ECE 5310: Quantum Optics for Photonics and Optoelectronics

Fall 2013

Homework 1 **Due on Sep. 10, 2013 by 5:00 PM (self-grade)**

Problem 1.1: (Commutators)

If the commutator of two operators $\hat{A}$ and $\hat{B}$ is $\left[\hat{A}\,,\hat{B}\right] = i\,\hbar$. Show that,

a) $\left[\hat{A}\,,\hat{B}^n\right] = i\,n\,\hbar\,\hat{B}^{n-1}$

b) $\left[\hat{A}\,,F\left(\hat{B}\right)\right] = i\,\hbar\,F'\left(\hat{B}\right)$ where F is any polynomial function and F' is its derivative.

c) $\exp\left(\hat{A}+\hat{B}\right) = \exp\left(-\frac{i\hbar}{2}\right)\exp\left(\hat{A}\right)\exp\left(\hat{B}\right)$

d) $\exp\left(\hat{A}+\hat{B}\right) = \exp\left(\frac{i\hbar}{2}\right)\exp\left(\hat{B}\right)\exp\left(\hat{A}\right)$

e) $\exp\left(-\frac{i}{\hbar}\alpha\hat{A}\right)\hat{B}\,\exp\left(\frac{i}{\hbar}\alpha\hat{A}\right) = \hat{B}+\alpha$, where α is a complex number.

Problem 1.2: (Measurement and wavefunction collapse)

Consider two physical quantities A and B, with corresponding Hermitian quantum operators $\hat{A}$ and $\hat{B}$, associated with an object. These operators have eigenvectors and eigenvalues given by:

$\hat{A}\left|a_k\right\rangle = \lambda_k\left|a_k\right\rangle$ (k=1,2,3,4,5,….)

$\hat{B}\left|b_k\right\rangle = \eta_k\left|b_k\right\rangle$ (k=1,2,3,4,5,….)

The inner products between the two sets of eigenvectors are given by $\left\langle a_k \middle| b_j\right\rangle = \chi_{kj}$. Suppose the quantum state of the object is $\left|\psi\right\rangle = \sum_k \alpha_k\left|a_k\right\rangle$, where $\left\langle\psi\middle|\psi\right\rangle = 1$. All eigenvalues of both the operators are distinct. Your answers for parts (a)-(d) must be expressed in terms of the expansion coefficients α_k and the inner products χ_{kj} .

a) A measurement of quantity is A is made. What is the probability of obtaining the result λ_j ? What is the wavefunction just after the measurement given that the result λ_j was obtained ?

b) A measurement of quantity is B is made. What is the probability of obtaining the result η_j ? What is the wavefunction just after the measurement given that the result η_j was obtained ?

c) A measurement of quantity B is made and the result η_j is obtained. Immediately after the first measurement a second measurement is made of quantity A. What is the probability of obtaining the result λ_j ?

Now suppose that the first three eigenvalues of $\hat{A}$ are the same (i.e. $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$) and the first three eigenvalues of $\hat{B}$ are also same ((i.e. $\eta_1 = \eta_2 = \eta_3 = \eta$). As before, the quantum state of the object is given as $|\psi\rangle = \sum_k \alpha_k |a_k\rangle$ and $\langle\psi|\psi\rangle = 1$.

c) A measurement of quantity is A is made. What is the probability of obtaining the result λ ? What is the wavefunction just after the measurement given that the result λ was obtained ?
d) A measurement of quantity is B is made. What is the probability of obtaining the result η ? What is the wavefunction just after the measurement given that the result η was obtained ?

Problem 1.3: (Time dependent two-level system)

The object of this problem is to show that with a suitable transformation the time *dependent* Hamiltonian of a two level system interacting with a classical electromagnetic field becomes a time *independent* Hamiltonian of the form studied in class. In the presence of a time dependent electric field the Hamiltonian of a two level system is,

$$\hat{H}(t) = \varepsilon_1 |e_1\rangle\langle e_2| \; + \; \varepsilon_2 |e_1\rangle\langle e_2| \; - q\,\hat{x}\,E_o\,\cos(\omega t)$$

In the two dimensional Hilbert space, the above Hamiltonian can be written as,

$$\hat{H}(t) = \varepsilon_1 |e_1\rangle\langle e_1| \; + \; \varepsilon_2 |e_2\rangle\langle e_2| \; - \; \frac{\hbar\Omega_R}{2}\left[2\cos(\omega t)|e_1\rangle\langle e_2| \; + 2\cos(\omega t)|e_2\rangle\langle e_1|\right]$$

where $\hbar\Omega_R = q\,E_o\,\langle e_2|\hat{x}|e_1\rangle = q\,E_o\,\langle e_1|\hat{x}|e_2\rangle$.

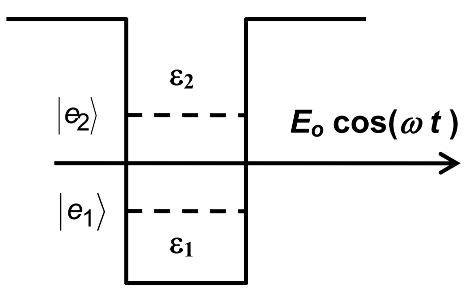

Fig. p1.3: Two level system

Note that $\cos(\omega t) = \frac{1}{2}\left[\exp(i\omega t) + \exp(-i\omega t)\right]$. In the so called *rotating wave approximation* only the important resonant term in each cosine term is retained (we will discuss this later in the course) and one obtains,

$$\hat{H}(t) = \varepsilon_1 |e_1\rangle\langle e_1| \;+\; \varepsilon_2 |e_2\rangle\langle e_2| \;-\; \frac{\hbar\Omega_R}{2}\left[\exp(i\omega t)\,|e_1\rangle\langle e_2| \;+\; \exp(-i\omega t)\,|e_2\rangle\langle e_1|\right]$$

$$= \varepsilon_1 \hat{N}_1 \;+\; \varepsilon_2 \hat{N}_2 \;-\; \frac{\hbar\Omega_R}{2}\left[\exp(i\omega t)\,\hat{\sigma}_- \;+\; \exp(-i\omega t)\,\hat{\sigma}_+\right]$$

The Hamiltonian above is used to describe the interaction of a classical electromagnetic field with a two level system. The quantum state $|\psi(t)\rangle$ of the system obeys the Schrodinger equation with a time *dependent* Hamiltonian,

$$i\hbar \frac{\partial |\psi(t)\rangle}{\partial t} = \hat{H}(t)\,|\psi(t)\rangle$$

a) Show that the following relation holds,

$$\exp\left(-i\omega\, \hat{N}_1\, t\right) = 1 + \hat{N}_1 \left[\exp(-i\omega\, t) - 1\right]$$

b) Show that the state $|\phi(t)\rangle$ defined by the relation,

$$|\phi(t)\rangle = \exp\left(-i\omega\, \hat{N}_1\, t\right)\, |\psi(t)\rangle$$

obeys the Schrodinger equation with a time *independent* Hamiltonian, i.e.

$$i\hbar \frac{\partial |\phi(t)\rangle}{\partial t} = \hat{H}_R\, |\phi(t)\rangle$$

where $\hat{H}_R$ is given by,

$$\hat{H}_R = (\varepsilon_1 + \hbar\omega)\hat{N}_1 \;+\; \varepsilon_2 \hat{N}_2 \;-\; \frac{\hbar\Omega_R}{2}\left[\,\hat{\sigma}_- \;+\; \hat{\sigma}_+\right]$$

For parts (c)-(f) assume $\varepsilon_2 - (\varepsilon_1 + \hbar\omega) = 0$ (i.e. the electromagnetic field frequency is perfectly tuned with the energy level spacing of the two level system).

c) Given the initial condition $|\psi(t=0)\rangle = |\phi(t=0)\rangle = |e_1\rangle$, calculate $|\phi(t)\rangle$.

d) Using $|\phi(t)\rangle$ determined in part (c), calculate $|\psi(t)\rangle$.

e) Using $|\psi(t)\rangle$ determined in part (d), evaluate $|\langle e_1|\psi(t)\rangle|^2$.

f) Using the relation given earlier, $\hbar\Omega_R = q\,E_o\,\langle e_2|\hat{x}|e_1\rangle = q\,E_o\,\langle e_1|\hat{x}|e_2\rangle$, calculate the mean position of the particle (defined as $\langle\psi(t)|\hat{x}|\psi(t)\rangle$). What is the frequency of oscillation of the mean particle position? What kind of motion is the particle doing and is it what you expected?

Problem 1.4: (A detuned two level system)

The Hamiltonian of a coupled two level system is as follows,

$$\hat{H} = \varepsilon_1 |e_1\rangle\langle e_1| \; + \; \varepsilon_2 |e_2\rangle\langle e_2| \; - \; U \left[|e_1\rangle\langle e_2| \; + \; |e_2\rangle\langle e_1| \right]$$

Or equivalently,

$$\hat{H} = \varepsilon_1 \hat{N}_1 \; + \; \varepsilon_2 \hat{N}_2 \; - \; U \left[\hat{\sigma}_- \; + \; \hat{\sigma}_+ \right]$$

where $\varepsilon_2 - \varepsilon_1 = \Delta$ (i.e. the two energy levels don't have the same energy – they are detuned)

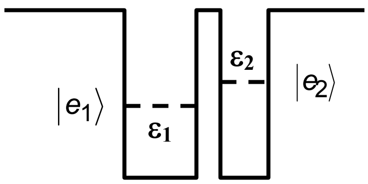

Fig. p1.4: Detuned coupled quantum well system

The object of this problem is to understand the effects of detuning on the temporal dynamics and also get some practice with Schrodinger and Heisenberg pictures.

a) Find the energy eigenvalues of the hamiltonian $\hat{H}$ in terms of the parameters ε_2 , ε_1 , Δ , U .

b) Fine the energy eigenvectors of the hamiltonian $\hat{H}$ in terms of the vectors $|e_1\rangle$ and $|e_2\rangle$.

Suppose the quantum state at time t=0 is given as $|\psi(t=0)\rangle = |e_1\rangle$.

c) Find $|\langle e_1 | \psi(t)\rangle|^2$ using the Schrodinger picture of time evolution.

d) Without doing any major computation, and using your result from part (c), find $|\langle e_2 | \psi(t)\rangle|^2$.

e) What happens to the temporal dynamics when the energy level detuning is very large?

Again, suppose the quantum state at time t=0 is given as $|\psi(t=0)\rangle = |e_1\rangle$

f) Find $|\langle e_1 | \psi(t)\rangle|^2$ using the Heisenberg picture of time evolution. This can be a long and hard problem.

Problem 1.5: (Trace operation)

The trace of an operator is defined as,

$$\text{Trace}\left[\hat{A}\right] = \sum_k \langle v_k | \hat{A} | v_k \rangle$$

where the vectors $|v_k\rangle$ are any set of vectors that form a complete set (i.e. $\sum_k |v_k\rangle\langle v_k| = \hat{1}$). In the matrix representation, the trace is the sum of the diagonal elements of the matrix representing the operator.

a) Show that the trace of an operator does not depend on the basis set chosen to do the computation (i.e. $\mathrm{Trace}\left[\hat{A}\right] = \sum_k \langle v_k | \hat{A} | v_k \rangle = \sum_j \langle e_j | \hat{A} | e_j \rangle$, where $|e_j\rangle$ is some other complete basis set).

b) Show that $\mathrm{Trace}\left[\hat{A}\,\hat{B}\,\hat{C}\right] = \mathrm{Trace}\left[\hat{C}\,\hat{A}\,\hat{B}\right] = \mathrm{Trace}\left[\hat{B}\,\hat{C}\,\hat{A}\right]$ (cyclic permutation of operators under trace)

Problem 1.6: (density operators)

The states $|e_+\rangle$ and $|e_-\rangle$ are defined as,

$$|e_+\rangle = \frac{1}{\sqrt{2}}\left(|e_1\rangle + |e_2\rangle\right) \qquad\qquad |e_-\rangle = \frac{1}{\sqrt{2}}\left(|e_1\rangle - |e_2\rangle\right)$$

where $|e_1\rangle$ and $|e_2\rangle$ are **energy eigenstates** with energies ε_1 and ε_2, respectively. Consider the following two sets of quantum states at time t=0:

Set A: A large number of identical copies of the state $|e_+\rangle$.

Set B: A mixture of a large number of states $|e_1\rangle$ and $|e_2\rangle$ in which the numbers of both the states are equal.

a) Write down the density matrices $\hat{\rho}_A$ and $\hat{\rho}_B$ (in 2x2 matrix representation in which column vectors $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ represent $|e_1\rangle$ and $|e_2\rangle$, respectively) for states belonging to set A and set B, respectively, at time t=0.

b) Suppose somebody gives you a set and you need to determine whether the set given to you is A or B by performing measurements. From your knowledge of the dynamics of two level systems can you think of measurements that will enable you to distinguish between set A and set B?

(Hint: try measuring the value of the operator $\hat{K}$ as a function of time, where $\hat{K} = |e_-\rangle\langle e_-|$. In other words, determine the mean value of the operator $\hat{K}$ as a function of time with respect to the density matrices $\hat{\rho}_A(t)$ and $\hat{\rho}_B(t)$ and see if you get different results.)

c) Now suppose there is a set C which includes a mixture of a large number of states $|e_+\rangle$ and $|e_-\rangle$ in which the numbers of both the states are equal. Write down the density matrix $\hat{\rho}_C(t)$ for a state belonging to the set C. Can you think of any measurements that can distinguish set B from set C?