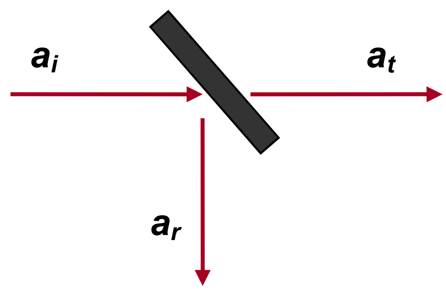
Chapter 9: Loss in Quantum Optics

9.1 Optical Beam Splitters: An Introduction

Describing photon loss in quantum optics is not as straight forward as in classical optics. In this section, we will see what happens when an optical beam is attenuated or when it is suffers a loss. The simplest consistent picture of loss is obtained with an optical beam splitter and the results can be used to model linear optical losses of any kind.

9.1.1 Classical Description of a Beam Splitter

Suppose a continuous wave optical beam is incident on a beam splitters as shown below,



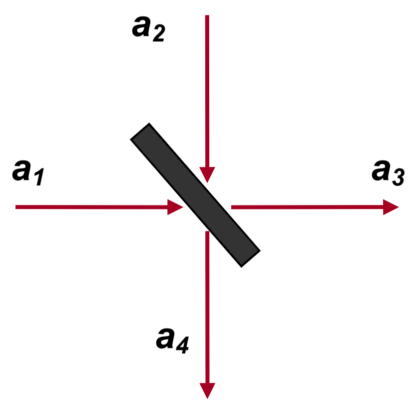
If the amplitude of the incident beam is a_i, and its power is P_i (where $P_i = |a_i|^2$), then the amplitude and power of the transmitted beam are,

$$a_t = t\,a_i \qquad P_t = |t|^2 P_i$$

and for the reflected beam we have,

$$a_r = r\,a_i \qquad P_r = |r|^2 P_i$$

A beam splitter is actually a four-port system, with two input and two output ports, as shown below.



The amplitudes at the output parts are related to the amplitudes at the input parts in the most general way by the scattering matrix (also called the S-matrix),

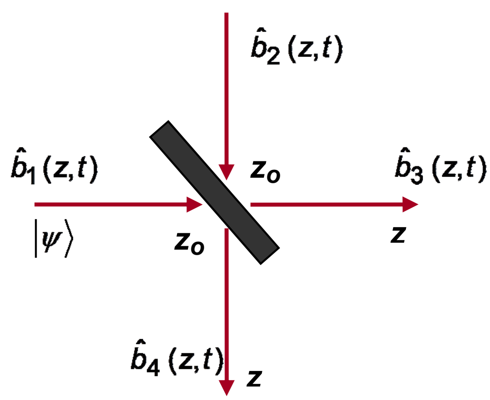
$$\begin{bmatrix} a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} t & r \\ -r^* & t^* \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

The form of the matrix above conserves power and preserves the time reversal symmetry,

$$
\begin{aligned}
\text{Outgoing power} &= |a_3|^2 + |a_4|^2 \\
&= |ta_1 + ra_2|^2 + |-r^* a_1 + t^* a_2|^2 \\
&= |t|^2 |a_1|^2 + |r|^2 |a_2|^2 + t^* r\, a_1^* a_2 + tr^* + a_1 a_2^* \\
&\quad + |r|^2 |a_1|^2 + |t|^2 |a_2|^2 - t\, r^* a_1 a_2^* - t^* r\, a_1^* a_2 \\
&= \left(|t|^2 + |r|^2\right)\left(|a_1|^2 + |a_2|^2\right) \\
&= |a_1|^2 + |a_2|^2 \qquad \left\{\text{since } |t|^2 + |r|^2 = 1\right\} \\
&= \text{Incoming power}
\end{aligned}
$$

9.2 Quantum Description of a Beam Splitter

Quantum description of a beam splitter is a little more complicated than the classical description. Suppose we have a quantum state $|\psi\rangle$ of light coming in from the input port 1, as show below,



We want to see what the beam splitter does to $|\psi\rangle$. Looking at quantum states implies working in the Schrodinger picture. Beam splitters are handled in a simpler way if one works with operators instead.

9.2.1 Beam Splitter in the Heisenberg Picture

In the Figure above, we have four ports coming together at a beam splitter. There are two input ports and two output ports. For simplicity, the vertical as well as the horizontal axes are labeled with z and the beam splitter is located $z = z_o$ on both axes. The destruction operators for the two input ports are $\hat{b}_1(z,t)$ and $\hat{b}_2(z,t)$, and the operators for the two output ports are $\hat{b}_3(z,t)$ and $\hat{b}_4(z,t)$.

The initial state $|\psi(t=0)\rangle$ is coming in on port 1. $|\psi(t=0)\rangle$ can be any "photon packet" created at time $t=0$ and localized at $z = z_1$ where $z_1 < z_o$. For example, a single-photon packet would be,

$$|\psi(t=0)\rangle = \int_{-\infty}^{\infty} dz'\, A(z')\, \hat{b}_1^+(z',0)|0\rangle = |1\rangle_1$$

where,

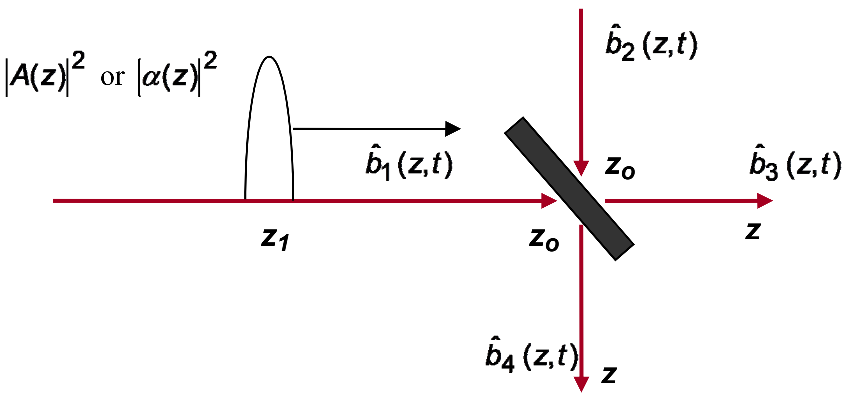
$$\int_{-\infty}^{\infty} dz\, |A(z)|^2 = 1$$

$|A(z)|^2$ is completely localized in the input port and is depicted below. A coherent state packet, again localized at $z = z_1$ at $t = 0$ would be,

$$|\psi(t=0)\rangle = e^{\int_{-\infty}^{\infty} dz' \left\{ \alpha(z')\,\hat{b}^{+}(z',0) - \alpha^{*}(z')\,\hat{b}(z',0) \right\}} |0\rangle = |\alpha(z)\rangle_1$$

where,

$$\int_{-\infty}^{\infty} |\alpha(z)|^2\, dz = N_o = \text{average number of particles}$$



At $z = z_o$, the relation between the input and output operators for all times t is,

$$\begin{bmatrix} \hat{b}_3(z_o,t)\, e^{i\beta_o z_o - i\omega_o t} \\ b_4(z_o,t)\, e^{i\beta_o z_o - i\omega_o t} \end{bmatrix} = \begin{bmatrix} t & r \\ -r^{*} & t^{*} \end{bmatrix} \begin{bmatrix} \hat{b}_1(z_o,t)\, e^{i\beta_o z_o - i\omega_o t} \\ \hat{b}_2(z_o,t)\, e^{i\beta_o z_o - i\omega_o t} \end{bmatrix}$$

Since we have judiciously chosen the location of the beam splitter to be z_o on both the vertical and the horizontal axes, the phase factors will cancel out and one is left with the simpler matrix relation,

$$\begin{bmatrix} \hat{b}_3(z_o,t) \\ b_4(z_o,t) \end{bmatrix} = \begin{bmatrix} t & r \\ -r^{*} & t^{*} \end{bmatrix} \begin{bmatrix} \hat{b}_1(z_o,t) \\ \hat{b}_2(z_o,t) \end{bmatrix}$$

For $z \geq z_o$, for all time t,

$$\hat{b}_3(z,t) = \hat{b}_3\left(z_o + (z - z_o), t\right) = \hat{b}_3\left(z_o, t - \frac{z - z_o}{v_g}\right)$$

$$\Rightarrow \hat{b}_3(z,t) = t\,\hat{b}_1\left(z_o, t - \frac{z - z_o}{v_g}\right) + r\,\hat{b}_2\left(z_o, t - \frac{z - z_o}{v_g}\right)$$

One can verify that the commutation relation at the output ports are preserved,

$$\left[\hat{b}_3(z,t),\hat{b}_3(z',t)\right]=|t|^2\left[\hat{b}_1\left(z_o,t-\frac{z-z_o}{v_g}\right),\hat{b}_1^+\left(z_o,t-\frac{z'-z_o}{v_g}\right)\right]$$
$$+|r|^2\left[\hat{b}_2\left(z_o,t-\frac{z-z_o}{v_g}\right),\hat{b}_2^+\left(z_o,t-\frac{z'-z_o}{v_g}\right)\right]$$
$$=|t|^2\left[\hat{b}_1\left(z_o-(z'-z),t-\frac{z'-z_o}{v_g}\right),\hat{b}_1^+\left(z_o,t-\frac{z'-z_o}{v_g}\right)\right]$$
$$+|r|^2\left[\hat{b}_2\left(z_o-(z'-z),t-\frac{z'-z_o}{v_g}\right),\hat{b}_2^+\left(z_o,t-\frac{z'-z_o}{v_g}\right)\right]$$
$$=|t|^2\,\delta(z-z')+|r|^2\,\delta(z-z')$$
$$=\delta(z-z')$$

Therefore, the matrix relation above for the beam splitter also preserves the commutation relations for the output ports when they hold for the input ports.

The photon flux operators for all channels are,

$$\hat{F}_k(z,t)=v_g\,\hat{b}_k^+(z,t)\hat{b}_k(z,t)\qquad\{k=1,2,3,4\}$$

Suppose the incoming quantum state at time $t=0$ is $|\psi\rangle$ where,

$$|\psi(t=0)\rangle=|\phi\rangle_1\otimes|0\rangle_2$$

Where $|\phi\rangle_1$ is the state in port 1, and port 2 is in the vacuum state, This means,

$$\langle\psi(t=0)|\hat{F}_2(z,t)|\psi(t=0)\rangle=0$$

And,

$$\langle\psi(t=0)|\hat{F}_1(z,t)|\psi(t=0)\rangle$$

is the average incident flux in port 1 at location z $(z\le z_o)$ at time t.

9.2.2 Outgoing Photon Flux

We need to find the average photon fluxes in the two output ports at time t that is large enough that the quantum state has gone past the beam splitter.

We first evaluate, $\langle\psi(t=0)|\hat{F}_3(z,t)|\psi(t=0)\rangle$ for $z\ge z_o$ and $z-v_g t\le z_o$,

$$\langle\psi(t=0)|\hat{F}_3(z,t)|\psi(t=0)\rangle$$
$$=v_g\langle\psi(t=0)|\hat{b}_3^+(z,t)\hat{b}_3(z,t)|\psi\rangle=v_g\langle\psi|\hat{b}_3^+(z_o+(z-z_o),t)\hat{b}_3(z_o+(z-z_o),t)|\psi(t=0)\rangle$$
$$=v_g\langle\psi(t=0)|\hat{b}_3^+\left(z_o,t-\frac{z-z_o}{v_g}\right)\hat{b}_3\left(z_o,t-\frac{z-z_o}{v_g}\right)|\psi(t=0)\rangle$$

But,

$$\hat{b}_3^+\left(z_o,t-\frac{z-z_o}{v_g}\right)=t\,\hat{b}_1\left(z_o,t-\frac{z-z_o}{v_g}\right)+r\,\hat{b}_2\left(z_o,t-\frac{z-z_o}{v_g}\right)$$

Therefore,

$$\langle\psi(t=0)|\hat{F}_3(z,t)\,|\psi(t=0)\rangle = v_g\,\langle\psi(t=0)|\left[t^*\hat{b}_1^+\left(z_o,t-\frac{z-z_o}{v_g}\right)+r^*\,\hat{b}_2^+\left(z_o,t-\frac{z-z_o}{v_g}\right)\right]$$

$$\left[t\hat{b}_1\left(z_o,t-\frac{z-z_o}{v_g}\right)+r\,\hat{b}_2\left(z_o,t-\frac{z-z_o}{v_g}\right)\right]|\psi(t=0)\rangle$$

$$= v_g\,|t|^2\,{}_1\langle\phi|\,\hat{b}_1^+\left(z_o,t-\frac{z-z_o}{v_g}\right)\hat{b}_1\left(z_o,t-\frac{z-z_o}{v_g}\right)|\phi\rangle_1$$

$$= v_g|t|^2\,{}_1\langle\phi|\,\hat{b}_1^+(z-v_g t,0)\hat{b}_1(z-v_g t,0)|\phi\rangle_1$$

$$\langle\psi|\,\hat{F}_3(z,t)\,|\psi\rangle = |t|^2\,{}_1\langle\phi|\,\hat{F}_1(z-v_g t,0)|\phi\rangle_1 \qquad \left\{ z-v_g t\le z_o\right.$$

Similarly, one can show that,

$$\langle\psi(t=0)|\hat{F}_4(z,t)\,|\psi(t=0)\rangle = |r|^2\,{}_1\langle\phi|\,\hat{F}_1(z-v_g t,0)|\phi\rangle_1$$

9.2.3 Outgoing Photon Flux Correlations

As in the classical splitter, the noise introduced by the beam splitter can be studied by looking at the photon flux correlation functions. We first calculate the photon flux correlation function,

$$\langle\psi(t=0)|\hat{F}_3(z,t_1)\hat{F}_3(z,t_2)|\psi(t=0)\rangle$$

in the output port 3 for $z\ge z_o$ and $z-v_g t_1\le z_o$ and $z-v_g t_2\le z_o$.

$$\langle\psi(t=0)|\hat{F}_3(z,t_1)\hat{F}_3(z,t_2)|\psi(t=0)\rangle = \langle\psi(t=0)|\hat{F}_3\left(z_o,t_1-\frac{z-z_o}{v_g}\right)\hat{F}_3\left(z_o,t_2-\frac{z-z_o}{v_g}\right)|\psi(t=0)\rangle$$

$$= v_g^2\langle\psi(t=0)|\left[t^*\hat{b}_1^+\left(z_o,t_1-\frac{z-z_o}{v_g}\right)+r^*\hat{b}_2^+\left(z_o,t_1-\frac{z-z_o}{v_g}\right)\right]$$

$$\left[t\,\hat{b}_1\left(z_o,t_1-\frac{z-z_o}{v_g}\right)+r\,\hat{b}_2\left(z_o,t_1-\frac{z-z_o}{v_g}\right)\right]$$

$$\left[t^*\hat{b}_1^+\left(z_o,t_2-\frac{z-z_o}{v_g}\right)+r^*\hat{b}_2^+\left(z_o,t_2-\frac{z-z_o}{v_g}\right)\right]$$

$$\left[t\hat{b}_1\left(z_o,t_2-\frac{z-z_o}{v_g}\right)+r\,\hat{b}_2\left(z_o,t_2-\frac{z-z_o}{v_g}\right)\right]|\psi(t=0)\rangle$$

The only non-zero terms are,

$$= |t|^4 \langle \psi(t=0)| \hat{F}_1(z - v_g t_1, 0)\, \hat{F}_1(z - v_g t_2, 0) |\psi(t=0)\rangle$$
$$+ v_g^2 |t|^2 |r|^2 \langle \psi(t=0)| \hat{b}_1^+ (z - v_g t_1) \hat{b}_1 (z - v_g t_2) \hat{b}_2 (z - v_g t_1, 0) \hat{b}_2^+ (z - v_g t_2, 0) |\psi(t=0)\rangle$$
$$= |t|^4 \langle \psi(t=0)| \hat{F}_1(z - v_g t_1, 0)\, \hat{F}_1(z - v_g t_2, 0) |\psi(t=0)\rangle$$
$$+ v_g^2 |t|^2 |r|^2 \,{}_1\langle \phi | \hat{b}_1^+ (z - v_g t_1, 0) \hat{b}_1 (z - v_g t_2, 0) |\phi\rangle_1 \,{}_2\langle 0 | \hat{b}_2 (z - v_g t_1, 0) \hat{b}_2^+ (z - v_g t_2, 0) | 0\rangle_2$$
$$= |t|^4 \langle \psi(t=0)| \hat{F}_1(z - v_g t_1, 0)\, \hat{F}_1(z - v_g t_2, 0) |\psi(t=0)\rangle$$
$$+ v_g |t|^2 |r|^2 \,{}_1\langle \phi | \hat{b}_1^+ (z - v_g t_1, 0) \hat{b}_1 (z - v_g t_2, 0) |\phi\rangle_1 \; \delta(t_1 - t_2)$$

We have used the fact that,

$$_2\langle 0 | \hat{b}_2 (z - v_g t_1, 0) \hat{b}_2^+ (z - v_g t_2, 0) | 0 \rangle_2$$
$$= {}_2\langle 0 | \left[\hat{b}_2 (z - v_g t_1, 0),\, \hat{b}_2^+ (z - v_g t_2, 0) \right] + \hat{b}_2^+ (z - v_g t_2, 0) \hat{b}_2 (z - v_g t_1, 0) | 0 \rangle_2$$
$$= \delta\big(-v_g (t_1 - t_2)\big)\, {}_2\langle 0 | 0 \rangle_2 = \frac{1}{v_g} \delta(t_1 - t_2)$$

Note the final result is,

$$\langle \psi(t=0)| \hat{F}_3 (z, t_1) \hat{F}_3 (z, t_2) |\psi(t=0)\rangle = |t|^4 \,{}_1\langle \phi | \hat{F}_1(z - v_g, t_1, 0) \hat{F}_2 (z - v_g\, t_2, 0) |\phi\rangle_1$$
$$+ |t|^2 |r|^2 \,{}_1\langle \phi | \hat{F}_1(z - v_g\, t_1, 0) |\phi\rangle_1 \; \delta(t_1 - t_2).$$

The noise in the photon flux, defined as,

$$\Delta \hat{F}_k (z,t) = \hat{F}_k (z,t) - \left\langle \hat{F}_k (z,t) \right\rangle \qquad \{k = 1,2,3,4\}$$

can now be computed for output port 3,

$$\langle \psi(t=0)| \Delta\hat{F}_3 (z, t_1) \Delta\hat{F}_3 (z, t_2) |\psi(t=0)\rangle = |t|^4 \,{}_1\langle \phi | \Delta\hat{F}_1(z - v_g t_1, 0) \Delta\hat{F}_1 (z - v_g t_2, 0) |\phi\rangle_1$$
$$+ |t|^2 |r|^2 \,{}_1\langle \phi | \hat{F}_1(z - v_g t_1, 0) |\phi\rangle_1 \; \delta(t_1 - t_2)$$

The first term on the right hand side represents the photon flux noise in the input port 1 attenuated by the beam splitter. The second term represents the noise added by the beam splitter. Notice that this second term is non-zero only because of the vacuum fluctuations that come in from the input port 2 of the beam splitter. Therefore, the noise added by the beam splitter can be understood as due to the interference between the input signal in port 1 and the vacuum fluctuations entering the beam splitter from the input port 2. The spectral density of the photon flux noise in port 3 is,

$$S_{\Delta F_3\, \Delta F_3}(\omega) = |t|^4 S_{\Delta F_1 \Delta F_1}(\omega) + |r|^2 |t|^2 \,{}_1\langle \phi | \hat{F}_1(z - v_g t_1, 0) |\phi\rangle_1$$

If we let, $|t|^2 = \eta$, $|r|^2 = 1 - \eta$, then we get the same result as obtained in the classical analysis,

$$S_{\Delta F_3\, \Delta F_3}(\omega) = \eta\, (1-\eta)\, {}_1\langle \phi | \hat{F}_1(z - v_g t_1, 0) |\phi\rangle_1 + \eta^2\, S_{\Delta F_1 \Delta F_1}(\omega)$$

Similary, for the output port 4 one obtains,

$$\langle \psi | \hat{F}_4 (z, t_1) \hat{F}_4 (z, t_2) | \psi \rangle$$
$$= |r|^4 \,{}_1\langle \phi | \hat{F}_1(z - v_g t_1, 0) \hat{F}_1(z - v_g t_2, 0) | \phi \rangle_1$$
$$+ |r|^2 |t|^2 \,{}_1\langle \phi | \hat{F}_1(z - v_g t_1, 0) |\phi\rangle_1 \; \delta(t_1 - t_2)$$

and

$$\langle\psi(t=0)|\,\Delta\hat{F}_4(z,t_1)\Delta\hat{F}_4(z,t_2)\,|\psi(t=0)\rangle$$
$$= |r|^4 \;{}_1\langle\phi|\,\Delta\hat{F}_1(z-v_g t_1,0)\Delta\hat{F}_1(z-v_g t_2,0)\,|\phi\rangle_1$$
$$+ |t|^2|r|^2 \;{}_1\langle\phi|\hat{F}_1(z-v_g t_1,0)|\phi\rangle_1 \;\delta(t_1-t_2).$$
$$\Rightarrow S_{\Delta F_4\;\Delta F_4}(\omega) = |r|^4 S_{\Delta F_1\,\Delta F_1}(\omega) + |r|^2|t|^2 \;{}_1\langle\phi|\hat{F}_1(z-v_g t_1,0)|\phi\rangle_1$$

Example of a Continuous Wave Coherent State: Suppose,

$$|\psi(t=0)\rangle = |\phi\rangle_1 \otimes |0\rangle_2$$

where $|\phi\rangle_1$ is a continuous wave coherent state,

$$|\phi\rangle_1 = e^{\int_{-\infty}^{\infty} dz'\left\{\alpha(z')\hat{b}_1^+(z',0) - \alpha^*(z')\hat{b}_1(z',0)\right\}}|0\rangle_1 = |\alpha(z)\rangle_1$$

and,

$$|\alpha(z)|^2 = \frac{P_o}{v_g\hbar\omega_o}$$

The average photon flux is equal to,

$$\langle\psi(t=0)|\hat{F}_1(z,t)|\psi(t=0)\rangle = \frac{P_o}{\hbar\omega_o} \qquad \text{for } z \le z_o$$

The photon flux noise correlation for the input is,

$$\langle\psi(t=0)|\,\Delta\hat{F}_1(z,t_1)\Delta\hat{F}_1(z,t_2)\,|\psi(t=0)\rangle = \frac{P_o}{\hbar\omega_o}\delta(t_1-t_2) \qquad \text{for } z \le z_o$$

The average output flux in port 3 is,

$$\langle\psi(t=0)|\hat{F}_3(z,t)|\psi(t=0)\rangle = |\,t\,|^2 \;{}_1\langle\phi|\hat{F}_1(z-v_g t,o)|\phi\rangle_1 = |\,t|^2\frac{P_o}{\hbar\omega_o}.$$

The noise in the flux in port 3 is,

$$\langle\psi(t=0)|\Delta\hat{F}_3(z,t_1)\Delta\hat{F}_3(z,t_2)|\psi(t=0)\rangle = |t|^4 \;{}_1\langle\phi|\Delta\hat{F}_1(z-v_g t_1,0)\Delta\hat{F}_1(z-v_g t_1,0)|\phi\rangle$$
$$+ |r|^2|t|^2 \;{}_1\langle\phi|\hat{F}_1(z-v_g t_1,0)|\phi\rangle_1\delta(t_1-t_2)$$

$$\langle\psi(t=0)|\Delta\hat{F}_3(z,t_1)\Delta\hat{F}_3(z,t_2)|\psi(t=0)\rangle = |t|^4\left(\frac{P_o}{\hbar\omega_o}\right)\delta(t_1-t_2) + |r|^2|t|^2\left(\frac{P_o}{\hbar\omega_o}\right)\delta(t_1-t_2)$$

The spectral density of the photon flux noise in port 3 is,

$$S_{\Delta F_3\;\Delta F_3}(\omega) = |t|^4\frac{P_o}{\hbar\omega_o} + |r|^2|t|^2\frac{P_o}{\hbar\omega_o}$$

If we let $|\,t|^2 = \eta$, $|\,r\,|^2 = 1-\eta$, then we get the result familiar from the classical analysis in the case when the input had shot noise,

$$S_{\Delta F_3\;\Delta F_3}(\omega) = \eta(1-\eta)\frac{P_o}{\hbar\omega_o} + \eta^2 S_{\Delta F_1\;\Delta F_1}(\omega)$$
$$= \eta(1-\eta)\frac{p_o}{\hbar\omega_o} + \eta^2\frac{p_o}{\hbar\omega_o} = \eta\frac{p_o}{\hbar\omega_o}$$

We see that when the input has shot noise the output also has shot noise.

Example of a Coherent State Packet: Suppose,

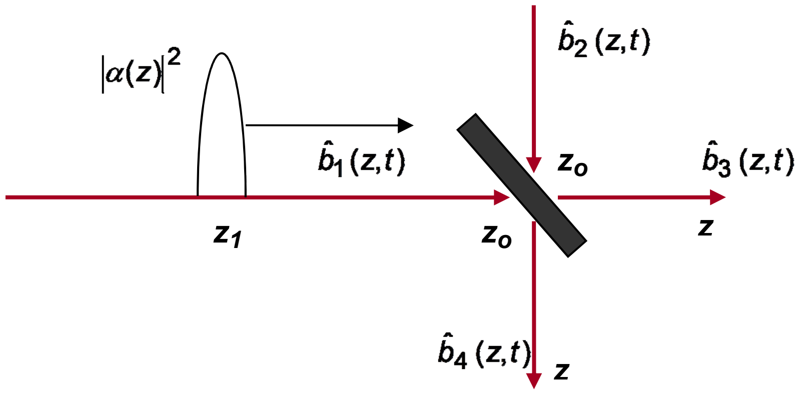
$$|\psi(t=0)\rangle = |\phi\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3 \otimes |0\rangle_4 = |\phi\rangle_1 \otimes |0\rangle_2$$

where,

$$|\phi\rangle_1 = e^{\int_{-\infty}^{\infty} dz' \left\{ \alpha(z')\hat{b}_1^{+}(z',0) - \alpha^*(z')\hat{b}_1(z',0) \right\}} |0\rangle_1 = |\alpha(z)\rangle_1$$

and,

$$\int_{-\infty}^{\infty} dz' |\alpha(z')|^2 = N_o = \text{average number of photons in the packet}$$



The photon number operator for packets at location z is,

$$\hat{N}_k(z) = \int_{-\infty}^{\infty} dt\, \hat{F}_k(z,t)$$

We have,

$${}_1\langle\phi|\hat{N}_1(z)|\phi\rangle_1 = N_o \qquad \text{for } z \le z_o$$

$${}_1\langle\phi|\Delta\hat{N}_1^2(z)|\phi\rangle_1 = N_o \qquad \text{for } z \le z_o$$

Also, since,

$$\langle\psi(t=0)|\hat{F}_3(z,t)|\psi(t=0)\rangle = |t|^2\, {}_1\langle\phi|\hat{F}_1(z - v_g t, 0)|\phi\rangle_1 \quad \begin{cases} z \ge z_o \\ z - v_g t \le z_o \end{cases}$$

One can integrate over time t from $\dfrac{z - z_o}{v_g}$ to $+\infty$ to get the average photon number in the transmitted packet in port 3,

$$\langle\psi(t=0)|\hat{N}_3(z)|\psi(t=0)\rangle = |t|^2\, {}_1\langle\phi| \int_{\frac{z-z_o}{v_g}}^{\infty} \hat{F}_1(z - v_g t, 0)\, dt\, |\phi\rangle_1$$

$$= |t|^2\, {}_1\langle\phi| \int_{-\infty}^{z_o} dz\, \hat{F}_1(z,0) |\phi\rangle_1$$

$$= |t|^2 \int_{-\infty}^{z_o} dz\, |\alpha(z)|^2 = |t|^2 N_o$$

As expected, one losses $\left(1-|t|^2\right)N_o$ photons in the splitter. What about fluctuations in the photon number of the output packet in port 3?

$$\langle\psi(t=0)|\hat{N}_3^{\,2}(z)|\psi(t=0)\rangle = \langle\psi(t=0)|\int_{\frac{z-z_o}{v_g}}^{\infty} dt_1 \int_{\frac{z-z_o}{v_g}}^{\infty} dt_2\, \hat{F}_3(z,t_1)\hat{F}_3(z,t_2)|\psi(t=0)\rangle$$

$$= \int_{\frac{z-z_o}{v_g}}^{\infty} dt_1 \int_{\frac{z-z_o}{v_g}}^{\infty} dt_2 \left\{ \begin{array}{l} |t|^4 \,{}_1\langle\phi|\hat{F}_1(z-v_g t_1,0)\hat{F}_1(z-v_g t_2,0)|\phi\rangle_1 \\ +|r|^2|t|^2 \,{}_1\langle\phi|\hat{F}_1(z-v_g t_1,0)|\phi\rangle_1 \delta(t_1-t_2) \end{array} \right\}$$

$$\langle\psi(t=0)|\hat{N}_3^{\,2}(z)|\psi(t=0)\rangle = |t|^4 N_o(1+N_o)+|r|^2|t|^2 N_o$$

Let,

$$|t|^2 = \eta$$

$$|r|^2 = 1-\eta$$

then,

$$\langle\psi(t=0)|\Delta\hat{N}_3^{\,2}(z)|\psi(t=0)\rangle = \eta(1-\eta)N_o + \eta^2 N_o = \eta N_o$$

The fluctuation in the photon number in the transmitted packet is also equal to the mean photon number. The properties of both continuous wave coherent states and coherent state packets subjected to loss from the beam splitter point to the fact that the output states are also continuous wave coherent states and coherent state packets, respectively. This property will be proven next.

9.2.4 Beam Splitter in the Schrodinger Picture

Example of a Coherent State Packet: Suppose,

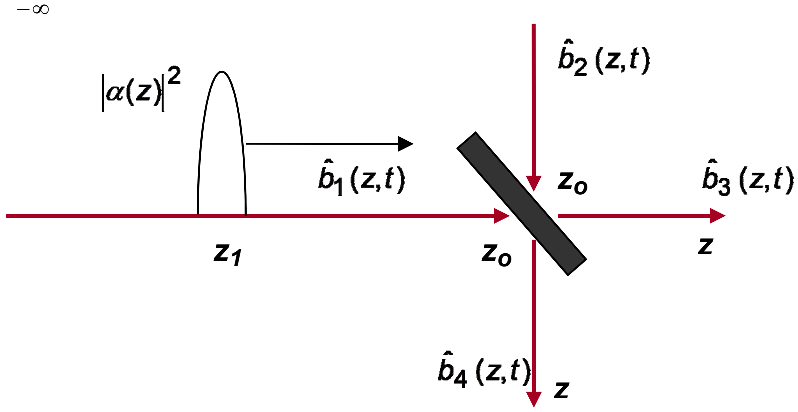
$$|\psi(t=0)\rangle = |\phi\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3 \otimes |0\rangle_4 = |\phi\rangle_1 \otimes |0\rangle_2$$

where,

$$|\phi\rangle_1 = e^{\int_{-\infty}^{\infty} dz' \left\{ \alpha(z')\hat{b}_1^{+}(z',0) - \alpha^*(z')\hat{b}_1(z',0) \right\}} |0\rangle_1 = |\alpha(z)\rangle_1$$

and,

$$\int_{-\infty}^{z_o} dz' |\alpha(z')|^2 = N_o = \text{average number of photons in the packet}$$



We want to find the state $|\psi(t)\rangle$ at a later time t which is large enough such that the packet has by then gone past the beam splitter. Recall that,

$$\hat{b}_k(z,t)e^{-i\omega_o t} = e^{i\frac{\hat{H}}{\hbar}t}\,\hat{b}_k(z,0)\,e^{-i\frac{\hat{H}}{\hbar}t}$$

We get,

$$|\psi(t)\rangle = e^{-i\frac{\hat{H}}{\hbar}t}|\psi(t=0)\rangle$$
$$= e^{-i\frac{\hat{H}}{\hbar}t}\,e^{\int_{-\infty}^{\infty}dz'\left\{\alpha(z')\hat{b}_1^{+}(z',0)-\alpha^*(z')\hat{b}_1(z',0)\right\}}|0\rangle$$
$$= e^{-i\frac{\hat{H}}{\hbar}t}\,e^{\int_{-\infty}^{z_o}dz'\left\{\alpha(z')\hat{b}_1^{+}(z',0)-\alpha^*(z')\hat{b}_1(z',0)\right\}}e^{+i\frac{\hat{H}}{\hbar}t}\,e^{-i\frac{\hat{H}}{\hbar}t}|0\rangle$$
$$= e^{\int_{-\infty}^{\infty}dz'\left\{\alpha(z')\hat{b}_1^{+}(z',-t)e^{-i\omega_o t}-\alpha^*(z')\hat{b}_1(z',-t)e^{i\omega_o t}\right\}}|0\rangle$$
$$= e^{\int_{-\infty}^{\infty}dz'\left\{\alpha(z')e^{-i\omega_o t}\hat{b}_1^{+}\left(z_o,-t-\frac{z'-z_o}{v_g}\right)-\alpha^*(z')e^{i\omega_o t}\hat{b}_1\left(z_o,-t-\frac{z'-z_o}{v_g}\right)\right\}}|0\rangle$$

The infinite vacuum energy has been ignored above. The beam splitter relation,

$$\begin{bmatrix}\hat{b}_3(z_o,t)\,e^{i\beta_o z_o-i\omega_o t}\\ b_4(z_o,t)\,e^{i\beta_o z_o-i\omega_o t}\end{bmatrix}=\begin{bmatrix}t & r\\ -r^* & t^*\end{bmatrix}\begin{bmatrix}\hat{b}_1(z_o,t)\,e^{i\beta_o z_o-i\omega_o t}\\ \hat{b}_2(z_o,t)\,e^{i\beta_o z_o-i\omega_o t}\end{bmatrix}$$

gives,

$$\hat{b}_1(z_o,t)=t^*\,\hat{b}_3(z_o,t)-r\,\hat{b}_4(z_o,t)$$

Therefore,

$$|\psi(t)\rangle = e^{\int_{-\infty}^{\infty}dz'\left\{\alpha(z')e^{-i\omega_o t}\hat{b}_1^{+}\left(z_o,-t-\frac{z'-z_o}{v_g}\right)-\alpha^*(z')e^{i\omega_o t}\hat{b}_1\left(z_o,-t-\frac{z'-z_o}{v_g}\right)\right\}}|0\rangle$$
$$= e^{\int_{-\infty}^{\infty}dz'\left\{\begin{array}{l}\alpha(z')e^{-i\omega_o t}\left[t\hat{b}_3^{+}\left(z_o,-t-\frac{z'-z_o}{v_g}\right)-r^*\hat{b}_4^{+}\left(z_o,-t-\frac{z'-z_o}{v_g}\right)\right]\\ \qquad -\alpha^*(z')e^{i\omega_o t}\left[t^*\hat{b}_3\left(z_o,-t-\frac{z'-z_o}{v_g}\right)-r\,\hat{b}_4^{+}\left(z_o,-t-\frac{z'-z_o}{v_g}\right)\right]\end{array}\right\}}|0\rangle$$

$$|\psi(t)\rangle = e^{\int_{-\infty}^{\infty}dz'\left\{\alpha(z')e^{-i\omega_o t}\left[t\hat{b}_3^{+}(z'+v_g t,0)-r^*\hat{b}_4^{+}(z'+v_g t,0)\right]-\alpha^*(z')e^{i\omega_o t}\left[t^*\hat{b}_3(z'+v_g t,0)-r\,\hat{b}_4^{+}(z'+v_g t,0)\right]\right\}}|0\rangle$$

We can write the above result as,

$$|\psi(t)\rangle = e^{\int_{-\infty}^{\infty} dz' \left\{ \alpha(z')e^{-i\omega_o t}\left[t\hat{b}_3^+(z'+v_g t,0) - r^*\hat{b}_4^+(z'+v_g t,0)\right] - \alpha^*(z')e^{i\omega_o t}\left[t^*\hat{b}_3(z'+v_g t,0) - r\,\hat{b}_4^+(z'+v_g t,0)\right]\right\}}|0\rangle_3 \otimes |0\rangle_4$$

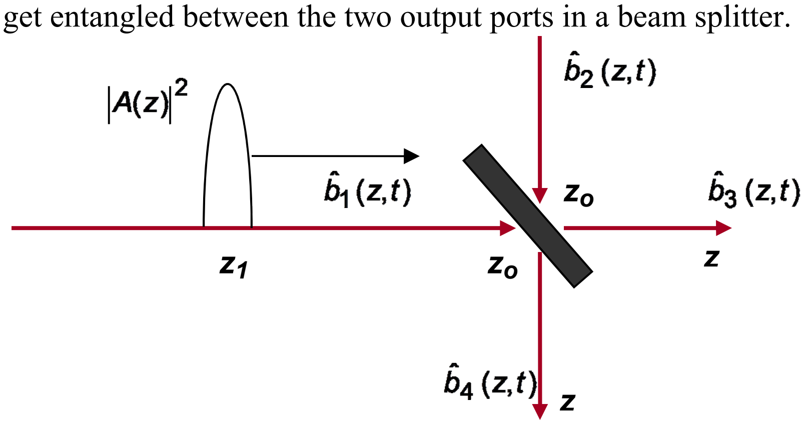
$$= e^{\int_{-\infty}^{\infty} dz' \left\{ t\alpha(z')e^{-i\omega_o t}\hat{b}_3^+(z'+v_g t,0) - t^*\alpha^*(z')e^{i\omega_o t}\hat{b}_3(z'+v_g t,0)\right\}}|0\rangle_3 \otimes e^{\int_{-\infty}^{\infty} dz' \left\{ -r^*\alpha(z')e^{-i\omega_o t}\hat{b}_4^+(z'+v_g t,0) + r\alpha^*(z')e^{i\omega_o t}\hat{b}_4(z'+v_g t,0)\right\}}|0\rangle_4$$

$$= e^{\int_{-\infty}^{\infty} dz' \left\{ t\alpha(z'-v_g t)e^{-i\omega_o t}\hat{b}_3^+(z',0) - t^*\alpha^*(z'-v_g t)e^{i\omega_o t}\hat{b}_3(z',0)\right\}}|0\rangle_3 \otimes e^{\int_{-\infty}^{\infty} dz' \left\{ -r^*\alpha(z'-v_g t)e^{-i\omega_o t}\hat{b}_4^+(z',0) + r\alpha^*(z'-v_g t)e^{i\omega_o t}\hat{b}_4(z',0)\right\}}|0\rangle_4$$

$$= \left| t\,\alpha(z-v_g t)e^{-i\omega_o t}\right\rangle_3 \otimes \left| -r^*\,\alpha(z-v_g t)e^{-i\omega_o t}\right\rangle_4$$

The above result shows that the state at a later time t is a coherent state packet in the output port 3 and a coherent state packet in the output port 4. The amplitude of the transmitted packet is scaled by the transmission coefficient and the amplitude of the reflected packet is scaled by the reflection coefficient. Coherent states thus remain coherent states when undergoing optical loss.

Example of A Number State Packet: Most quantum optical states, with the exception of coherent states, get entangled between the two output ports in a beam splitter.



This is best illustrated by considering a single photon packet incident from input port 1,

$$|\psi(t=0)\rangle = \int_{-\infty}^{\infty} dz'\, A(z')\,\hat{b}_1^+(z',0)|0\rangle = |1\rangle_1$$

where,

$$\int_{-\infty}^{\infty} dz\, |A(z)|^2 = 1$$

We want to find the state $|\psi(t)\rangle$ at a later time t which is large enough such that the packet has by then gone past the beam splitter.

$$|\psi(t)\rangle = e^{-i\frac{\hat{H}}{\hbar}t}|\psi(t=0)\rangle = e^{-i\frac{\hat{H}}{\hbar}t}\int_{-\infty}^{\infty} dz'\, A(z')\hat{b}_1^{+}(z',0)|0\rangle$$

$$= e^{-i\frac{\hat{H}}{\hbar}t}\int_{-\infty}^{\infty} dz'\, A(z')\hat{b}_1^{+}(z',0)\; e^{+i\frac{\hat{H}}{\hbar}t}\, e^{-i\frac{\hat{H}}{\hbar}t}|0\rangle$$

$$= \int_{-\infty}^{\infty} dz'\, A(z')e^{-i\omega_o t}\hat{b}_1^{+}(z',-t)|0\rangle = \int_{-\infty}^{\infty} dz'\, A(z')e^{-i\omega_o t}\,\hat{b}_1^{+}\left(z_o,-t-\frac{z'-z_o}{v_g}\right)|0\rangle$$

The vacuum energy has been ignored above. The beam splitter relation,

$$\begin{bmatrix}\hat{b}_3(z_o,t)\,e^{i\beta_o z_o - i\omega_o t}\\ b_4(z_o,t)\,e^{i\beta_o z_o - i\omega_o t}\end{bmatrix} = \begin{bmatrix} t & r\\ -r^* & t^*\end{bmatrix}\begin{bmatrix}\hat{b}_1(z_o,t)\,e^{i\beta_o z_o - i\omega_o t}\\ \hat{b}_2(z_o,t)\,e^{i\beta_o z_o - i\omega_o t}\end{bmatrix}$$

gives,

$$\hat{b}_1(z_o,t) = t^*\,\hat{b}_3(z_o,t) - r\,\hat{b}_4(z_o,t)$$

Therefore,

$$|\psi(t)\rangle = \int_{-\infty}^{\infty} dz'\, A(z')e^{-i\omega_o t}\,\hat{b}_1^{+}\left(z_o,-t-\frac{z'-z_o}{v_g}\right)|0\rangle$$

$$= \int_{-\infty}^{\infty} dz'\, A(z')e^{-i\omega_o t}\left[t\hat{b}_3^{+}\left(z_o,-t-\frac{z'-z_o}{v_g}\right) - r^*\,\hat{b}_4^{+}\left(z_o,-t-\frac{z'-z_o}{v_g}\right)\right]|0\rangle$$

$$= \int_{-\infty}^{\infty} dz'\, A(z')e^{-i\omega_o t}\left[t\hat{b}_3^{+}\left(z'+v_g t,0\right) - r^*\,\hat{b}_4^{+}\left(z'+v_g t,0\right)\right]|0\rangle_3\otimes|0\rangle_4$$

$$= t\left[\int_{-\infty}^{\infty} dz'\, A(z'-v_g t)e^{-i\omega_o t}\hat{b}_3^{+}(z',0)\right]|0\rangle_3\otimes|0\rangle_4 - r^*\left[\int_{-\infty}^{\infty} dz'\, A(z'-v_g t)e^{-i\omega_o t}\hat{b}_4^{+}(z',0)\right]|0\rangle_3\otimes|0\rangle_4$$

$$= t\,|1\rangle_3\otimes|0\rangle_4 - r^*\,|0\rangle_3\otimes|1\rangle_4$$

The quantum state at a later time is a linear superposition of the photon in the output port 3 (with vacuum in the output port 4) with a probability $|t|^2$, and the photon in the output port 4 (with vacuum in the output port 3) with a probability $|r|^2$. The final state is in fact an entangled state. Note that coherent state packets do not get entangled between the two output ports, as shown in the previous example.

9.2.5 Some Comments on Loss in Quantum Optics

In the previous Sections, we saw that whenever a beam suffers loss and is attenuated noise also gets added to the beam in the process. For example, the beam splitter relation,

$$\begin{bmatrix}\hat{b}_3(z_o,t)\,e^{i\beta_o z_o - i\omega_o t}\\ b_4(z_o,t)\,e^{i\beta_o z_o - i\omega_o t}\end{bmatrix} = \begin{bmatrix} t & r\\ -r^* & t^*\end{bmatrix}\begin{bmatrix}\hat{b}_1(z_o,t)\,e^{i\beta_o z_o - i\omega_o t}\\ \hat{b}_2(z_o,t)\,e^{i\beta_o z_o - i\omega_o t}\end{bmatrix}$$

implied that for a signal coming in on port 1, the signal leaving on port 3 is given by,

$$\hat{b}_3(z_o,t) = t\hat{b}_1(z_o,t) + r\hat{b}_2(z_o,t)$$

The last term on the left hand side represented noise due to the vacuum fluctuations going into the output port 3 from the input port 2. We also saw that the addition of the noise was necessary in order to the preserve the operator commutation relations at the output. These observations suggest that a quantum mechanically consistent model for loss in quantum optics can be developed that is independent of the microscopic details of the loss mechanism by ensuring the preservation of commutation relations. In the next Section, we discuss such a model.

9.3 Quantum Description of Loss for Propagating States

For a non-dispersive and a non-interacting waveguide (or fiber of free-space) we derived the travelling wave equation,

$$\left(\frac{\partial}{\partial z}+\frac{1}{v_g}\frac{\partial}{\partial t}\right)\hat{b}(z,t)=0$$

Suppose the propagation is taking place in a medium that has loss. In order to model the loss, we can add a term on the right hand side,

$$\left(\frac{\partial}{\partial z}+\frac{1}{v_g}\frac{\partial}{\partial t}\right)\hat{b}(z,t)=-\alpha\;\hat{b}(z,t)$$

where 2α is the power loss coefficient (units: per unit length). The above equation, although classically correct, is quantum mechanically inconsistent. We know that,

$$\left[\hat{b}(z,t),\hat{b}(z',t)\right]\;=\delta(z-z')$$

Let's see if the travelling wave equation with loss preserves the commutation relation. Define a change of variables,

$$z'=z-v_g t$$

We get,

$$\left(\frac{\partial}{\partial z}+\frac{1}{v_g}\frac{\partial}{\partial t}\right)\hat{b}(z,t)=\frac{1}{v_g}\frac{\partial}{\partial t}\hat{b}(z'+v_g t,t)=-\alpha\,\hat{b}(z'+v_g t,t)$$

$$\Rightarrow\hat{b}(z'+v_g t,t)=\hat{b}(z',0)e^{-\alpha v_g t}$$

$$\Rightarrow\hat{b}(z,t)=\hat{b}(z-v_g t,0)e^{-\alpha v_g t}$$

We can now find the equal-time commutation relation at time t,

$$\left[\hat{b}(z,t),\hat{b}^{+}(z',t)\right]\;=\left[\hat{b}(z-v_g t,0),\hat{b}^{+}(z'-v_g t,0)\right]e^{-2\alpha v_g t}$$

If at time $t=0$,

$$\left[\hat{b}(z,t=0),\hat{b}(z',t=0)\right]\;=\delta(z-z')$$

then at time t,

$$\left[\hat{b}(z,t),\hat{b}^{+}(z',t)\right]=\delta(z-z')e^{-2\alpha v_g t}.$$

We see that the commutation relation decays and is not preserved.

Irrespective of the microscopic details of the loss mechanism, noise is introduced during propagation in any lossy medium. To model this noise, we introduce a quantum Langevin noise operator $\hat{S}(z,t)$ as follows,

$$\left[\frac{\partial}{\partial z}+\frac{1}{v_g}\frac{\partial}{\partial t}\right]\hat{b}(z,t)=-\alpha\,\hat{b}(z,t)+\sqrt{A}\,\hat{S}(z,t)$$

We impose the following commutation relation on the noise operator,

$$\left[\hat{S}(z,t),\hat{S}^{+}(z',t')\right] = \delta(z-z')\,\delta(t-t')$$

The noise operators have zero mean values,

$$\left\langle \hat{S}(z,t)\right\rangle = \left\langle \hat{S}^{+}(z,t)\right\rangle = 0$$

The averaging above is with respect to the density operator describing the noise sources. Solution is,

$$\hat{b}(z,t) = \hat{b}(z-v_g t,0)\,e^{-\alpha v_g t} + v_g\sqrt{A}\int_0^t dt_1\, e^{-\alpha v_g (t-t_1)}\,\hat{S}(z-v_g(t-t_1),t_1)$$

The commutation relation is,

$$\left[\hat{b}(z,t),\hat{b}^{+}(z',t)\right] = \delta(z-z')\,e^{-2\alpha v_g t} + v_g^2\,A\,\delta(z-z')\frac{\left(1-e^{-2\alpha v_g t}\right)}{2\alpha v_g}$$

The commutation relation is preserved if $A = 2\alpha/v_g$. Thus, the correct quantum equation in the presence of loss is,

$$\left[\frac{\partial}{\partial z}+\frac{1}{v_g}\frac{\partial}{\partial t}\right]\hat{b}(z,t) = -\alpha\,\hat{b}(z,t) + \sqrt{\frac{2\alpha}{v_g}}\,\hat{S}(z,t)$$

In most cases, one imposes additional conditions on the Langevin noise sources that accompany loss,

$$\left\langle \hat{S}^{+}(z,t)\hat{S}(z',t')\right\rangle = 0$$

$$\left\langle \hat{S}(z,t)\,\hat{S}^{+}(z',t')\right\rangle = \delta(z-z')\,\delta(t-t')$$

These conditions ensure that the average photon flux obeys,

$$\left\langle \hat{F}(z,t)\right\rangle = \left\langle \hat{F}(z-v_g t,0)\right\rangle e^{-2\alpha v_g t}$$

as one might intuitively expect.

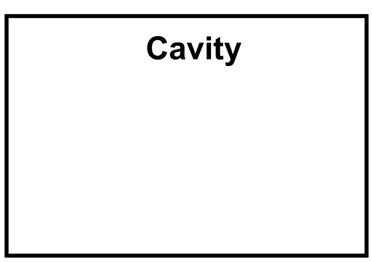
Comment on Averages: In order to make the averaging procedure in the presence of Langevin sources explicit, suppose the initial state of the radiation mode is $|\phi(t=0)\rangle$ and the corresponding density operator is $\hat{\rho}_S(t=0) = |\phi(t=0)\rangle\langle\phi(t=0)|$. Suppose the initial state describing the "reservoir" associated with the loss is $\hat{\rho}_R(t=0)$. The reservoir represents those degrees of freedom into which the energy lost from the radiation mode is going. The reservoir degrees of freedom are also responsible for injecting noise into the radiation mode. The state of the complete system is then given by the density operator,

$$\hat{\rho}(t=0) = \hat{\rho}_S(t=0)\otimes\hat{\rho}_R(t=0).$$

All averaging, in the Heisenberg picture, is performed with respect to the density operator above.

9.4 Quantum Theory of Loss for Cavity Modes

Consider a cavity, as shown below.



We assume that it contains only a single radiation mode and the Hamiltonian is,

$$\hat{H} = \hbar\omega_o \; (\hat{a}^+\hat{a} + \frac{1}{2})$$

The creation and destruction obey the equal-time commutation relation,

$$\left[\hat{a}(t), \hat{a}^+(t)\right] \; = 1$$

The time development is given by the Heisenberg equations,

$$\frac{d\hat{a}(t)}{dt} = -i\omega_o \; \hat{a}(t)$$

$$\frac{d\hat{a}^+(t)}{dt} = i\omega_o \; \hat{a}^+(t)$$

It follows that

$$\frac{d}{dt}\hat{n}(t) = \frac{d}{dt}\left[\hat{a}^+(t)\hat{a}(t)\right] = 0$$

$$\Rightarrow \hat{n}(t) = \hat{n}(0)$$

Now we assume that the cavity contains loss. In the presence of loss, irrespective of the microscopic details of the loss mechanism, the average photon number should decrease as,

$$\langle \hat{n}(t) \rangle = \langle \hat{n}(0) \rangle e^{-2\gamma \, t}$$

To model loss, we try by adding decay terms to the operator equations,

$$\frac{d\hat{a}(t)}{dt} = -i\omega_o \; \hat{a}(t) - \gamma\hat{a}(t)$$

$$\frac{d\hat{a}^+(t)}{dt} = i\omega_o \; \hat{a}^+(t) - \gamma\hat{a}^+(t)$$

Now,

$$\frac{d\hat{n}(t)}{dt} = -2\gamma \, \hat{n}(t)$$

and,

$$\hat{n}(t) = \hat{n}(0) e^{-2\gamma \, t}$$

$$\Rightarrow \langle \hat{n}(t) \rangle = \langle \hat{n}(0) \rangle e^{-2\gamma \, t}$$

The creation and destruction operators at time t are,

$$\hat{a}(t) \; = \hat{a}(t=0) e^{(-i\,\omega_o - \gamma)\, t}$$

$$\hat{a}^+(t) = \hat{a}^+(t=0) e^{(i\,\omega_o - \gamma)\, t}$$

It follows that the equal-time commutation relation at time t is,

$$\left[\hat{a}(t), \hat{a}^+(t)\right] = \left[\hat{a}(t=0), \hat{a}^+(t=0)\right] \; e^{-2\gamma \, t}$$

If at time $t = 0$,

$$\left[\hat{a}(t=0), \hat{a}^+(t=0)\right] \; = 1$$

then at time t,

$$\left[\hat{a}(t), \hat{a}^+(t)\right] = e^{-2\gamma \, t}$$

The commutation relation decays with time. Therefore, our method of introducing loss is quantum mechanically inconsistent. Equal-time commutation relations are laws of nature. If they are found to be violated, it means that one has made a mistake. To model loss correctly, we need to introduce the noise that comes with the loss. We modify the creation and destruction operator equations and introduce Langevin noise sources,

$$\frac{d\hat{a}(t)}{dt} = -i\omega_o\,\hat{a}(t) - \gamma\hat{a}(t) + \sqrt{A}\,\hat{S}(t)$$

$$\frac{d\hat{a}^+(t)}{dt} = i\omega_o\,\hat{a}^+(t) - \gamma\hat{a}^+(t) + \sqrt{A}\,\hat{S}^+(t)$$

$\hat{S}(t)$ and $\hat{S}^+(t)$ are Langevin noise operators and the only requirements we impose on them is that they satisfy,

$$\left[\hat{S}(t),\hat{S}^+(t')\right] = \delta(t-t')$$

$$\left\langle\hat{S}(t)\right\rangle = \left\langle\hat{S}^+(t)\right\rangle = 0$$

Solving the Equations above one obtains,

$$\hat{a}(t) = \hat{a}(t=0)e^{(-i\omega_o-\gamma)t} + \sqrt{A}\int_o^t dt'\,e^{(-i\,\omega_o-\gamma)(t-t')}\hat{S}(t')$$

$$\hat{a}^+(t) = \hat{a}^+(t=0)e^{(+i\omega_o-\gamma)t} + \sqrt{A}\int_o^t dt'\,e^{(i\,\omega_o-\gamma)(t-t')}\hat{S}^+(t')$$

The equal-time commutation relation is,

$$\begin{aligned}\left[\hat{a}(t),\hat{a}^+(t)\right] &= \left[\hat{a}(t=0),\hat{a}^+(t=0)\right]e^{-2\gamma\,t} + A\int_o^t dt_1\int_o^t dt_2\,e^{(-i\,\omega_o-\gamma)(t-t_1)}\,e^{(i\,\omega_o-\gamma)(t-t_2)}\,\delta(t_1-t_2)\\ &= e^{-2\gamma\,t} + A\int_o^t dt_1\,e^{-2\gamma\,(t-t_1)}\\ &= e^{-2\gamma\,t} + \frac{A}{2\gamma}\left(1-e^{-2\gamma\,t}\right)\end{aligned}$$

If $A = 2\gamma$, then $\left[\hat{a}(t),\hat{a}^+(t)\right] = 1$ for all time t. Therefore, in the presence of loss the correct operator equations are,

$$\frac{d\hat{a}(t)}{dt} = -i\omega\,\hat{a}(t) - \gamma\hat{a}(t) + \sqrt{2\gamma}\,\hat{S}(t)$$

$$\frac{d\hat{a}^+(t)}{dt} = i\omega\,\hat{a}^+(t) - \gamma\hat{a}^+(t) + \sqrt{2\gamma}\,\hat{S}^+(t)$$

where,

$$\left[\hat{S}(t),\hat{S}^+(t')\right] = \delta(t-t')$$

$$\left\langle\hat{S}(t)\right\rangle = \left\langle\hat{S}^+(t)\right\rangle = 0$$

Suppose we want to find the average photon number at time t. We have,

$$\hat{n}(t) = \hat{n}(t=0)e^{-2\gamma\,t} + 2\gamma\int_o^t dt_1\int_o^t dt_2\;e^{(-i\,\omega-\gamma)(t-t_1)}\,e^{\,(i\,\omega-\gamma)(t-t_1)}\hat{S}^+(t_2)\hat{S}(t_1)$$

The average photon number is,

$$\left\langle\hat{n}(t)\right\rangle = T_r\left\{\,\hat{\rho}(t=0)\,\hat{n}(t)\,\right\} \qquad \left\{\,\hat{\rho}(t=0) = \hat{\rho}_S(t=0)\otimes\hat{\rho}_R(t=0)\right)$$

In order to obtain the intuitive result,

$$\left\langle\hat{n}(t)\right\rangle = \left\langle\hat{n}(0)\right\rangle e^{-2\gamma\,t}$$

one must impose the following additional condition on the noise sources,

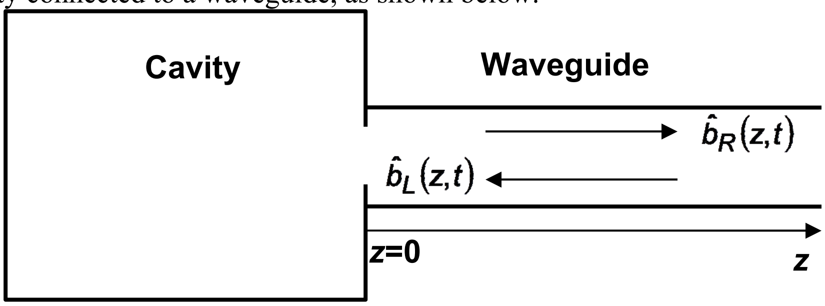
$$\left\langle\hat{S}^+(t)\hat{S}(t')\right\rangle = 0$$

The relation above and the commutation relation for the noise sources implies that we must also have,

$$\left\langle \hat{S}(t)\, \hat{S}^{+}(t') \right\rangle = \delta(t - t')$$

9.5 A Waveguide Model for Loss in Cavities

Consider a cavity connected to a waveguide, as shown below.



We assume that the cavity has a single radiation mode. The Hamiltonian for the radiation mode of a closed cavity (i.e. cavity with no waveguide attached) is,

$$\hat{H}_o = \hbar\omega_o \left(\hat{a}^{+} a + \frac{1}{2} \right)$$

In the presence of the waveguide, the Hamiltonian must include the coupling between the cavity mode and the propagating mode inside the waveguide. The details of this coupling, as you will see, will turn out to be unimportant. We assume that in the presence of the waveguide, the cavity mode experiences loss since energy is transferred from the cavity mode to the propagating mode inside the waveguide. In other words, the energy leaks out from the cavity into the waveguide. We can model this loss by adding decay terms to the Heisenberg equations of the creation and destruction operators, $\hat{a}^{+}(t)$ and $\hat{a}(t)$, respectively,

$$\frac{d\hat{a}(t)}{dt} = -i\omega_o\, \hat{a}(t) - \gamma \hat{a}(t)$$

$$\frac{d\hat{a}^{+}(t)}{dt} = i\omega_o\, \hat{a}^{+}(t) - \gamma \hat{a}^{+}(t)$$

As discussed earlier, the loss must be accompanied by noise which can be modeled by introducing Langevin noise sources,

$$\frac{d\hat{a}(t)}{dt} = -i\omega_o \hat{a}(t) - \gamma \hat{a}(t) + \sqrt{2\gamma}\, \hat{S}(t)$$

$$\frac{d\hat{a}^{+}(t)}{dt} = i\omega_o \hat{a}^{+}(t) - \gamma \hat{a}^{+}(t) + \sqrt{2\gamma}\, \hat{S}^{+}(t)$$

The above equations are adequate for describing the cavity loss due to coupling with the waveguide. However, here we will discuss a microscopic model for the cavity loss due to coupling to the waveguide with the aim to clarify the origin of the noise sources.

As a result of the coupling between the cavity and the waveguide, photons inside the cavity can leak into the waveguide. The operators $\hat{b}_L(z,t)$ and $\hat{b}_R(z,t)$ stand for the waveguide modes moving in the left and right directions, respectively.

$$\hat{b}_L(z,t) = L \int_{-\beta_o-\frac{\Delta\beta}{2}}^{-\beta_o+\frac{\Delta\beta}{2}} \frac{d\beta}{2\pi}\ \hat{b}(\beta)\ \frac{e^{i(\beta-\beta_0)z}}{\sqrt{L}}\ e^{-i[\omega(\beta)-\omega(\beta_o)]t}$$

$$\hat{b}_R(z,t) = L \int_{\beta_o-\frac{\Delta\beta}{2}}^{\beta_o+\frac{\Delta\beta}{2}} \frac{d\beta}{2\pi}\ \hat{b}(\beta)\ \frac{e^{i(\beta-\beta_0)z}}{\sqrt{L}}\ e^{-i[\omega(\beta)-\omega(\beta_o)]t}$$

They satisfy the following equal-space and equal-time commutation relations,

$$\left[\hat{b}_L(z,t),\hat{b}_L^{\,+}(z',t)\right] = \delta(z-z') = \left[\hat{b}_R(z,t),\hat{b}_R^{\,+}(z',t)\right]$$

$$\left[\hat{b}_L(z,t),\hat{b}_L^{\,+}(z,t')\right] = \frac{1}{v_g}\ \delta(t-t') = \left[\hat{b}_R(z,t),\hat{b}_R^{\,+}(z,t')\right]$$

Since we are interested only in those waveguide modes whose frequency is close to the cavity frequency, we choose β_o such that $\omega(\beta_o) = \omega_o = \text{cavity frequency}$. The coupling between the cavity and the waveguide is taken into account by the following equation,

$$\frac{d\hat{a}(t)}{dt} = (-i\omega_o - \gamma)\,\hat{a}(t) + k\,\hat{b}_L(z=0,t)e^{-i\omega_o t}$$

The last term describes the coupling of the left-moving fields into the cavity. The $e^{-i\omega_o t}$ term has been added to make a Heisenberg operator out of the slowly varying envelope part $\hat{b}_L(z=0,t)$. The coupling constant k remains to be determined. The corresponding equation for the creation operator is,

$$\frac{d\hat{a}^+(t)}{dt} = (i\omega_o - \gamma)\,\hat{a}^+(t) + k\,\hat{b}_L^+(z=0,t)\,e^{i\omega_o t}$$

Comparing Equations (3) and (4) with (1) and (2), we see that they are equivalent provided the value of the coupling constant k is $\sqrt{2\gamma\, v_g}$. Therefore, the phenomenological Langevin noise sources are related to the waveguide operators as,

$$\hat{S}(t) = \sqrt{v_g}\ \hat{b}_L(z=0,t)\ e^{-i\omega_o t}$$

$$\hat{S}^+(t) = \sqrt{v_g}\ \hat{b}_L^+(z=0,t)\ e^{i\omega_o t}$$

One can verify that all commutation relations and averages involving operators $\hat{S}(t)$ and $\hat{S}^+(t)$ are satisfied with the above definitions. The noise that accompanies photon loss into the waveguide is due to the vacuum fluctuations coming into the cavity from the waveguide.

The final remaining question is how to describe the photons leaking out into the waveguide and moving to the right. We know that the average photon number inside the cavity decays as,

$$\frac{d\,\langle\hat{n}(t)\rangle}{dt} = -2\gamma\langle\hat{n}(t)\rangle$$

So the average photon flux in the waveguide at $z=0$ going in the right direction must equal the rate of photon loss in the cavity,

$$\langle F_R(z=0,t)\rangle = v_g\left\langle \hat{b}_R^+(z=0,t)\hat{b}_R(z=0,t)\right\rangle = 2\gamma\ \langle\hat{n}(t)\rangle$$

In order to obtain the above relation, one might conjecture that,

$$\hat{b}_R(z=0,t)e^{-i\omega_o t} = \sqrt{\frac{2\gamma}{v_g}}\;\hat{a}(t)$$

The above relation is incorrect. $\hat{b}_R(z=0,t)e^{-i\omega_o t}$ must also include the reflected part of the left-moving field (i.e. the part that does not make into the cavity). So we try,

$$\hat{b}_R(z=0,t)e^{-i\omega_o t} = \sqrt{\frac{2\gamma}{v_g}}\;\hat{a}(t) + c\,\hat{b}_L(z=0,t)e^{-i\omega_o t}$$

The parameter c needs to be determined. Solving the Equation,

$$\frac{d\hat{a}(t)}{dt} = (-i\omega_o - \gamma)\hat{a}(t) + \sqrt{2\gamma\, v_g}\;\hat{b}_L(z=0,t)e^{-i\omega_o t}$$

one obtains,

$$\hat{a}(t) = \hat{a}\,e^{(-i\omega_o - \gamma)t} + \sqrt{2\gamma\, v_g}\; e^{-i\omega_o t}\int_0^t dt'\, e^{-\gamma\,(t-t')}\;\hat{b}_L(z=0,t')$$

After substituting the above result in the expression for $\hat{b}_R(z=0,t)e^{-i\omega_o t}$ we get,

$$\hat{b}_R(z=0,t)e^{-i\omega_o t} = \sqrt{\frac{2\gamma}{v_g}}\;\hat{a}\,e^{(-i\omega_o - \gamma)t} + 2\gamma\; e^{-i\omega_o t}\int_0^t dt'\, e^{-\gamma\,(t-t')}\;\hat{b}_L(z=0,t')$$
$$+ c\,\hat{b}_L(z=0,t)e^{-i\omega_o t}$$

From the above expression, the equal-space commutation relation for $\hat{b}_R(z=0,t)$ is,

$$\left[\hat{b}_R(z=0,t_1),\hat{b}_R^{+}(z=0,t_2)\right] = \frac{2\gamma}{v_g}\,e^{-\gamma(t_1+t_2)} + \frac{2\gamma}{v_g}\left(e^{-\gamma|t_1-t_2|} - e^{-\gamma(t_1+t_2)}\right) + \frac{2\gamma\, c}{v_g}\,e^{-\gamma|t_1-t_2|}.$$
$$+ \frac{c^2}{v_g}\,\delta(t_1-t_2)$$

The left hand side must equal $\delta(t_1-t_2)/v_g$. Therefore, c must equal -1. Finally, we can write the two equations describing the coupling between the waveguide and the cavity as,

$$\frac{d\hat{a}(t)}{dt} = (-i\omega_o - \gamma)\hat{a}(t) + \sqrt{2\gamma\, v_g}\;\hat{b}_L(z=0,t)e^{-i\omega_o t}$$

$$\hat{b}_R(z=0,t)e^{-i\omega_o t} = \sqrt{\frac{2\gamma}{v_g}}\;\hat{a}(t) - \hat{b}_L(z=0,t)e^{-i\omega_o t}$$

The corresponding equations for the adjoints are,

$$\frac{d\hat{a}^{+}(t)}{dt} = (i\omega_o - \gamma)\hat{a}^{+}(t) + \sqrt{2\gamma\, v_g}\;\hat{b}_L^{+}(z=0,t)e^{i\omega_o t}$$

$$\hat{b}_R^{+}(z=0,t)e^{i\omega_o t} = \sqrt{\frac{2\gamma}{v_g}}\;\hat{a}^{+}(t) - \hat{b}_L^{+}(z=0,t)e^{i\omega_o t}$$

Note that these equations depend on only one parameter – the loss rate γ. The value of γ is determined by the details of the coupling between the cavity and the waveguide.

In general, a cavity can have more than one source of loss, e.g. material loss inside the cavity, radiation loss, waveguide loss. For each distinct loss mechanism, one can introduce an independent Langevin noise source.