrum.

Now we will let go the assumption that,

$$\hat{N}_2(t) - \hat{N}_1(t) = -1$$

Instead, you will first derive the equation for the populations in steady state. If you repeat part (a), without the above assumption, then you would obtain that in steady state,

$$\langle \hat{\sigma}_-(t) \rangle = -k \frac{[N_2 - N_1] b e^{-i\omega t}}{(\hbar\omega - \Delta\varepsilon) + i\hbar/T_2} \qquad \langle \hat{\sigma}_+(t) \rangle = -k^* \frac{[N_2 - N_1] b^* e^{i\omega t}}{(\hbar\omega - \Delta\varepsilon) - i\hbar/T_2}$$

where we have assumed that in steady state,

$$\left\langle \left(\hat{N}_2(t) - \hat{N}_1(t)\right)\hat{a}(t) \right\rangle = (N_2 - N_1)\langle a(t) \rangle = (N_2 - N_1) b e^{-i\omega t}$$

In steady state, the averages of the population equations are,

$$0 = \frac{d\langle \hat{N}_2(t) \rangle}{dt} = -\frac{N_2}{T_1} - \frac{i}{\hbar}\left[k\langle \hat{\sigma}_+(t) \rangle b e^{-i\omega t} - k^* b^* e^{i\omega t} \langle \hat{\sigma}_-(t) \rangle\right]$$

$$0 = \frac{d\langle \hat{N}_1(t) \rangle}{dt} = +\frac{N_2}{T_1} + \frac{i}{\hbar}\left[k\langle \hat{\sigma}_+(t) \rangle b e^{-i\omega t} - k^* b^* e^{i\omega t} \langle \hat{\sigma}_-(t) \rangle\right]$$

The population relaxation time, assumed to be entirely due to spontaneous emission, is $T_1 \left(= T_2/2\right)$. We also have that,

$$(N_2 + N_1) = 1$$

f) Find the populations, and the population difference $N_d = N_2 - N_1$, in steady state as a function of the average photon number inside the cavity.

g) If, on the average, you have lots of photons inside the cavity (because a strong light source was used to do the experiment) what is the population difference in steady state (still -1, or +1, or close to zero)? Hint: don't forget that,

$$\frac{2|k|}{\hbar} >> \frac{1}{\tau_p}, \frac{1}{T_2}$$

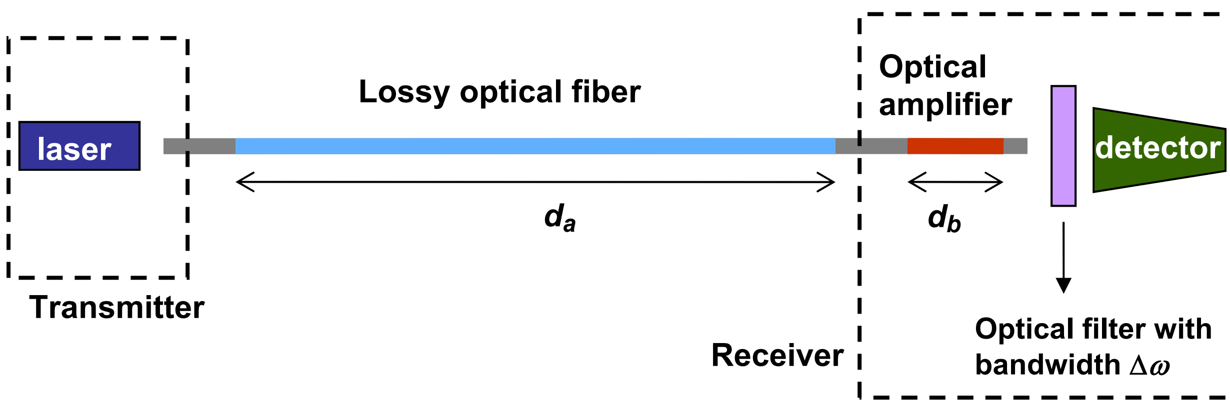
h) From what you (hopefully) discovered in parts (f)-(g), if you were to repeat parts (a) through (c) assuming lots of photons inside the cavity, would you still see a null in the transmission spectrum at the frequency at which you previously saw the null in part (c)? Why or why not? Present analysis and don not assume that T_2 is infinite. If one would like to have a small transmission (at the null frequency) even when the average number of photons inside the cavity approaches ~1 then show that one needs to be in the Purcell regime where,

$$\frac{1}{\tau_p} >> \frac{1}{T_2}$$

Problem 10.2 (Photonic/Optical Communication Systems)

In this problem you will analyze a long haul fiber optical communication link, and determine the bit error probabilities. You will determine the minimum average number of photons for logical 1's **that must reach the receiver** so that they can be amplified and detected with a BEP (Bit Error Probability) better than 10^{-9}. The answer will tell you how many photons are necessary to transmit one bit of information on a state-of-the-art fiber link. A simple model for a fiber optical link is shown below.

Optical signal from a laser goes into an optical fiber which has a lossy region of length d_a. The signal at the receiver side is too attenuated to be directly detected with a photodetector. Even the smallest noise intrinsic to the photodetector (e.g. thermal noise) can overwhelm the signal. The signal is therefore first amplified with an amplifier of length d_b (which in practice is also a piece of fiber but with gain – called an EDFA, which stands for Erbium Doped Fiber Amplifier). The amplified signal passes through an optical filter and then it goes into a photodetector.

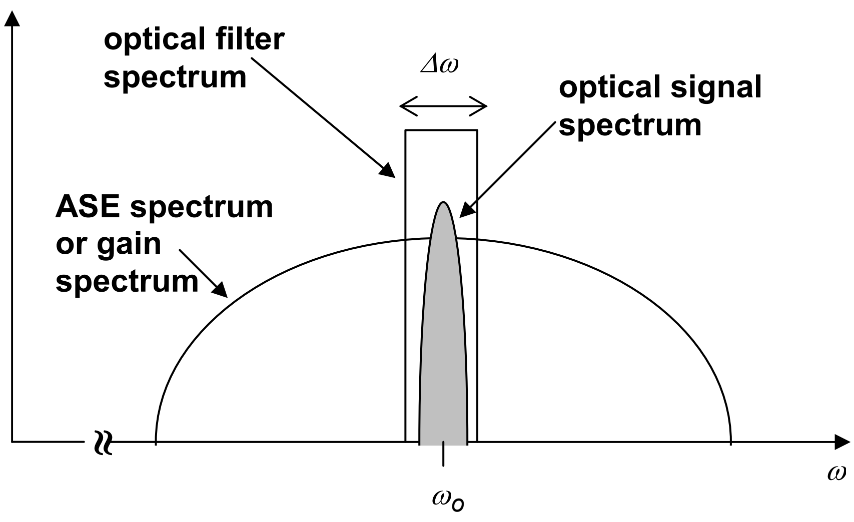


Optical Filter:

The optical filter is placed in front of the photodetector because, as discussed in the lecture, the amplified spontaneous emission (ASE) flux coming out of the amplifier is non-zero over a very wide frequency range (actually the entire gain bandwidth of the amplifier). The signal from the laser on the other hand generally has a much narrower frequency bandwidth. The optical filter only lets those photons go through that are in the signal bandwidth, thereby cutting down on the unnecessary ASE photon flux entering the photodetector. This is shown in the figure above. Since the filter rejects the photon outside its transmission bandwidth, it makes sense to expand the field operator in this bandwidth only right from the start,

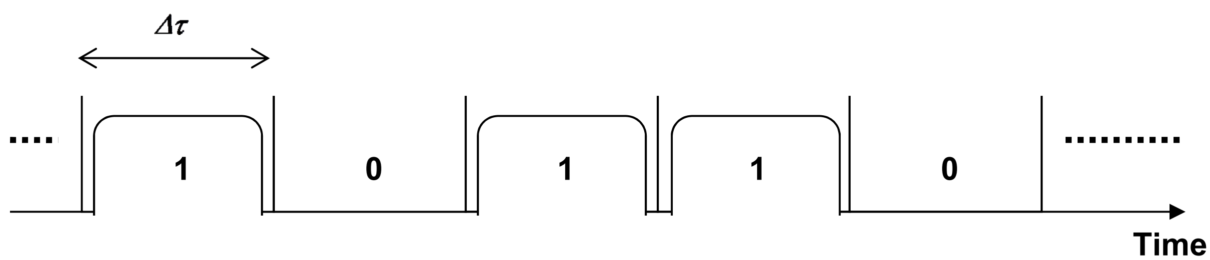
$$\hat{b}(z,t) = d \int_{\beta_o - \Delta\beta/2}^{\beta_o + \Delta\beta/2} \frac{d\beta}{2\pi} \; \hat{a}(\beta) \frac{\exp[i(\beta - \beta_o)\, z]}{\sqrt{d}} \exp[-i(\omega(\beta) - \omega(\beta_o))\, t]$$

where $\Delta\beta = \frac{d\beta}{d\omega}\Delta\omega = \frac{\Delta\omega}{v_g}$, and $\omega_o = \omega(\beta_o)$, and d is the relevant length (of the fiber or of the gain medium etc).

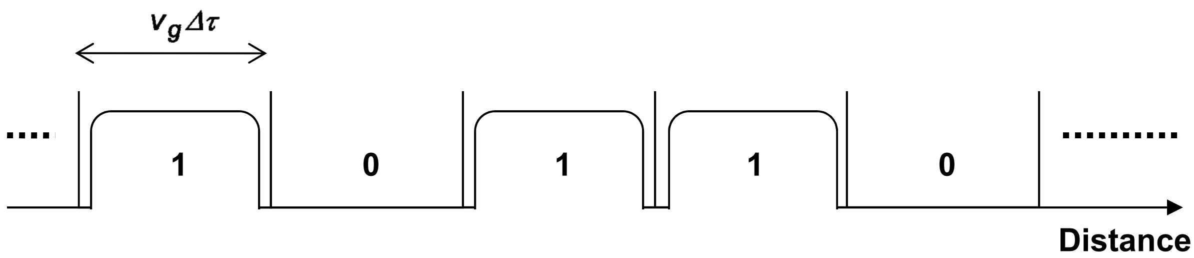


Input Signal:

The signal input to the fiber from the laser consists of 0's and 1's. Each logical '1' is a **coherent-state photon packet** (or an optical pulse) containing p_o average number of photons and occupying a duration in time of $\Delta\tau$ and a duration in space of $v_g\ \Delta\tau$. A logical '0' means nothing in a duration in time of $\Delta\tau$ and a duration in space of $v_g\ \Delta\tau$. So if you stand at one location in the fiber you will observe a signal in time that looks like,



And if you take a snapshot of the fiber at any time you will observe a signal in space that looks like,



The data rate is clearly one bit of information per $\Delta\tau$ seconds, or $1/\Delta\tau$ bits/sec. In state of the art commercial optical systems data rates are between 10-40 Gbits/sec, so the duration $\Delta\tau$ is between 100 ps and 25 ps.

Bit Errors at the Receiver:

At the receiver the photodetector counts the number of photon in each time bin of duration $\Delta\tau$ and sees if the result crosses a certain threshold. If it does it declares a logical '1', and if it does not it declares a logical '0'. Bit errors can occur at the photodetector in the following way. A logical '1' may loose enough photons while traveling in the optical link and be wrongly interpreted as a logical '0'. And a logical '0', as a result of ASE photons emerging from the amplifier, may end up having enough photons to cross the detector threshold and be wrongly interpreted as a logical '1'.In this problem you will analyze these errors and calculate the probability of the detector making an error. Practical communication systems demand a bit error probability (BEP) as low as 10^{-9}. This means only 1 error on the average in every 10^9 bits received by the detector.

Part I: Lossy Fiber

In this part you will only consider the lossy region of the fiber and forget everything else. The propagation in the lossy part of the optical fiber (which is of length d_a) is described by the following equations,

$$\left(\frac{\partial}{\partial z}+\frac{1}{v_g}\frac{\partial}{\partial t}\right)\hat{b}(z,t)=-\gamma\,\hat{b}(z,t)+\sqrt{\frac{2\gamma}{v_g}}\,\hat{F}_L(z,t)$$

$$\left(\frac{\partial}{\partial z}+\frac{1}{v_g}\frac{\partial}{\partial t}\right)\hat{b}^{+}(z,t)=-\gamma\,\hat{b}^{+}(z,t)+\sqrt{\frac{2\gamma}{v_g}}\,\hat{F}_L^{+}(z,t)$$

The constant γ describes the loss in the fiber. The Langevin noise sources that model the noise coming from the loss have the following correlations (**make sure you don't confuse the noise source $\hat{F}_L(z,t)$ with the photon flux operator $\hat{F}(z,t)$**),

$$\left\langle \hat{F}_L^{+}(z,t)\right\rangle=\left\langle \hat{F}_L(z,t)\right\rangle=0 \qquad \left\langle \hat{F}_L^{+}(z,t)\,\hat{F}_L(z',t')\right\rangle=0$$

$$\left\langle \hat{F}_L(z,t)\,\hat{F}_L^{+}(z',t')\right\rangle=\delta(z-z')\,\delta(t-t')$$

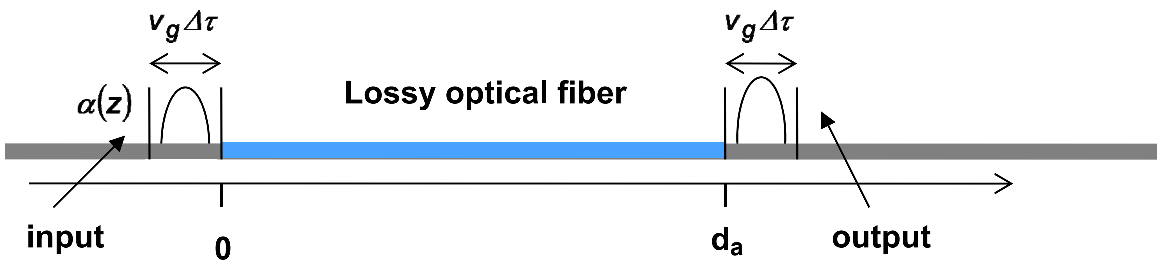
$\left\langle \ldots\ldots\hat{F}_L(z,t)\right\rangle=\left\langle F_L^{+}(z,t)\ldots\ldots\right\rangle=0$ (the dots stand for any sequence of operators)

$\left\langle \hat{F}_L^{a}(z_1,t_1)\hat{F}_L^{b}(z_2,t_2)\ldots\ldots\hat{F}_L^{c}(z_n,t_n)\right\rangle=0$ for n odd (the alphabets 'a', 'b', and 'c' mean that the operator is an adjoint if the alphabet is '+', or not an adjoint if the alphabet is ' ').

The input to the lossy fiber is a coherent-state packet defined at $t = 0$ by the equation,

$$|\psi(t=0)\rangle = |\alpha\rangle = \exp\left[\int_{-v_g\Delta\tau}^{0} dz' \left\{\alpha(z')\,\hat{b}^{+}(z',0) - \alpha^{*}(z')\,\hat{b}(z',0)\right\}\right]|0\rangle$$

The function $\alpha(z)$ is localized to the left of the lossy region of the fiber in a slot of length $v_g\ \Delta\tau$ as shown below:



I have adjusted the spatial co-ordinate system so that the zero of the co-ordinate system is the beginning of the lossy region of the fiber. The input photon number operator is defined as:

$$\hat{N}_{in} = \frac{1}{v_g}\int_{-v_g\Delta\tau}^{0} dz\ \hat{F}(z,0)$$

and so,

$$\langle\psi(t=0)|\hat{N}_{in}|\psi(t=0)\rangle = \langle\hat{N}_{in}\rangle = \int_{-v_g\Delta\tau}^{0} dz\ \ |\alpha(z)|^2 = p_o$$

$$\langle\psi(t=0)|\Delta\hat{N}_{in}^2|\psi(t=0)\rangle = \langle\Delta\hat{N}_{in}^2\rangle = p_o$$

The output photon number operator is defined as,

$$\hat{N}_{out} = \frac{1}{v_g}\int_{d_a}^{d_a+v_g\Delta\tau} dz\ \ \hat{F}\left(z, \frac{d_a + v_g\Delta\tau}{v_g}\right)$$

Solution:

The solution of the lossy fiber equations can be written as (see if you can derive it),

$$\hat{b}(z,t) = \hat{b}(z - v_g t,\, 0)\exp(-\gamma\ d_a) + \sqrt{\frac{2\gamma}{v_g}}\int_{0}^{d_a} dz'\ \ \hat{F}_L\left(z', t - \frac{(z-z')}{v_g}\right)\exp[-\gamma(d_a - z')]$$

(and the corresponding adjoint, of course)

The solution given above is valid provided,

i) the location z is greater than d_a (i..e to the right side of the lossy region), and

ii) the time t is sufficiently large so that the location $z - v_g t$ is before the start of the lossy region of the fiber (i.e. $z - v_g t < 0$).

a) Calculate the following quantities (remember the averages are with respect to the initial state):

$\left\langle \hat{N}_{out} \right\rangle$

$\left\langle \Delta\hat{N}_{out}^2 \right\rangle$

and express your answers in terms of the power attenuation factor L, defined as $L = \exp(-2\gamma\, d_a)$, and the quantities: $\left\langle \hat{N}_{in} \right\rangle$ and $\left\langle \Delta\hat{N}_{in}^2 \right\rangle$

Hint: answers should be (and this should not come as a surprise),

$$\left\langle \hat{N}_{out} \right\rangle = \left\langle \hat{N}_{in} \right\rangle L = p_o\, L$$

$$\left\langle \Delta\hat{N}_{out}^2 \right\rangle = \left\langle \Delta\hat{N}_{in}^2 \right\rangle L^2 + \left\langle \hat{N}_{in} \right\rangle L\,(1-L) = p_o\, L$$

Part II: Amplifier

In this part you will only consider the amplifier and forget everything else. The propagation in the amplifier (which is of length d_b) is described by the following equations:

$$\left(\frac{\partial}{\partial z} + \frac{1}{v_g}\frac{\partial}{\partial t}\right)\hat{b}(z,t) = g\,\hat{b}(z,t) + \hat{F}_{sp}(z,t) \qquad \left(\frac{\partial}{\partial z} + \frac{1}{v_g}\frac{\partial}{\partial t}\right)\hat{b}^{+}(z,t) = g\,\hat{b}^{+}(z,t) + \hat{F}_{sp}^{+}(z,t)$$

where g describes the amplifier gain (units: length^{-1}) and the Langevin noise sources that model the noise coming from the gain have the following correlations,

$$\left\langle \hat{F}_{sp}^{+}(z,t) \right\rangle = \left\langle \hat{F}_{sp}(z,t) \right\rangle = 0 \qquad \left\langle \hat{F}_{sp}^{+}(z,t)\,\hat{F}_{sp}(z',t') \right\rangle = \frac{2g}{v_g}\, n_{sp}\;\delta(z-z')\,\delta(t-t')$$

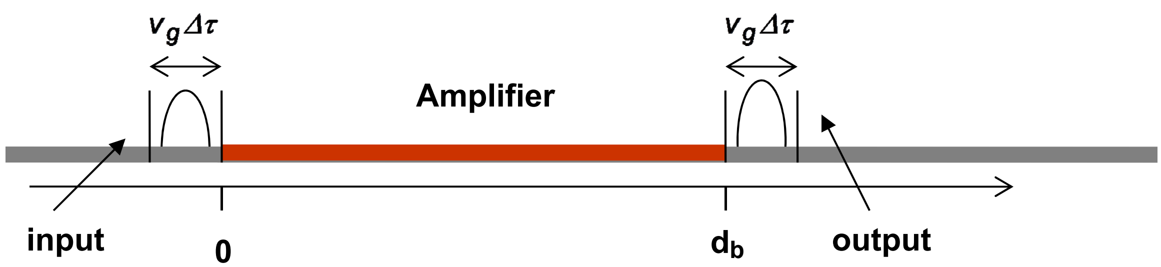
$$\left\langle \hat{F}_{sp}(z,t)\,\hat{F}_{sp}^{+}(z',t') \right\rangle = \frac{2g}{v_g}\left(n_{sp}-1\right)\delta(z-z')\,\delta(t-t')$$

$\left\langle \hat{F}_{sp}^{a}(z_1,t_1)\hat{F}_{sp}^{b}(z_2,t_2)\ldots\ldots\hat{F}_{sp}^{c}(z_n,t_n) \right\rangle = 0$ for n odd (the alphabets 'a', 'b', and 'c' mean that the operator is an adjoint if the alphabet is '+', or not an adjoint if the alphabet is ' '). And finally, if n is even then we can brake up the correlation function into a sum of products of all non-zero pairings (paying due regard to the operator orderings), e.g.,

$$\left\langle \hat{F}_{sp}^{+}(z_1,t_1)\,\hat{F}_{sp}(z_2,t_2)\,\hat{F}_{sp}^{+}(z_3,t_3)\,\hat{F}_{sp}(z_4,t_4) \right\rangle = \left\langle \hat{F}_{sp}^{+}(z_1,t_1)\,\hat{F}_{sp}(z_2,t_2) \right\rangle \left\langle \hat{F}_{sp}^{+}(z_3,t_3)\,\hat{F}_{sp}(z_4,t_4) \right\rangle$$
$$+ \left\langle \hat{F}_{sp}^{+}(z_1,t_1)\,\hat{F}_{sp}(z_4,t_4) \right\rangle \left\langle \hat{F}_{sp}(z_2,t_2)\,\hat{F}_{sp}^{+}(z_3,t_3) \right\rangle$$

In this course I never showed you a full microscopic derivation for the gain (units: length^{-1}) for a traveling wave amplifier. I showed you the derivation for the gain (units: time^{-1}) in the case of a cavity field interacting with two level systems. The nice thing that quantum mechanics provides is that one can get the form of the noise sources in the traveling wave amplifier without knowing or having to model the full microscopic details of the amplifier.

The input to the amplifier is some **unknown packet state** localized at $t = 0$ to the left of the amplifier in a slot of length $v_g \; \Delta\tau$ as shown in the figure below,



I have again adjusted the spatial co-ordinate system so that the zero of the co-ordinate system is the beginning of the amplifier. The input photon number operator is defined as:

$$\hat{N}_{in} = \frac{1}{v_g} \int_{-v_g\Delta\tau}^{0} dz \; \hat{F}(z,0)$$

The only two things we know about the input state are the following: $\langle \hat{N}_{in} \rangle$ and $\langle \Delta\hat{N}_{in}^2 \rangle$. The output photon number operator is defined as,

$$\hat{N}_{out} = \frac{1}{v_g} \int_{d_b}^{d_b + v_g\Delta\tau} dz \;\; \hat{F}\left(z, \frac{d_b + v_g\Delta\tau}{v_g} \right)$$

Solution:
The solution of the amplifier equations can be written as,

$$\hat{b}(z,t) = \hat{b}(z - v_g t, 0)\exp(\; g \; d_b) + \int_0^{d_b} dz' \;\; \hat{F}_{sp}\left(z', t - \frac{(z - z')}{v_g} \right) \exp[\; g(d_b - z')]$$

(and the corresponding adjoint, of course)

The solution given above is valid provided,

i) the location z is greater than d_b (i..e to the right side of the amplifier), and
ii) the time t is sufficiently large so that the location $z - v_g t$ is to the left side of the amplifier (i.e. $z - v_g t < 0$).

b) Calculate the following quantities (remember the averages are with respect to the initial state),

$$\langle \hat{N}_{out} \rangle$$

$$\left\langle \Delta\hat{N}_{out}^2 \right\rangle$$

and express your answers in terms of the power gain factor G, defined as $G = \exp(2g\ d_b)$, and the two input quantities that are known $\left\langle \hat{N}_{in} \right\rangle$ and $\left\langle \Delta\hat{N}_{in}^2 \right\rangle$.

Hint: answers should be:

$$\left\langle \hat{N}_{out} \right\rangle = \left\langle \hat{N}_{in} \right\rangle G + \frac{\Delta\omega\ \Delta\tau}{2\pi} n_{sp}(G-1)$$

$$\left\langle \Delta\hat{N}_{out}^2 \right\rangle = \left\langle \Delta\hat{N}_{in}^2 \right\rangle G^2 + \left\langle \hat{N}_{in} \right\rangle (2n_{sp}-1)G(G-1) + \frac{\Delta\omega\ \Delta\tau}{2\pi} n_{sp}(G-1)[n_{sp}(G-1)+1]$$

Putting it All Together:

We know that in the system under consideration the photon packet state that goes into the amplifier is the packet state that comes out from the lossy fiber. Therefore, in the expressions obtained in part (b) you can replace $\left\langle \hat{N}_{in} \right\rangle$ and $\left\langle \Delta\hat{N}_{in}^2 \right\rangle$ by the values $\left\langle \hat{N}_{out} \right\rangle$ and $\left\langle \Delta\hat{N}_{out}^2 \right\rangle$, respectively, that were calculated in part (a).

Also, since the packet state that goes into the photodetector is the packet state that comes out of the amplifier, the average number of photons $\left\langle \hat{N}_d \right\rangle$ counted by the photodetector in duration $\Delta\tau$ and the mean square fluctuations in the photodetector photon counts $\left\langle \Delta\hat{N}_d^2 \right\rangle$ are therefore equal to $\left\langle \hat{N}_{out} \right\rangle$ and $\left\langle \Delta\hat{N}_{out}^2 \right\rangle$, respectively, that were calculated in part (b).

c) Using the above facts show that for the coherent state packet state generated by the transmitter the values of $\left\langle \hat{N}_d \right\rangle$ and $\left\langle \Delta\hat{N}_d^2 \right\rangle$ are,

$$\left\langle \hat{N}_d \right\rangle = p_o\ L\ G + \frac{\Delta\omega\ \Delta\tau}{2\pi} n_{sp}(G-1)$$

$$\left\langle \Delta\hat{N}_d^2 \right\rangle = p_o\ L\ G^2 + p_o\ L\ (2n_{sp}-1)G(G-1) + \frac{\Delta\omega\ \Delta\tau}{2\pi} n_{sp}(G-1)[n_{sp}(G-1)+1]$$

Part III: Bit Error Rates

When logical 1's enter the photodetector we have (as calculated in parts a-c):

$$n_1 = \left\langle \hat{N}_d \right\rangle = p_o\ L\ G + \frac{\Delta\omega\ \Delta\tau}{2\pi} n_{sp}(G-1)$$

$$\sigma_1^2 = \left\langle \Delta\hat{N}_d^2 \right\rangle = p_o\ L\ G^2 + p_o\ L\ (2n_{sp}-1)G(G-1) + \frac{\Delta\omega\ \Delta\tau}{2\pi} n_{sp}(G-1)[n_{sp}(G-1)+1]$$

When logical 0's enter the photodetector we have (set p_o equal to zero in the equations above for logical 1's)

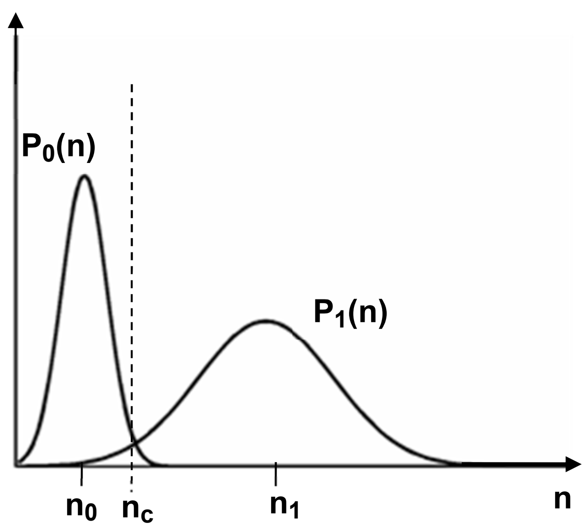
$$n_0 = \langle \hat{N}_d \rangle = \frac{\Delta\omega\,\Delta\tau}{2\pi} n_{sp}(G-1)$$

$$\sigma_0^2 = \langle \Delta\hat{N}_d^2 \rangle = \frac{\Delta\omega\,\Delta\tau}{2\pi} n_{sp}(G-1)\left[n_{sp}(G-1)+1\right]$$

Note that the non-zero values in case of logical 0's are due to the ASE photons coming out from the amplifier. ASE photons are continuously coming out. This is why it is important to have an optical filter that limits the number of spontaneously emitted photons entering the photodetector to the minimum possible values.

We don't know the actual probability distributions of the photons in logical 1's and logical 0's. We have calculated only the average values and the variances. It turns out that it is not a bad approximation to assume the probability distributions to be Gaussians (this assumption has an accuracy of 90% if the results are compared with the complicated actual distributions). The probability distributions $P_1(n)$ and $P_0(n)$ for the photon counts by the detector for logical 1's and logical 0's, respectively, are therefore,

$$P_0(n) = \frac{1}{\sqrt{2\pi\,\sigma_0^2}} \exp\left(-\frac{(n-n_0)^2}{2\sigma_0^2}\right) \qquad P_1(n) = \frac{1}{\sqrt{2\pi\,\sigma_1^2}} \exp\left(-\frac{(n-n_1)^2}{2\sigma_1^2}\right)$$



These probability distributions are sketched in the figure above. A system designer must decide upon a threshold photon number n_c so that if the number of photons counted in any one bit interval of duration $\Delta\tau$ comes out to be larger than n_c then a logical 1 is declared, and if the number of photons counted is smaller than n_c a logical 0 is declared. These decisions are made by electronic circuitry in the receiver.

The bit error probabilities are defined as follows:

i) The probability $P(0|1)$ for a logical 1 to be wrongly interpreted as a logical 0 is:

$$P(0|1) = \int_0^{n_c} P_1(n)\,dn \approx \int_{-\infty}^{n_c} P_1(n)\,dn$$

ii) The probability $P(1|0)$ for a logical 0 to be wrongly interpreted as a logical 1 is:

$$P(1|0) = \int_{n_c}^{\infty} P_0(n)\,dn$$

If the threshold photon number n_c is chosen too large then $P(0|1)$ will become too large and $P(1|0)$ will become too small. If the threshold photon number n_c is chosen too small then $P(0|1)$ will become too small and $P(1|0)$ will become too large. So the best way to choose the threshold photon number n_c is such that $P(0|1)$ is equal to $P(1|0)$. You can convince yourselves that for Gaussian distributions $P(0|1)$ equals $P(1|0)$ if n_c is chosen such that $P_0(n_c) = P_1(n_c)$. The value n_c is indicated in the figure, and it comes out to be,

$$n_c = \frac{n_o\sigma_1 + n_1\sigma_0}{(\sigma_1 + \sigma_0)}$$

We then have the final result for the bit error probability (BEP),

$$\text{BEP} = P(1|0) = P(0|1) = \int_{n_c}^{\infty} P_0(n)\,dn = \int_{\frac{n_1 - n_0}{\sigma_1 + \sigma_0}}^{\infty} dx\,\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right) = \frac{1}{2}\text{erfc}\left(\frac{1}{\sqrt{2}}\frac{n_1 - n_0}{\sigma_1 + \sigma_0}\right)$$

d) The average number of photons for logical 1's that reach the receiver is $p_o\,L$. In this problem you will determine the minimum average number of photons for logical 1's **that must reach the receiver** so that they can be amplified and detected with a BEP better than 10^{-9}. Assume the following values (these are typical for fiber optical communication systems),

$$G = 200 \quad n_{sp} = 1 \quad \Delta\tau = 25 \text{ ps (corresponding to a data rate of 40 Gb/s)} \quad \Delta\omega = \frac{8\pi}{\Delta\tau}$$

Plot $\log_{10}$(BEP) as a function of $p_o\,L$ (choose values of $p_o\,L$ between 0 and 250 for the plot). What is the minimum number of photons that must reach the receiver in order to get a BEP less than 10^{-9}? Hint: Matlab has a **erfc** function that you can use.

The experimentally measured minimum number of photons required for a BEP of 10^{-9} for parameter values close to what we assumed is typically between 100 and 150 photons.

e) Repeat part (d) for a non-ideal amplifier (that is not fully inverted) and assume the following values:

$$G = 200 \quad n_{sp} = 3 \quad \Delta\tau = 25 \text{ ps (corresponding to a data rate of 40 Gb/s)} \quad \Delta\omega = \frac{8\pi}{\Delta\tau}$$

What is the minimum number of photons that must reach the receiver in order to get a BEP less than 10^{-9} in this case? **Lesson: the level of inversion in the amplifier affects the receiver performance.**

f) Assume that at the transmitter the average number of photons generated by the laser in the coherent-state packet for transmitting logical 1 is one million (i.e. $p_o = 10^6$). Typical power losses in the fiber are 0.2 dB/km, and this corresponds to a value of 0.023 km^{-1} for the constant γ for the lossy fiber. From the $p_o L$ value calculated in part (d) for a BEP of 10^{-9}, figure out the maximum distance that the data can travel on the lossy fiber before it gets too attenuated to be detected with a BEP of 10^{-9}. For the maximum distance you calculate, what is the value of the attenuation factor L ?

g) Suppose Alice and Bob come to you and tell you that they just developed a super light source that generates number-state photon packets (as opposed to coherent state photon packets). They claim that since number-state packets have zero variance in the photon number, their new light source should work much better in optical links and should reduce the BEP at the receiver compared to conventional lasers that (supposedly) produce coherent-state packets. Your job is to decide whether what Alice and Bob are saying makes sense.

Assume that the super light source generates number-state photon packets with the number equal to $p_o = 10^6$. Use the value for L determined in part (f), and use,

$G = 200 \quad n_{sp} = 1 \quad \Delta\tau = 25$ ps (corresponding to a data rate of 40 Gb/s) $\quad \Delta\omega = \dfrac{8\pi}{\Delta\tau}$

and calculate BEP when the super light source is used in the transmitter. Is the BEP much less than 10^{-9} – the value that coherent state packets achieved for the same parameter values and for the same average number of photons in the logical 1 state coming out of the transmitter? Is the light source of Alice and Bob really super?

Hint: The answer can be written in one line (i.e. no computation required!). Of course, you can also quickly plug in the values and get the answer since you have all the formulas.