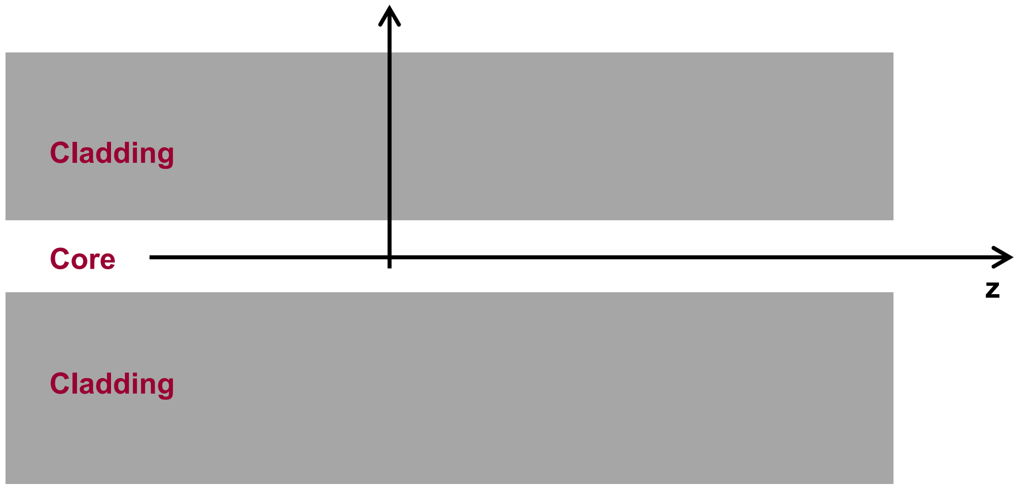
Chapter 8: Propagating Quantum States of Radiation

8.1 Electromagnetic Modes in a Waveguide

In this chapter we will consider propagating quantum states of radiation in waveguides, or fibers, and free space. Consider the Figure shown below for a dielectric waveguide.



We assume that the waveguide has a total length equal to L in the z-direction, and L will be assumed to be very large. The dielectric constant is a function of x, y only and the wave propagates in the $\pm z$ directions. The wave equation is,

$$\frac{1}{\varepsilon(x,y)}\nabla\times\nabla\times\vec{A}(x,y,z,t) = -\frac{1}{c^2}\frac{\partial^2\vec{A}(x,y,z,t)}{\partial t^2}$$

We first need to find the eigenmodes of the differential operator,

$$\frac{1}{\varepsilon(x,y)}\nabla\times\nabla\times$$

We assume eigenmodes of the form,

$$\vec{\phi}_n(x,y,\beta)\frac{e^{i\beta z}}{\sqrt{L}}$$

where,

$$\frac{1}{\varepsilon(x,y)}\nabla\times\nabla\times\vec{\phi}_n(x,y,\beta)e^{i\beta z} = \frac{\omega_n^2(\beta)}{c^2}\vec{\phi}_n(x,y,\beta)e^{i\beta z}$$

The modes are labeled by the transverse mode index, n, and the propagation vector β. Note that,

$$\nabla = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$$

Define,

$$\nabla' = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + i\beta\hat{z}$$

Then $\vec{\phi}_n(x,y,\beta)$ satisfies,

$$\nabla'(\nabla' . \vec{\phi}_n(x,y,\beta)) - \nabla'^2 \vec{\phi}_n(x,y,\beta) = \frac{\omega_n^{\,2}(\beta)}{c^2} \varepsilon(x,y) \vec{\phi}_n(x,y,\beta)$$

Normalization of the Transverse Modes:

$$\vec{\phi}_n^*(x,y,\beta)).\vec{\phi}_n(x,y,\beta) dxdy = 1$$

Orthogonality of the Transverse Modes:

$$\int\int \varepsilon(x,y) \vec{\phi}_n^*(x,y,\beta).\vec{\phi}_m(x,y,\beta) dxdy = \varepsilon_n \, \delta_{nm}$$

Orthogonality of the Modes:

$$\int\int\int \varepsilon(x,y) \vec{\phi}_m^*(x,y,\beta') \frac{e^{-i\beta' z}}{\sqrt{L}} . \vec{\phi}_n(x,y,\beta) \frac{e^{i\beta z}}{\sqrt{L}} dxdydz = \varepsilon_n \, \delta_{nm} \; \delta_{\beta\beta'}$$

One can expand the vector potential $\vec{A}(r,t)$ in terms of the eigenmodes as follows,

$$\vec{A}(x,y,z,t) = \sum_n L \int_{-\infty}^{\infty} \frac{d\beta}{2\pi} \frac{q_n(\beta,t)}{\sqrt{\varepsilon_o \varepsilon_n}} \vec{\phi}_n(x,y,\beta) \frac{e^{i\beta z}}{\sqrt{L}}$$

The reality of $\vec{A}(x,y,z,t)$ implies,

$$q_n(-\beta,t) = q_n^{\,*}(\beta,t)$$

$$\vec{\phi}_n(x,y,-\beta) = \vec{\phi}_n^*(x,y,\beta)$$

Assuming periodic boundary condition in the z-direction, one finds the following equivalence,

$$\sum_\beta \quad \rightarrow \quad L \int_{-\infty}^{\infty} \frac{d\beta}{2\pi}$$

Let,

$$\frac{d\, q_n(\beta,t)}{dt} = p_n(\beta,t)$$

Then,

$$\vec{E}(x,y,z,t) = -\frac{\partial \vec{A}}{\partial t} = -\sum_n L \int_{-\infty}^{\infty} \frac{d\beta}{2\pi} \frac{p_n(\beta,t)}{\sqrt{\varepsilon_o \varepsilon_n}} \vec{\phi}_n(x,y,\beta) \frac{e^{i\beta z}}{\sqrt{L}}$$

$$\vec{H}(x,y,z,t) = \frac{1}{\mu_o} \nabla \times \vec{A}(x,y,z,t)$$

$$= \sum_n L \int_{-\infty}^{\infty} \frac{d\beta}{2\pi} \frac{q_n(\beta,t)}{\mu_o \sqrt{\varepsilon_\circ \varepsilon_n}} \frac{e^{i\beta z}}{\sqrt{L}} \nabla' \times \vec{\phi}_n(x,y,\beta)$$

Total energy of the field is,

$$H = \int\int\int \frac{1}{2} \varepsilon(x,y) \vec{E}.\vec{E} + \frac{1}{2} \mu_o \vec{H}.\vec{H} \; dxdydz$$

$$= \sum_n L \int_{-\infty}^{\infty} \frac{d\beta}{2\pi} \left\{ \left| \frac{p_n(\beta,t)}{2} \right|^2 + \frac{1}{2} \omega_n^2(\beta) |q_n(\beta,t)|^2 \right\}$$

8.2 Field Quantization

As before, we let the fields become operators, and impose the following commutation relations on the field amplitudes,

$$\left[\hat{q}_n(\beta,t),\hat{p}^+{}_m(\beta',t)\right] = i\hbar\delta_{nm}\ \delta_{\beta\beta'}$$

Define,

$$\hat{a}_n(\beta,t) = \frac{1}{\sqrt{2\hbar\omega_n(\beta)}}\left[\omega_n(\beta)\hat{q}_n(\beta,t) + i\,\hat{p}_n(\beta,t)\right]$$

$$\hat{a}_n^+(\beta,t) = \frac{1}{\sqrt{2\hbar\omega_n(\beta)}}\left[\omega_n(\beta)\hat{q}_n^+(\beta,t) - i\,\hat{p}_n^+(\beta,t)\right]$$

Note that,

$$\hat{q}_n(-\beta,t) = \hat{q}_n^+(\beta,t)$$

$$\hat{p}_n(-\beta,t) = \hat{p}_n^+(\beta,t)$$

We get the following commutation relation for the creation and destruction operators,

$$\left[\hat{a}_n(\beta,t),\hat{a}_m^+(\beta',t)\right] = \delta_{nm}\ \delta_{\beta\,\beta'}$$

The Hamiltonian becomes,

$$\hat{H} = \sum_n L\int_{-\infty}^{\infty}\frac{d\beta}{2\pi}\ \hbar\omega_n(\beta)\left[\hat{a}_n^+(\beta)\hat{a}_n(\beta) + \frac{1}{2}\right]$$

The above Hamiltonian gives the following time dependence of the field creation and destruction operators,

$$\hat{a}_n^+(\beta,t) = \hat{a}_n^+(\beta,t=0)e^{i\omega_n(\beta)t}$$

$$\hat{a}_n(\beta,t) = \hat{a}_n(\beta,t=0)e^{-i\omega_n(\beta)t}$$

8.3 Slowly Varying Envelope Approximation

In many cases of practical interest, one is interested in fields $\hat{\vec{A}}(\vec{r},t)$ that,

(i) Have a bandwidth centered around a particular wavevecter β_o

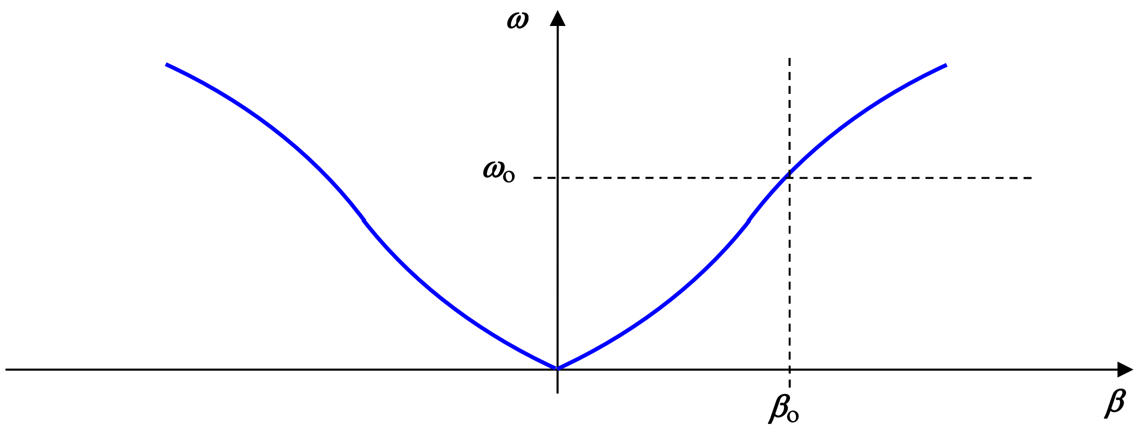
(ii) And depend on only a single transverse mode $\vec{\phi}_n(x,y,\beta)$. Typically, the transverse mode is the lowest mode $\vec{\phi}_0(x,y,\beta)$

In this Chapter we will deal with only a single transverse mode, and so we will drop the transverse mode index from now onwards.

For the single mode we are considering there is usually a one-to-one relationship between the wavevecter β and the frequency ω in the neighbourhood of β_o given by the dispersion relation $\omega(\beta)$ or equivalently by the inverse relation $\beta(\omega)$, as shown in the Figure. Suppose,

$$\omega(\beta_o) = \omega_o$$

$$\Rightarrow \beta(\omega_o) = \beta_o$$



One can Taylor expand the wavevector $\beta(\omega)$ in the neighborhood of β_o,

$$\beta(\omega) = \beta(\omega_o) + \left.\frac{\partial \beta}{\partial \omega}\right|_{\omega_o}(\omega-\omega_o) + \frac{1}{2}\left.\frac{\partial^2 \beta}{\partial \omega^2}\right|_{\omega_o}(\omega-\omega_o)^2 + \frac{1}{3!}\left.\frac{\partial^3 \beta}{\partial \omega^3}\right|_{\omega_o}(\omega-\omega_o)^3 +$$

$$\beta(\omega) = \beta_o + \frac{(\omega-\omega_o)}{v_g} + \frac{1}{2}\beta_2(\omega-\omega_o)^2 + \frac{1}{3!}\beta_3(\omega-\omega_o)^3 + \ldots$$

where the group velocity v_g, the dispersion β_2, and all the higher order dispersions are defined as,

$$v_g = \left.\frac{\partial \omega}{\partial \beta}\right|_{\omega_o} = \frac{1}{\left.\frac{\partial \beta}{\partial \omega}\right|_{\omega_o}}$$

$$\beta_2 = \left.\frac{\partial^2 \beta}{\partial \omega^2}\right|_{\omega_o}$$

$$\beta_n = \left.\frac{\partial^n \beta}{\partial \omega^n}\right|_{\omega_o}$$

Since the bandwidth is centered around β_o (or, equivalently, ω_o) we can write $\hat{\vec{A}}(\vec{r},t)$ as,

$$\hat{\vec{A}}(\vec{r},t) = \sqrt{\frac{\hbar}{2\omega(\beta_o)\varepsilon_o\varepsilon}}\, L \int_{\beta_o-\frac{\Delta\beta}{2}}^{\beta_o+\frac{\Delta\beta}{2}} \frac{d\beta}{2\pi}\left[\hat{a}(\beta,t)\,\vec{\phi}(x,y,\beta_o)\frac{e^{i\beta z}}{\sqrt{L}} + \hat{a}^+(\beta,t)\,\vec{\phi}^*(x,y,\beta_o))\frac{e^{-i\beta z}}{\sqrt{L}}\right]$$

It is convenient to factor out the fast time dependence of the operator $\hat{a}(\beta,t)$ and define,

$$\hat{a}(\beta,t) = \hat{b}(\beta,t)e^{-i\omega(\beta)t}$$

For non-interacting free fields the operator $\hat{b}(\beta,t) = \hat{a}(\beta)$ is independent of time. We then get,

$$\left[\hat{a}(\beta,t),\hat{a}^+(\beta',t)\right] = \left[\hat{b}(\beta,t),\hat{b}^+(\beta',t)\right] = \delta_{\beta\,\beta'}$$

One can write,

$$\hat{\vec{A}}(\vec{r},t) = \hat{\vec{A}}_+(\vec{r},t)e^{i\beta_o z - i\omega(\beta_o)t} + \hat{\vec{A}}_-(\vec{r},t)e^{-i\beta_o z + i\omega(\beta_o)t}$$

where

$$\hat{\vec{A}}_+(\vec{r},t)=\sqrt{\frac{\hbar}{2\omega(\beta_o)\varepsilon_o\varepsilon}}\,\vec{\phi}(x,y,\beta_o)L\int_{\beta_o-\frac{\Delta\beta}{2}}^{\beta_o+\frac{\Delta\beta}{2}}\frac{d\beta}{2\pi}\hat{b}(\beta,t)\frac{e^{i(\beta-\beta_o)z}}{\sqrt{L}}\,e^{-i[\omega(\beta)-\omega(\beta_o)]t}$$

$$\hat{\vec{A}}_-(\vec{r},t)=\left(\hat{\vec{A}}_+(\vec{r},t)\right)^+$$

Similarly,

$$\hat{\vec{E}}(\vec{r},t)=\hat{\vec{E}}_+(\vec{r},t)e^{i\beta_o z-i\omega(\beta_o)t}+\hat{\vec{E}}_-(\vec{r},t)e^{-i\beta_o z+i\omega(\beta_o)t}$$

$$\hat{\vec{E}}_+(\vec{r},t)=i\sqrt{\frac{\hbar\omega(\beta_o)}{2\varepsilon_o\varepsilon}}\,\vec{\phi}(x,y,\beta_o)L\int_{\beta_o-\frac{\Delta\beta}{2}}^{\beta_o+\frac{\Delta\beta}{2}}\frac{d\beta}{2\pi}\hat{b}(\beta,t)\frac{e^{i(\beta-\beta_o)z}}{\sqrt{L}}\,e^{-i[\omega(\beta)-\omega(\beta_o)]t}$$

$$\hat{\vec{E}}_+(\vec{r},t)=\left(\hat{\vec{E}}_-(\vec{r},t)\right)^+$$

Also,

$$\hat{\vec{H}}(\vec{r},t)=\hat{\vec{H}}_+(\vec{r},t)e^{i\beta_o z-i\omega(\beta_o)t}+\hat{\vec{H}}_-(\vec{r},t)\,e^{-i\beta_o z+i\omega(\beta_o)t}$$

$$\hat{\vec{H}}_+(\vec{r},t)=\frac{1}{\mu_o}\sqrt{\frac{\hbar}{2\omega(\beta_o)\varepsilon_o\varepsilon}}\nabla'\times\vec{\phi}(x,y,\beta_o)L\int_{\beta_o-\frac{\Delta\beta}{2}}^{\beta_o+\frac{\Delta\beta}{2}}\frac{d\beta}{2\pi}\hat{b}(\beta,t)\frac{e^{i(\beta-\beta_o)z-i[\omega(\beta)-\omega(\beta_o)]t}}{\sqrt{L}}$$

$$\hat{\vec{H}}_-(\vec{r},t)=\left[\hat{\vec{H}}_+(\vec{r},t)\right]^+$$

The quantities $\hat{\vec{A}}_+(\vec{r},t),\hat{\vec{E}}_+(\vec{r},t),\hat{\vec{H}}_+(\vec{r},t)$ and their adjoints are called the slowly varying envelops since the fast spatial and time dependencies have been factored out. We can also write the fields as,

$$\hat{\vec{A}}_+(\vec{r},t)=\sqrt{\frac{\hbar}{2\omega(\beta_o)\varepsilon_o\varepsilon}}\,\vec{\phi}(x,y,\beta_o)\,\hat{b}(z,t)$$

$$\hat{\vec{E}}_+(\vec{r},t)=i\sqrt{\frac{\hbar\omega(\beta_o)}{2\varepsilon_o\varepsilon}}\vec{\phi}(x,y,\beta_o)\hat{b}(z,t)$$

$$\hat{\vec{H}}_+(\vec{r},t)=\frac{1}{\mu_o}\sqrt{\frac{\hbar}{2\omega(\beta_o)\varepsilon_o\varepsilon}}\nabla'\times\vec{\phi}(x,y,\beta_o)\hat{b}(z,t)$$

Where the operator $\hat{b}(z,t)$ is,

$$\hat{b}(z,t)=L\int_{\beta_o-\frac{\Delta\beta}{2}}^{\beta_o+\frac{\Delta\beta}{2}}\frac{d\beta}{2\pi}\hat{b}(\beta,t)\frac{e^{i(\beta-\beta_o)z}}{\sqrt{L}}\,e^{-i[\omega(\beta)-\omega(\beta_o)]t}$$

8.3.1 Power Flow

The power (energy flow per second) in the z-direction is given by the Poynting operator $\hat{S}(z,t)$,

$$\hat{S}(z,t)=\int dxdy\left[\hat{\vec{E}}(\vec{r},t)\times\hat{\vec{H}}(\vec{r},t)\right].\hat{z}$$
$$=\int dxdy\left[\hat{\vec{E}}_{-}(\vec{r},t)\times\hat{\vec{H}}_{+}(\vec{r},t)+\hat{\vec{E}}_{+}(\vec{r},t))\times\hat{\vec{H}}_{-}(\vec{r},t)\right].\hat{z}$$

Using the identity,

$$\mathrm{Re}\int\int i\vec{\phi}(x,y,\beta)\times\left[\nabla'\times\vec{\phi}(x,y,\beta)\right]^{*}.\hat{z}\,dxdy=\frac{\omega(\beta)}{c^{2}}v_{g}\iint\varepsilon(x,y)\,\vec{\phi}^{*}(x,y,\beta)\,\vec{\phi}(x,y,\beta)\,dxdy$$
$$=\frac{\omega(\beta)}{c^{2}}v_{g}\,\varepsilon$$

we get,

$$\hat{S}(z,t)=\frac{\hbar\omega(\beta_{o})}{2}v_{g}\,L\int_{\beta_{o}-\frac{\Delta\beta}{2}}^{\beta_{o}+\frac{\Delta\beta}{2}}\frac{d\beta}{2\pi}L\int_{\beta_{o}-\frac{\Delta\beta}{2}}^{\beta_{o}+\frac{\Delta\beta}{2}}\frac{d\beta'}{2\pi}\left[\begin{array}{l}\hat{b}(\beta,t)\hat{b}^{+}(\beta' t)\frac{e^{i(\beta-\beta')z}}{\sqrt{L}}e^{-i\left[\omega(\beta)-\omega(\beta')\right]t}\\+\hat{b}^{+}(\beta,t)\hat{b}(\beta',t)\frac{e^{-i(\beta-\beta')z}}{\sqrt{L}}e^{i\left[\omega(\beta)-\omega(\beta')\right]t}\end{array}\right]$$

If one ignores the vacuum contribution to $\hat{S}(z,t)$, then $\hat{S}(z,t)$ can be written as,

$$\hat{S}(z,t)=v_{g}\,\hbar\omega(\beta_{o})\,\hat{b}^{+}(z,t)\,\hat{b}(z,t)$$

where as before,

$$\hat{b}(z,t)=L\int_{\beta_{o}-\frac{\Delta\beta}{2}}^{\beta_{o}+\frac{\Delta\beta}{2}}\frac{d\beta}{2\pi}\,\hat{b}(\beta,t)\,\frac{e^{i(\beta-\beta_{o})z}}{\sqrt{L}}\,e^{-i[\omega(\beta)-\omega(\beta_{o})]t}$$

Since $\hat{S}(z,t)$ is the operator for energy flow per second, the quantity $\hat{b}^{+}(z,t)\hat{b}(z,t)$ must be the operator for photon density (i.e. number photons per unit length). Next, we look at the properties of the operator $\hat{b}(z,t)$ in more detail.

8.3.2 The Creation and Destruction Operators $\hat{b}^{+}(z,t)$ and $\hat{b}(z,t)$

The operator $\hat{b}(z,t)$ satisfies the equal time commutation relation,

$$[\hat{b}(z,t),\hat{b}^{+}(z',t)]=L\int_{\beta_{o}-\frac{\Delta\beta}{2}}^{\beta_{o}+\frac{\Delta\beta}{2}}\frac{d\beta}{2\pi}L\int_{\beta_{o}-\frac{\Delta\beta}{2}}^{\beta_{o}+\frac{\Delta\beta}{2}}\frac{d\beta'}{2\pi}\,\delta_{\beta\beta'}\,\frac{e^{i(\beta-\beta_{o})z}}{\sqrt{L}}\,\frac{e^{-i(\beta'-\beta_{o})z'}}{\sqrt{L}}\,e^{-i[\omega(\beta)-\omega(\beta')]t}$$
$$=\int_{\beta_{o}-\frac{\Delta\beta}{2}}^{\beta_{o}+\frac{\Delta\beta}{2}}\frac{d\beta}{2\pi}\,e^{i(\beta-\beta_{o})(z-z')}$$

$$[\hat{b}(z,t),\hat{b}^{+}(z',t)]==\int_{\beta_{o}-\frac{\Delta\beta}{2}}^{\beta_{o}+\frac{\Delta\beta}{2}}\frac{d\beta}{2\pi}\,e^{i(\beta-\beta_{o})(z-z')}=\frac{\Delta\beta}{2\pi}\frac{\sin\left[\frac{\Delta\beta}{2}(z-z')\right]}{\frac{\Delta\beta}{2}(z-z')}\approx\delta(z-z')$$

Although, the commutation relation is not exactly a delta function in space, it can be approximated as one provided one keeps it in mind that the width of the approximate "delta function" in space is of the order of $2\pi/\Delta\beta$. Sometimes the expression for $\hat{b}(z,t)$ is written not as an integral over β, as in,

$$\hat{b}(z,t) = L \int_{\beta_o - \frac{\Delta\beta}{2}}^{\beta_o + \frac{\Delta\beta}{2}} \frac{d\beta}{2\pi} \hat{b}(\beta,t) \frac{e^{i(\beta-\beta_o)z - i[\omega(\beta)-\omega(\beta_o)]t}}{\sqrt{L}}$$

but as an integral over ω. This can be done since there is one-to-one relationship between ω and β (given by $\omega(\beta)$ or $\beta(\omega)$). To convert to an integral over ω, note that,

$$d\omega = \left.\frac{\partial\omega}{\partial\beta}\right|_{\beta_o} d\beta = \frac{d\beta}{v_g}$$

$$\omega(\beta_o) = \omega_o$$

$$\beta_o = \beta(\omega_o)$$

Therefore,

$$\int_{\beta_o - \frac{\Delta\beta}{2}}^{\beta_o + \frac{\Delta\beta}{2}} \frac{d\beta}{2\pi} \quad \rightarrow \quad \frac{1}{v_g} \int_{\omega_o - \frac{\Delta\omega}{2}}^{\omega_o + \frac{\Delta\omega}{2}} \frac{d\omega}{2\pi}$$

$$\hat{b}(\beta,t) = \hat{b}(\beta(\omega),t) = \hat{b}(\omega,t)$$

and,

$$\hat{b}(z,t) = \frac{L}{v_g} \int_{\omega_o - \frac{\Delta\omega}{2}}^{\omega_o + \frac{\Delta\omega}{2}} \frac{d\omega}{2\pi} \hat{b}(\omega,t) \frac{e^{i[\beta(\omega)-\beta(\omega_o)]z}}{\sqrt{L}} e^{-i(\omega-\omega_o)t}$$

Note that,

$$\sum_{\beta} = L\int \frac{d\beta}{2\pi} = \frac{L}{v_g}\int \frac{d\omega}{2\pi} = \sum_{\omega}$$

and,

$$\left[\hat{b}(\beta,t), \hat{b}^{+}(\beta',t)\right] = \left[\hat{a}(\beta,t)\hat{a}^{+}(\beta',t)\right] = \delta_{\beta\beta'}$$

$$\hat{b}(\beta,t) = \hat{b}(\beta(\omega),t) = \hat{b}(\omega,t)$$

$$\Rightarrow \left[\hat{b}(\omega,t), \hat{b}^{+}(\omega',t)\right] = \delta_{\omega\omega'}$$

Now argue that the Schrodinger operator $\hat{b}^{+}(z,0)$ creates a photon approximately at the location z and the operator $\hat{b}(z,0)$ destroys a photon at the location z. Recall that,

$$[\hat{b}(z,t), \hat{b}^{+}(z',t)] = \delta(z-z')$$

The quantum state $|\psi\rangle$ given by,

$$|\psi\rangle = \hat{b}^{+}(z,0)|0\rangle$$

corresponds to a single photon localized at the location z. First note that the above operation creates a single photon state in a superposition of different wavevectors,

$$|\psi\rangle = \hat{b}^{+}(z,0)|0\rangle = L \int_{\beta_o - \frac{\Delta\beta}{2}}^{\beta_o + \frac{\Delta\beta}{2}} \frac{d\beta}{2\pi} \hat{b}^{+}(\beta,0) \frac{e^{-i(\beta-\beta_o)z}}{\sqrt{L}} |0\rangle = L \int_{\beta_o - \frac{\Delta\beta}{2}}^{\beta_o + \frac{\Delta\beta}{2}} \frac{d\beta}{2\pi} \frac{e^{-i(\beta-\beta_o)z}}{\sqrt{L}} |1\rangle_{\beta}$$

Now we will see what happens when one takes the state $|\psi\rangle$ and tries to destroy a photon at a different location z',

$$\hat{b}(z',0)|\psi\rangle$$

Suppose,

$$|\chi\rangle = \hat{b}(z',0)|\psi\rangle$$

If one successfully destroyed the photon created in the first step then $|\chi\rangle$ should be the same as the vacuum state $|0\rangle$, and the inner product $\langle 0|\chi\rangle$ should be nonzero. So we evaluate $\langle 0|\chi\rangle$,

$$\begin{aligned}\langle 0|\chi\rangle = \langle 0|\hat{b}(z',0)|\psi\rangle &= \langle 0|\hat{b}(z',0)\,\hat{b}^{+}(z,0)|0\rangle \\ &= \langle 0|\left[\hat{b}(z',0),\hat{b}^{+}(z,0)\right] + \hat{b}^{+}(z,0),\hat{b}(z',0)|0\rangle \\ &= \langle 0|\delta(z-z')|0\rangle = \delta(z-z')\end{aligned}$$

The above result tells us that one can destroy a photon only at the same location at which it was created. The operators $\hat{b}^{+}(z)$ and $\hat{b}(z)$ do indeed create and destroy a single photon at the location z. Recall that the delta function that comes from the commutation relation has in fact some finite spatial width of the order of the inverse bandwidth $2\pi/\Delta\beta$. So the photon created by $\hat{b}^{+}(z,0)$ is localized in space in a region of size of the order of the inverse bandwidth $2\pi/\Delta\beta$.

8.3.3 Time-Dependence: The Travelling Wave Equation

We now study the time dependence of the operator $\hat{b}^{+}(z,t)$. If one evaluates,

$$\frac{\partial}{\partial z}\hat{b}(z,t) + \frac{1}{v_g}\frac{\partial \hat{b}(z,t)}{\partial t}$$

one obtains,

$$\left[\frac{\partial}{\partial z} + \frac{1}{v_g}\frac{\partial}{\partial t}\right]\hat{b}(z,t) = \frac{L}{v_g}\int_{\omega_o-\frac{\Delta\omega}{2}}^{\omega_o+\frac{\Delta\omega}{2}}\frac{d\omega}{2\pi}\,\hat{b}(\omega,t)\left[i[\beta(\omega)-\beta(\omega_o)] - i\frac{(\omega-\omega_o)}{v_g}\right]\frac{e^{i[\beta(\omega)-\beta(\omega_o)]z - i(\omega-\omega_o)t}}{\sqrt{L}}$$

$$+\frac{L}{v_g}\int_{\omega_o-\frac{\Delta\omega}{2}}^{\omega_o+\frac{\Delta\omega}{2}}\frac{d\omega}{2\pi}\,\frac{1}{v_g}\,\frac{\partial \hat{b}(\omega,t)}{\partial t}\,\frac{e^{i[\beta(\omega)-\beta(\omega_o)]z - i(\omega-\omega_o)\,t}}{\sqrt{L}}$$

The last term is non-zero only if the operator $\hat{b}(\omega,t)$ has time dependence. In the case of non-interacting free fields, $\hat{b}(\omega,t)$ has no time-dependence. But in the case of fields interacting with matter, $\hat{b}(\omega,t)$ can be time-dependent. In the first term on the right hand side, using the Taylor expansion of the wavevector,

$$\beta(\omega) - \beta(\omega_o) = \frac{(\omega-\omega_o)}{v_g} + \frac{1}{2}\,\beta_2(\omega-\omega_o)^2 + \frac{1}{3!}\,\beta_3(\omega-\omega_o)^3 + \cdots$$

and then inverse Fourier transforming, we obtain,

$$\left[\frac{\partial}{\partial z}+\frac{1}{v_g}\frac{\partial}{\partial t}\right]\hat{b}(z,t)=\left[-i\frac{\beta_2}{2}\frac{\partial^2\hat{b}(z,t)}{\partial t^2}+\frac{\beta_3}{3!}\frac{\partial^3\hat{b}(z,t)}{\partial t^3}+\cdots\right]$$
$$+\frac{L}{v_g}\int_{\omega_o-\frac{\Delta\omega}{2}}^{\omega_o+\frac{\Delta\omega}{2}}\frac{d\omega}{2\pi}\frac{1}{v_g}\frac{\partial\hat{b}(\omega,t)}{\partial t}\frac{e}{\sqrt{L}}^{-i[\beta(\omega)-\beta(\omega_o)]-i(\omega-\omega_o)t}$$

The terms inside the brackets on the right hand side describe the effects of dispersion (to all orders) on $\hat{b}(z,t)$, and the second term describes the effect of interactions.

In the case of a non-interacting and non-dispersive waveguide (or a fiber), we obtain,

$$\left[\frac{\partial}{\partial z}+\frac{1}{v_g}\frac{\partial}{\partial t}\right]\hat{b}(z,t)=0$$

Let,

$$z'=z-v_g t$$
$$b(z,t)=b(z'+v_g t,t)$$

Also,

$$\left[\frac{\partial}{\partial z}+\frac{1}{v_g}\frac{\partial}{\partial t}\right]\hat{b}(z,t)=\frac{1}{v_g}\frac{\partial\hat{b}(z'+v_g t,t)}{\partial t}=\frac{1}{v_g}\frac{\partial b(z'+v_g t,t)}{\partial t}$$

So our equation becomes,

$$\frac{1}{v_g}\frac{\partial\hat{b}(z'+v_g t,t)}{\partial t}=0$$

Solution is,

$$\hat{b}(z'+v_g t_2,t_2)=\hat{b}(z'+v_g t_1,t_1)$$

Change the dummy variable z' to $z-v_g t_2$,

$$\hat{b}(z,t_2)=\hat{b}(z-v_g(t_2-t_1),t_1)$$

One can also write the solution as,

$$\hat{b}(z,t)=\hat{b}(z-x,t-\frac{x}{v_g})$$

The above expression implies that the solution at a later time is a translated version of a solution at an earlier time.

8.3.4 Time-Dependence: The Heisenberg Approach

One can also use the Heisenberg equation to see how the operator $\hat{b}(z,t)$ behaves in time. We have,

$$\hat{b}(z,t)=L\int_{\beta_o-\frac{\Delta\beta}{2}}^{\beta_o+\frac{\Delta\beta}{2}}\frac{d\beta}{2\pi}\hat{b}(\beta,t)\frac{e^{i(\beta-\beta_o)z}}{\sqrt{L}}e^{-i[\omega(\beta)-\omega(\beta_o)]t}$$

and at time $t=0$,

$$\hat{b}(z,0)=L\int_{\beta_o-\frac{\Delta\beta}{2}}^{\beta_o+\frac{\Delta\beta}{2}}\frac{d\beta}{2\pi}\hat{b}(\beta,t=0)\frac{e^{i(\beta-\beta_o)z}}{\sqrt{L}}$$

The time-dependence of the operator $\hat{b}(z,t)$ is not given by the Heisenberg relation. To see this, recall that the Heisnberg operator $\hat{a}(\beta,t)$ is,

$$\hat{a}(\beta,t) = \hat{b}(\beta,t)e^{-i\omega(\beta)t}$$

The corresponding Schrodinger operators are,

$$\hat{a}(\beta,t=0) = \hat{a}(\beta) = \hat{b}(\beta,t=0) = \hat{b}(\beta)$$

The time-dependence of the destruction operator is,

$$\hat{a}(\beta,t) = e^{i\frac{\hat{H}}{\hbar}t}\,\hat{a}(\beta)e^{-i\frac{\hat{H}}{\hbar}t}$$

Therefore,

$$\hat{b}(\beta,t) = \hat{a}(\beta,t)e^{i\omega(\beta)t} = e^{i\frac{\hat{H}}{\hbar}t}\,\hat{b}(\beta)e^{i\omega(\beta)t}\,e^{-i\frac{\hat{H}}{\hbar}t}$$

Or equivalently,

$$\hat{b}(\beta,t)e^{-i\omega(\beta)t} = e^{i\frac{\hat{H}}{\hbar}t}\,\hat{b}(\beta)\,e^{-i\frac{\hat{H}}{\hbar}t}$$

Note that,

$$\hat{b}(\beta,t) \neq e^{i\frac{\hat{H}}{\hbar}t}\,\hat{b}(\beta)\,e^{-i\frac{\hat{H}}{\hbar}t}$$

This is because we had inserted a time-dependent exponential in the definition of the operator $\hat{b}(\beta,t)$. Now we can write,

$$\hat{b}(z,t) = e^{i\frac{\hat{H}}{\hbar}t}\,\hat{b}(z,0)\;e^{i\omega(\beta_o)t}\;e^{-i\frac{\hat{H}}{\hbar}t}$$

$$\Rightarrow \hat{b}(z,t)\,e^{-i\omega(\beta_o)t} = e^{i\frac{\hat{H}}{\hbar}t}\,\hat{b}(z,0)\,e^{-i\frac{\hat{H}}{\hbar}t}$$

So the Heisenberg operator corresponding to the operator $\hat{b}(z,0)$ is in fact $\hat{b}(z,t)\,e^{-i\omega(\beta_o)t}$.

For a **non-interacting** waveguide (or a fiber) (where, $\hat{b}(\beta,t) = \hat{b}(\beta)$) the above relation gives,

$$\hat{b}(z,t) = L\int_{\beta_o-\frac{\Delta\beta}{2}}^{\beta_o+\frac{\Delta\beta}{2}}\frac{d\beta}{2\pi}\,\hat{b}(\beta)\frac{e^{i(\beta-\beta_o)z}}{\sqrt{L}}e^{-i[\omega(\beta)-\omega(\beta_o)]t}$$

And for a **non-interacting** and a **non-dispersive** waveguide (or a fiber) (where we also have, $\omega(\beta)-\omega(\beta_o) = v_g(\beta-\beta_o)$) we get,

$$\hat{b}(z,t) = L\int_{\beta_o-\frac{\Delta\beta}{2}}^{\beta_o+\frac{\Delta\beta}{2}}\frac{d\beta}{2\pi}\,\hat{b}(\beta)\frac{e^{i(\beta-\beta_o)(z-v_g t)}}{\sqrt{L}}$$

and it is easy to see that,

$$\hat{b}(z,t) = \hat{b}(z-v_g t,0)$$

The above is the same result as the one obtained earlier by solving the travelling wave equation for a non-interacting and non-dispersive waveguide,

$$\left(\frac{\partial}{\partial z}+\frac{1}{v_g}\frac{\partial}{\partial t}\right)\hat{b}(z,t) = 0$$

8.3.5 Equal-Space Commutation Relations

We know that,

$$\left[\hat{b}(z,t),\hat{b}^{+}(z',t)\right] = \delta(z-z')$$

For a non-dispersive and non-interacting waveguide, we can find a simple expression for the equal-space commutation relation,

$$\left[\hat{b}(z,t_1),\hat{b}^{+}(z,t_2)\right] = \left[\hat{b}(z-v_g t_1,0),\hat{b}^{+}(z-v_g t_2,0)\right]$$

Using the equal time commutation relation we get,

$$= \delta(-v_g(t_1-t_2))$$

$$= \frac{1}{v_g}\delta(t_1-t_2)$$

8.3.6 Free-Space Generalization

The discussion above is also relevant to free-space propagation if we assume the dispersion relation for free space,

$$\beta = \frac{\omega}{c}$$

$$\Rightarrow v_g = c$$

$$\Rightarrow \beta_2 = 0\,,\ \beta_3 = 0\,,\ \ldots\ldots\beta_n = 0\ldots\ldots$$

and note that in the case of free space,

$$\varepsilon(x,y) = \varepsilon_o$$

The transverse modes can be chosen as constants and the modes will then represent plane waves with infinite transverse extension propagating in the $\pm z$ direction. If we want to consider a beam with a finite size in the transverse direction, then we would need to consider appropriate free space radiation modes, such as Hermite-Gaussian modes, in which case the beams would converge or diverge in the transverse direction with distance and if this convergence or divergence is small over distances of interest then one can simply replace the transverse mode $\vec{\phi}(x,y)$ in our previous analysis with the transverse beam profile.

8.3.7 Photon Flux Operator

We have seen that the energy flow per second is given by,

$$\hat{S}(z,t) = v_g\,\hbar\omega(\beta_o)\,\hat{b}^{+}(z,t)\,\hat{b}(z,t)$$

The photon flux operator $\hat{F}(z,t)$ is therefore,

$$\hat{F}(z,t) = v_g\,\hat{b}^{+}(z,t)\hat{b}(z,t)$$

The operator $\hat{F}(z,t)$ gives the number of photons passing per second at location z at time t.

8.3.8 Photon Density Operator

Since the photon flux operator is,

$$\hat{F}(z,t) = v_g\,\hat{b}^{+}(z,t)\hat{b}(z,t)$$

The photon density operator $\hat{n}(z,t)$ (operator for the number of photons per unit length) must therefore be,

$$\hat{n}(z,t) = \hat{b}^{+}(z,t)\hat{b}(z,t)$$

8.3.9 Photon Number Operator

One can define photon number operators for propagating radiation packets in two ways. The photon number operator at location z for photon packets is defined as,

$$\hat{N}(z)=\int_{-\infty}^{\infty} dt\,\hat{F}(z,t)=\int_{-\infty}^{\infty} dt\; v_g\;\hat{b}^{+}(z,t)\hat{b}(z,t)$$

The operator $\hat{N}(z)$ corresponds to counting photons in radiation packets while sitting at one fixed location z. The operator $\hat{N}(z)$ corresponds to the photon number measurements made by a photodetector.

The photon number operator at time t for radiation packets can also be defined as,

$$\hat{N}(t)=\int_{-\infty}^{\infty} dz\,\hat{n}(z,t)=\int_{-\infty}^{\infty} dz\;\hat{b}^{+}(z,t)\hat{b}(z,t)$$

The operator $\hat{N}(t)$ corresponds to the number of photons in the entire photon packet at one given instant. For a non-interacting, non-dispersive waveguide (or fiber), we have $\hat{b}(z,t)=\hat{b}(z-v_g t,0)$, and therefore,

$$\hat{N}(z)=\int_{-\infty}^{\infty} dt\,\hat{F}(z,t)=\int_{-\infty}^{\infty} dt\,\hat{F}\left(z-v_g\,t,0\right)=\frac{1}{v_g}\int_{-\infty}^{\infty} dz\;\hat{F}(z,0)=\hat{N}(t=0)$$

and,

$$\hat{N}(t)=\frac{1}{v_g}\int_{-\infty}^{\infty} dz\,\hat{F}(z,t)=\frac{1}{v_g}\int_{-\infty}^{\infty} dz\,\hat{F}(z-x,t-\frac{x}{v_g})$$

Since x is arbitrary, choose $x=z$ to get,

$$\hat{N}(t)=\frac{1}{v_g}\int_{-\infty}^{\infty} dz\,\hat{F}(0,t-\frac{z}{v_g})=\int_{-\infty}^{\infty} d\tau\,\hat{F}(0,\tau)=\hat{N}(z=0)$$

So for a non-interacting, no-dispersive waveguide, the photon number operator for a packet is the same whether one counts photons sitting in one position as the packet passes by or if one counts the photons in one instant in the entire packet. Since there is no loss or gain of photons in the absence of interactions, the photon number operator for photon packets does not depend upon how one does the counting.

8.4 Propagating Quantum States

In the following Sections, we will consider some simple propagating quantum states. We will always deal with a non-dispersive, non-interacting, medium for which,

$$\beta(\omega)=\beta(\omega_o)+\frac{1}{v_g}(\omega-\omega_o)$$

And $\hat{b}(z,t)$ satisfies,

$$\left[\frac{\partial}{\partial z}+\frac{1}{v_g}\frac{\partial}{\partial t}\right]\hat{b}(z,t)=0$$

Solution of the above equation is,

$$\hat{b}(z,t)=\hat{b}\left(z-v_g t,0\right)$$

8.4.1 Photon Number Packets

We defined $\hat{b}(z,t)$ as,

$$\hat{b}(z,t)=L\int_{\beta_o-\frac{\Delta\beta}{2}}^{\beta_o+\frac{\Delta\beta}{2}}\frac{d\beta}{2\pi}\,\hat{b}(\beta)\frac{e^{i(\beta-\beta_o)z}}{\sqrt{L}}e^{-i[\omega(\beta)-\omega(\beta_o)]t}$$

The equal time commutation relations are,

$$\left[\hat{b}(z,t),\hat{b}^{+}(z',t)\right]=\delta(z-z')$$

The photon density operator is,

$$\hat{n}(z,t)=\hat{b}^{+}(z,t)\hat{b}(z,t)$$

Single-Photon Pockets: Consider the state,

$$|\psi\rangle=\hat{b}^{+}(z',0)|0\rangle$$

The state $|\psi\rangle$ represents a single photon localized at z' and moving with group velocity v_g. It is a photon packet because $|\psi\rangle$ is a linear superposition of single photon number states with different wavevectors,

$$\hat{b}^{+}(z',0)|0\rangle=L\int_{\beta_o-\frac{\Delta\beta}{2}}^{\beta_o+\frac{\Delta\beta}{2}}\frac{d\beta}{2\pi}\,\hat{b}^{+}(\beta)\frac{e^{-i(\beta-\beta_o)z'}}{\sqrt{L}}|0\rangle=L\int_{\beta_o-\frac{\Delta\beta}{2}}^{\beta_o+\frac{\Delta\beta}{2}}\frac{d\beta}{2\pi}\,\frac{e^{-i(\beta-\beta_o)z'}}{\sqrt{L}}|1\rangle_{\beta}$$

We can find out more about $|\psi\rangle=\hat{b}^{+}(z',0)|0\rangle$ by applying the density operator $\hat{n}(z,0)=\hat{b}^{+}(z,0)\hat{b}(z,0)$ on $|\psi\rangle$,

$$\begin{aligned}\hat{n}(z,0)|\psi\rangle&=\hat{b}^{+}(z,0)\hat{b}(z,0)\hat{b}^{+}(z',0)|0\rangle\\&=\left[\hat{b}^{+}(z,0)\hat{b}(z,0),\hat{b}^{+}(z',0)\right]|0\rangle+\hat{b}^{+}(z',0)\hat{n}(z,0)|0\rangle\\&=\delta(z-z')\hat{b}^{+}(z',0)|0\rangle\end{aligned}$$

$$\hat{n}(z,0)|\psi\rangle=\delta(z-z')|\psi\rangle$$

Therefore, $|\psi\rangle$ is an eigenstate of the photon density operator $\hat{n}(z,0)$ with an eigenvalue $\delta(z-z')$. The eigenvalue is exactly what one would have expected for a localized photon. The state $|\psi\rangle$ is not properly normalized. The state $|\psi\rangle$ is extremely localized. What if we want to construct a state corresponding to a relatively fat single photon packet , one that is not so sharply localized and also one whose state is properly normalized, and therefore more realistic. Consider the state,

$$|\phi\rangle=\left\{\int_{-\infty}^{\infty}dz'\,A(z')\hat{b}^{+}(z',0)\right\}|0\rangle$$

The norm of the state $|\phi\rangle$ is,

$$\langle\phi|\phi\rangle = \langle 0|\left\{\int_{-\infty}^{\infty} dz''\, A^*(z'')\hat{b}(z'',0)\right\}\left\{\int_{-\infty}^{\infty} dz'\, A(z')\hat{b}^+(z',0)\right\}|0\rangle$$

$$\langle 0|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dz'\,dz'' A^*(z'')A(z')\hat{b}(z'',0)\hat{b}^+(z',0)|0\rangle$$

$$= \langle 0|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dz'dz''\, A^*(z'')A(z')\left\{\delta(z''-z') + \hat{b}^+(z',0)\hat{b}(z'',0)\right\}|0\rangle$$

$$= \langle 0|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dz'\,dz''\, A^*(z'')A(z')\delta(z'-z'')|0\rangle$$

$$= \int_{-\infty}^{\infty} dz'|A(z')|^2\langle 0|0\rangle = \int_{-\infty}^{\infty} dz'|A(z')|^2$$

For $\langle\phi|\phi\rangle = 1$, we need $\int_{-\infty}^{\infty} dz'|A(z')|^2 = 1$. The state $|\phi\rangle$ is not a photon density operator eigenstate since it is a linear superposition of photon density eigenstates. We can calculate the average value of the photon density for the state $|\phi\rangle$ as follows,

$$\langle\phi|\hat{n}(z,0)|\phi\rangle$$
$$= \langle 0|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dz''\,dz'\, A^*(z'')A(z')\hat{b}(z'',0)\hat{n}(z,0)\hat{b}^+(z',0)|0\rangle$$

One way to obtain the final result is to move all the destruction operators appearing the expression above towards the right using the commutation relations. Note that,

$$\begin{aligned}\hat{b}(z'',0)\hat{n}(z,0)\hat{b}^+(z',0) &= \hat{b}(z'',0)\left[\hat{n}(z,0),\hat{b}^+(z',0)\right] + \hat{b}(z'',0)\hat{b}^+(z',0)\hat{n}(z,0)\\ &= \hat{b}(z'',0)\delta(z-z')\hat{b}^+(z',0) + \hat{b}(z'',0)\hat{b}^+(z',0)\hat{n}(z,0)\\ &= \delta(z-z')\delta(z'-z'') + \delta(z-z')\hat{b}^+(z',0)\hat{b}(z'',0)\\ &\quad + \hat{b}(z'',0)\hat{b}^+(z',0)\hat{n}(z,0)\end{aligned}$$

The above result implies,

$$\begin{aligned}\langle\psi|\hat{n}(z,0)|\psi\rangle &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dz''\,dz'\, A^*(z'')A(z')\delta(z-z')\delta(z'-z'')\langle 0|0\rangle\\ &= |A(z)|^2\end{aligned}$$

Thus, the photon density is distributed in space with a probability distribution given by $|A(z)|^2$. We already know that $\int_{-\infty}^{\infty} dz|A(z)|^2 = 1$. The average photon flux is therefore $v_g|A(z)|^2$ (which has the correct units of 1/sec).

$$\langle\psi|\hat{F}(z,0)|\psi\rangle = v_g|A(z)|^2$$

Also, note that,

$$|\phi\rangle = \int_{-\infty}^{\infty} dz'\, A(z')\hat{b}^{+}(z',0)|0\rangle$$

$$= L \int_{\beta_o-\frac{\Delta\omega}{2}}^{\beta_o-\frac{\Delta\omega}{2}} \frac{d\beta}{2\pi} \int_{-\infty}^{\infty} dz'\, A(z') \frac{e^{-i(\beta-\beta_o)z'}}{\sqrt{L}} \hat{b}^{+}(\beta)|0\rangle$$

$$|\phi\rangle = \sqrt{L} \int_{\beta_o-\frac{\Delta\omega}{2}}^{\beta_o-\frac{\Delta\omega}{2}} \frac{d\beta}{2\pi} \tilde{A}(\beta-\beta_o)\,|1\rangle_{\beta}$$

where, $\tilde{A}(\beta-\beta_o)$ is the Fourier transform of $A(z)$. Therefore, $|\phi\rangle$ is a linear superposition of single photon states of different wavevectors and each wavevector in the linear superposition is weighed by $\left|\tilde{A}(\beta-\beta_o)\right|^2$.

We will write these states as,

$$|\phi\rangle = \left\{ \int_{-\infty}^{\infty} dz'\, A(z')\hat{b}^{+}(z',0) \right\} |0\rangle = |A(z)\rangle$$

Time-Dependence: What if we evaluate $\langle\phi|\hat{n}(z,t)|\phi\rangle$? In time t, we expect the single photon packet to have travelled a distance $v_g t$. $\hat{b}(z,t)$ satisfies,

$$\left(\frac{\partial}{\partial z} + \frac{1}{v_g}\frac{\partial}{\partial t} \right) \hat{b}(z,t) = 0$$

$$\Rightarrow \hat{b}(z,t) = \hat{b}(z - v_g t, 0)$$

Similarly,

$$\hat{b}^{+}(z,t) = \hat{b}^{+}(z - v_g t, 0)$$

Therefore,

$$\hat{n}(z,t) = \hat{b}^{+}(z,t)\hat{b}(z,t) = \hat{b}^{+}(z - v_g t,0)\hat{b}(z - v_g t,0) = \hat{n}(z - v_g t,0)$$

$$\Rightarrow \langle\phi|\hat{n}(z,t)|\phi\rangle = \langle\phi|\hat{n}(z - v_g t,0)|\phi\rangle$$

Since for all z,

$$\langle\phi|\hat{n}(z,0)|\phi\rangle = |A(z)|^2$$

$$\Rightarrow \langle\phi|\hat{n}(z,t)|\phi\rangle = \langle\phi|\hat{n}(z - v_g t,0)|\phi\rangle = \left|A(z - v_g t)\right|^2$$

Therefore, at time t the photon density has moved forward by $v_g t$.

Multi-Photon Packets: In order to construct packets containing n photons, we can generalize as follows. Consider the state,

$$|n\rangle = \frac{\left(\int_{-\infty}^{\infty} A(z')\hat{b}^{+}(z',0)dz' \right)^n}{\sqrt{n!}} |0\rangle$$

One can show that $\langle n|n\rangle = 1$ if $\int_{-\infty}^{\infty} |A(z)|^2 dz = 1$. Let us calculate the average photon density,

$$\langle n|\hat{n}(z,0)|n\rangle = \frac{1}{n!}\langle 0| \left(\int dz'\, A(z')\hat{b}(z',0)\right)^n \hat{n}(z,0)\left(\int dz'\, A(z')\hat{b}^+(z',0)\right)^n|0\rangle$$

The computation is carried out by shifting $\hat{n}(z,0)$ to the extreme right (or left) using the commutation relations. The answer is,

$$\langle n|\hat{n}(z,0)|n\rangle = n|A(z)|^2$$

8.4.2 Coherent State Photon Packets

Consider the state defined below for any complex function $\alpha(z)$,

$$|\alpha\rangle = \exp\left[\int_{-\infty}^{\infty} dz'\left(\alpha(z')\hat{b}^+(z',0) - \alpha^*(z')\hat{b}(z',0)\right)\right]|0\rangle$$

$$= \hat{T}(\alpha)|0\rangle$$

Properties of $\hat{T}(\alpha)$ and the State $|\alpha\rangle$:

a) $\hat{T}^+(\alpha) = \hat{T}(-\alpha) = \hat{T}^{-1}(\alpha) \Rightarrow \hat{T}^+(\alpha)T(\alpha) = \hat{1}$

$\Rightarrow$ $|\alpha\rangle$ is normalized

$$\langle\alpha|\alpha\rangle = \langle 0|\hat{T}^+(\alpha)\hat{T}(\alpha)|0\rangle = \langle 0|0\rangle = 1$$

b) $\hat{T}^+(\alpha)\hat{b}(z,0)\hat{T}(\alpha) = \hat{b}(z,0) + \alpha(z)$

$\hat{T}^+(\alpha)\hat{b}^+(z,0)\hat{T}(\alpha) = \hat{b}(z,0) + \alpha^*(z)$

c) $\hat{b}(z,0)|\alpha\rangle = \hat{b}(z,0)\hat{T}(\alpha)|0\rangle = \left\{\left[\hat{b}(z,0), \hat{T}(\alpha)\right] + \hat{T}(\alpha)\hat{b}(z,0)\right\}|0\rangle$

$= \left\{\alpha(z)\hat{T}(\alpha) + \hat{T}(\alpha)\hat{b}(z,0)\right\}|0\rangle = \alpha(z)|\alpha\rangle$

$\Rightarrow \langle\alpha|\hat{b}^+(z,0) = \langle\alpha|\,\alpha^*(z)$

Therefore, $|\alpha\rangle$ is an eigenstate of the $\hat{b}(z,0)$ operator. $|\alpha\rangle$ is in fact a travelling coherent state packet. The average values of the photon density operator and the photon flux operator are,

$$\langle\alpha|n(z,0)|\alpha\rangle = \langle\alpha|\hat{b}^+(z,0)\hat{b}(z,0)|\alpha\rangle = |\alpha(z)|^2$$

$$\langle\alpha|\hat{F}(z,0)|\alpha\rangle = v_g\langle\alpha|\hat{b}^+(z,0)\hat{b}(z,0)|\alpha\rangle = v_g|\alpha(z)|^2$$

Just like closed cavity coherent states, coherent state packet $|\alpha\rangle$ is a linear superposition of photon number packets $|n\rangle$. There is no constrain on $\alpha(z)$ and $\int_{-\infty}^{\infty} dz|\alpha(z)|^2$ can be any number. Suppose $\int_{-\infty}^{\infty} dz|\alpha(z)|^2 = N_o$, then N_o is the average photon number in the state $|\alpha\rangle$. To see this we use the operator for the total number of photons at location z,

$$\hat{N}(z) = \int_{-\infty}^{\infty} dt\, \hat{F}(z,t)$$

and calculate the average photon number for the state $|\alpha\rangle$,

$$\langle\alpha|\,\hat{N}(z)|\alpha\rangle = \int_{-\infty}^{\infty} dt\,\langle\alpha|\hat{F}(z,t)|\alpha\rangle = \int_{-\infty}^{\infty} dt\langle\alpha|\hat{F}(z-v_g t,0)|\alpha\rangle$$

$$= v_g \int_{-\infty}^{\infty} dt\,|\alpha(z-v_g t)|^2 = \int_{-\infty}^{\infty} dy\,|\alpha(y)|^2 = N_o$$

Thus, $|\alpha\rangle$ is a coherent state packet with an average photon number equal to $\int_{-\infty}^{\infty}|\alpha(z)|^2\,dz = N_o$. What about photon number fluctuations? We start from,

$$\langle\alpha|\hat{N}^2(z)|\alpha\rangle = \langle\alpha|\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \hat{F}(z,t_1)\hat{F}(z,t_2)|\alpha\rangle$$

$$= \langle 0|\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2\, \hat{T}^+(\alpha)\hat{F}(z,t_1)\hat{F}(z,t_2)\hat{T}(\alpha)|0\rangle$$

$$= v_g^2\,\langle 0|\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \left[\hat{b}^+(z-v_g t_1,0)+\alpha^*(z-v_g t_1)\right]$$
$$\left[\hat{b}(z-v_g t_1,0)+\alpha(z-v_g t_1)\right]$$
$$\left[\hat{b}^+(z-v_g t_2,0)+\alpha^*(z-v_g t_2)\right]$$
$$\left[\hat{b}(z-v_g t_2,0)+\alpha(z-v_g t_2)\right]|0\rangle$$

$$= v_g^2 \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2\, v_g^2 |\alpha(z-v_g t_1)|^2\,|\alpha(z-v_g t_2)|^2$$

$$+ v_g^2 \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \left\{\alpha^*(z-v_g t_1)\alpha(z-v_g t_2)\right. \quad \times$$
$$\left.\langle 0|\hat{b}(z-v_g t_1,0)\hat{b}^+(z-v_g t_2,0)|0\rangle\right\}$$

$$= \left(\int_{-\infty}^{\infty} dy|\alpha(y)|^2\right)^2 + v_g \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2\, \alpha(z-v_g t_1)\alpha(z-v_g t_2)\times\delta(t_1-t_2)$$

$$= N_o^2 + v_g \int_{-\infty}^{\infty} dt_1 |\alpha(z-v_g t_1)|^2 = N_o^2 + N_o$$

Therefore,

$$\langle\alpha|\Delta\hat{N}^2(z)|\alpha\rangle = \langle\alpha|\hat{N}^2(z)|\alpha\rangle - \langle\alpha|\hat{N}(z)|\alpha\rangle^2 = N_o$$

The standard deviation in photon number is equal to the mean, as expected for coherent states.

8.4.3 Continuous-Wave (CW) Coherent States

A continuous beam of light is closer to common experience than ultrashort pulses (or packets). Suppose we want to construct a coherent state corresponding to a continuous electromagnetic wave of phase ϕ and power (i.e. energy flow/sec) equal to P_o. The photon flux is then $P_o/\hbar\omega_o$ and is independent of location. We define a complex function $\alpha(z)$ as,

$$\alpha(z) = \sqrt{\frac{P_o}{v_g \hbar\omega_o}}\, e^{i\phi}$$

Then $\int_{-\infty}^{\infty}|\alpha(z)|^2 dz=\infty$. A continuous wave of constant power will have infinite value for the average photon number. The average photon flux is,

$$\langle\alpha|\hat{F}(z,0)|\alpha\rangle=v_g|\alpha(z)|^2=\frac{P_o}{\hbar\omega_o}$$

Also,

$$\langle\alpha|\hat{F}(z,t)|\alpha\rangle=v_g\left|\alpha\left(z-v_g t\right)\right|^2=\frac{P_o}{\hbar\omega_o}$$

The flux is independent of the location.

Photon Flux Noise: We want to evaluate the noise in the photon flux for the continuous wave coherent state. We know the average flux is $P_o/\hbar\omega_o$. Below, we calculate the flux-flux correction function,

$$\begin{aligned}
&\langle\alpha|\hat{F}(z,t_1)\hat{F}(z,t_2)|\alpha\rangle\\
&=\langle\alpha|\hat{F}\left(z-v_g t_1,0\right)\hat{F}\left(z-v_g t_2,0\right)|\alpha\rangle\\
&=v_g^2\langle 0|\hat{T}^+(\alpha)\hat{b}^+\left(z-v_g t_{1,}0\right)\hat{b}\left(z-v_g t_1,0\right)\hat{b}^+\left(z-v_g t_{2,}0\right)\hat{b}\left(z-v_g t_2,0\right)\hat{T}(\alpha)|0\rangle
\end{aligned}$$

Insert $\hat{T}^+(\alpha)\hat{T}(\alpha)$ between each operator to get.

$$\begin{aligned}
&=v_g^2\langle 0|\left(\hat{b}^+(z-v_g t_{1,}0)+\alpha^*(z-v_g t_1)\right)\left(\hat{b}(z-v_g t_{1,}0)+\alpha(z-v_g t_1)\right)\\
&\qquad\left(\hat{b}^+(z-v_g t_2,0)+\alpha^*(z-v_g t_2,0)\right)\left(\hat{b}(z-v_g t_{2,}0+\alpha(z-v_g t_2)\right)|0\rangle\\
&=\left(\frac{P_o}{\hbar\omega_o}\right)^2+v_g\frac{P_o}{\hbar\omega_o}\langle 0|\hat{b}\left(z-v_g t_{1,}0\right)\hat{b}^+\left(z-v_g t_{2,}0\right)|0\rangle\\
&=\left(\frac{P_o}{\hbar\omega_o}\right)^2+v_g\frac{P_o}{\hbar\omega_o}\langle 0|\left[\hat{b}(z-v_g t_{1,}0),\hat{b}^+(z-v_g t_{2,}0)\right]\\
&\qquad\qquad+\hat{b}^+(z-v_g t_{1,}0),\hat{b}(z-v_g t_{2,}0)|0\rangle
\end{aligned}$$

The left hand side further simplifies to,

$$\begin{aligned}
&=\left(\frac{P_o}{\hbar\omega_o}\right)^2+v_g\frac{P_o}{\hbar\omega_o}\langle 0|\ \delta(z-v_g t_1-z+v_g t_2)\ |0\rangle\\
&=\left(\frac{P_o}{\hbar\omega_o}\right)^2+\frac{P_o}{\hbar\omega_o}\ v_g\ \delta\left(v_g(t_1-t_2)\right)\\
&=\left(\frac{P_o}{\hbar\omega_o}\right)^2+\frac{P_o}{\hbar\omega_o}\delta\left(t_1-t_2\right)
\end{aligned}$$

The photon flux noise correction function is then,

$$\langle\alpha|\ \Delta\hat{F}(z,t_1)\Delta\hat{F}(z,t_2)\ |\alpha\rangle$$

where,

$$\Delta\hat{F}(z,t_1)=\hat{F}(z,t_1)-\langle\alpha|\hat{F}(z,t_1)|\alpha\rangle=\hat{F}(z,t_1)-\frac{P_o}{\hbar\omega_o}$$

$$\Delta\hat{F}(z,t_2)=\hat{F}(z,t_2)-\langle\alpha|\hat{F}(z,t_2)|\alpha\rangle=\hat{F}(z,t_2)-\frac{P_o}{\hbar\omega_o}$$

and,

$$\langle\alpha|\,\Delta\hat{F}(z,t_1)\,\Delta\hat{F}(z,t_2)|\alpha\rangle = \frac{P_o}{h\omega_o}\,\delta(t_1-t_2)$$

The noise in the photon flux is delta correlated in time. The spectral density of the photon flux noise is,

$$S_{\Delta F\,\Delta F}(\omega)=\int_{-\infty}^{\infty} d\tau\, e^{i\omega\tau}\,\langle\alpha|\Delta\hat{F}(z,t+\tau)\Delta\hat{F}(z,t)|\alpha\rangle$$
$$=\frac{P_o}{\hbar\omega_o}$$

$S_{\Delta F\Delta F}(\omega)$ is flat, and has a value equal to the mean photon flux. Therefore, the photon flux for continuous wave coherent states has shot noise characteristics. In fact, it is not difficult to show that the noise in photon flux is indeed exactly shot noise. This is an interesting result. It says that a person sitting in one location and observing photons in a continuous wave coherent state of radiation will see that photon arrival times are completely uncorrelated.

8.4.4 Time Dependence of Propagating States in the Schrodinger Picture

Consider a single photon packet at time $t=0$ given by,

$$|\psi(t=0)\rangle=\int_{-\infty}^{\infty} dz'\; A(z')\,\hat{b}^{+}(z',0)|0\rangle = |A(z)\rangle$$

where,

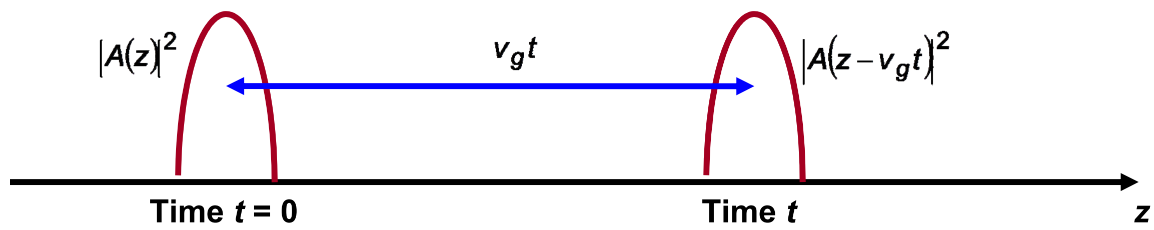
$$\int_{-\infty}^{\infty} dz\,|A(z)|^2 = 1$$

We want to find the state $|\psi(t)\rangle$ at a later time t. Recall that,

$$\hat{b}(z,t)e^{-i\omega_o t} = e^{i\frac{\hat{H}}{\hbar}t}\,\hat{b}(z,0)\,e^{-i\frac{\hat{H}}{\hbar}t}$$
$$\hat{b}^{+}(z,t)e^{i\omega_o t} = e^{i\frac{\hat{H}}{\hbar}t}\,\hat{b}^{+}(z,0)\,e^{-i\frac{\hat{H}}{\hbar}t}$$

The state $|\psi(t)\rangle$ is found as follows,

$$|\psi(t)\rangle = e^{-i\frac{\hat{H}}{\hbar}t}|\psi(t=0)\rangle = e^{-i\frac{\hat{H}}{\hbar}t}\int_{-\infty}^{\infty} dz'\; A(z')\,\hat{b}^{+}(z',0)\,|0\rangle$$
$$= e^{-i\frac{\hat{H}}{\hbar}t}\int_{-\infty}^{\infty} dz'\; A(z')\,\hat{b}^{+}(z',0)\; e^{+i\frac{\hat{H}}{\hbar}t}\,e^{-i\frac{\hat{H}}{\hbar}t}|0\rangle = \int_{-\infty}^{\infty} dz'\; A(z')e^{-i\omega_o t}\,\hat{b}^{+}(z',-t)\,|0\rangle$$
$$= \int_{-\infty}^{\infty} dz'\; A(z')e^{-i\omega_o t}\;\hat{b}^{+}\left(z'+v_g t,0\right)|0\rangle = \int_{-\infty}^{\infty} dz'\; A(z'-v_g t)e^{-i\omega_o t}\;\hat{b}_1^{+}(z',0)|0\rangle$$
$$= \left|A(z'-v_g t)e^{-i\omega_o t}\right\rangle$$



Note that we have ignored the infinite vacuum energy in the derivation above. The above result shows that at time t the wavepacket has moved forward in space by $v_g t$. As another example, consider a coherent state packet given at time $t = 0$ given by,

$$|\psi(t=0)\rangle = e^{\int_{-\infty}^{\infty} dz' \left\{ \alpha(z')\hat{b}_1^+(z',0) - \alpha^*(z')\hat{b}_1(z',0) \right\}} |0\rangle = |\alpha(z)\rangle$$

We want to find the state $|\psi(t)\rangle$ at a later time t. We get,

$$|\psi(t)\rangle = e^{-i\frac{\hat{H}}{\hbar}t} |\psi(t=0)\rangle = e^{-i\frac{\hat{H}}{\hbar}t} e^{\int_{-\infty}^{\infty} dz' \left\{ \alpha(z')\hat{b}^+(z',0) - \alpha^*(z')\hat{b}(z',0) \right\}} |0\rangle$$

$$= e^{-i\frac{\hat{H}}{\hbar}t} e^{\int_{-\infty}^{\infty} dz' \left\{ \alpha(z')\hat{b}^+(z',0) - \alpha^*(z')\hat{b}(z',0) \right\}} e^{+i\frac{\hat{H}}{\hbar}t} e^{-i\frac{\hat{H}}{\hbar}t} |0\rangle$$

$$= e^{\int_{-\infty}^{\infty} dz' \left\{ \alpha(z')\hat{b}^+(z',-t)e^{-i\omega_o t} - \alpha^*(z')\hat{b}(z',-t)e^{i\omega_o t} \right\}} |0\rangle$$

$$= e^{\int_{-\infty}^{\infty} dz' \left\{ \alpha(z')e^{-i\omega_o t}\hat{b}^+(z'+v_g t,0) - \alpha^*(z')e^{i\omega_o t}\hat{b}(z'+v_g t,0) \right\}} |0\rangle$$

$$= e^{\int_{-\infty}^{\infty} dz' \left\{ \alpha(z'-v_g t)e^{-i\omega_o t}\hat{b}^+(z',0) - \alpha^*(z'-v_g t)e^{i\omega_o t}\hat{b}(z',0) \right\}} |0\rangle$$

$$= \left| \alpha(z'-v_g t)e^{-i\omega_o t} \right\rangle$$

8.4.5 Quadrature Operators and Quadrature Noise for Propagating States

The quadrature operators for propagating states are defined as,

$$\hat{b}(z,t) = \left[\hat{x}_\theta(z,t) + i\,\hat{x}_{\theta+\pi/2}(z,t) \right] e^{i\theta}$$

$$\hat{b}^+(z,t) = \left[\hat{x}_\theta(z,t) - i\,\hat{x}_{\theta+\pi/2}(z,t) \right] e^{-i\theta}$$

Note that unlike the quadrature operators for cavity fields, quadrature operators for propagating fields have units of inverse square root of length.

As an example, consider a coherent state,

$$|\psi(t=0)\rangle = e^{\int_{-\infty}^{\infty} dz' \left\{ \alpha(z')\hat{b}_1^+(z',0) - \alpha^*(z')\hat{b}_1(z',0) \right\}} |0\rangle = |\alpha(z)\rangle$$

where,

$$\alpha(z) = |\alpha(z)| e^{i\phi(z)}$$

The average value of the quadrature $\hat{x}_1(z,t)$ is,

$$\langle\psi(t=0)|\hat{x}_1(z,t)|\psi(t=0)\rangle = \langle\psi(t=0)|\frac{\hat{b}(z,t)+\hat{b}^+(z,t)}{2}|\psi(t=0)\rangle$$

Recall that,

$$\langle\psi(t=0)|\hat{b}(z,t)|\psi(t=0)\rangle = \langle\psi(t=0)|\,\hat{b}(z-v_g t,0)\,|\psi(t=0)\rangle = \alpha(z-v_g t)$$

Similarly,

$$\langle\psi(z=0)|\hat{b}^+(z,t)|\psi(t=0)\rangle = \alpha^*(z-v_g t)$$

So we get,

$$\langle\psi(t=0)|\hat{x}_1(z,t)|\psi(t=0)\rangle = \langle\psi(t=0)|\frac{\hat{b}(z,t)+\hat{b}^+(z,t)}{2}|\psi(t=0)\rangle$$

$$= \frac{\alpha(z-v_g t)+\alpha^*(z-v_g t)}{2}$$

$$= |\alpha(z-v_g t)|\cos[\phi(z-v_g t)]$$

The average value of the quadrature $\hat{x}_2(z,t)$ is,

$$\langle\psi(t=0)|\hat{x}_2(z,t)|\psi(t=0)\rangle = \langle\psi(t=0)|\frac{\hat{b}(z,t)-\hat{b}^+(z,t)}{2i}|\psi(t=0)\rangle$$

$$= \frac{\alpha(z-v_g t)-\alpha^*(z-v_g t)}{2i}$$

$$= |\alpha(z-v_g t)|\sin[\phi(z-v_g t)]$$

The next question to ask is, what is the quadrature noise? As you will see, it is not meaningful to evaluate quantities such as, $\langle\Delta\hat{x}_1^2(z,t)\rangle$ or $\langle\Delta\hat{x}_2^2(z,t)\rangle$. Instead we look at the quadrature noise correlation functions, $\langle\Delta\hat{x}_1(z,t)\Delta\hat{x}_1(z,t')\rangle$ and $\langle\Delta\hat{x}_2(z,t)\Delta\hat{x}_2(z,t')\rangle$. We assume, as before, a coherent state,

$$|\psi(t=0)\rangle = e^{\int_{-\infty}^{\infty} dz'\left\{\alpha(z')\hat{b}_1^+(z',0)-\alpha^*(z')\hat{b}_1(z',0)\right\}}|0\rangle = |\alpha(z)\rangle$$

and find $\langle\hat{x}_1(z,t)\hat{x}_1(z,t')\rangle$,

$$\langle\psi(t=0)|\hat{x}_1(z,t)\hat{x}_1(z,t')|\psi(t=0)\rangle$$

$$= \langle\psi(t=0)|\left[\frac{\hat{b}(z,t)+\hat{b}^+(z,t)}{2}\right]\left[\frac{\hat{b}(z,t')+\hat{b}^+(z,t')}{2}\right]|\psi(t=0)\rangle$$

$$= \langle\psi(t=0)|\left[\frac{\hat{b}(z-v_g t,0)+\hat{b}^+(z-v_g t,0)}{2}\right]\left[\frac{\hat{b}(z-v_g t',0)+\hat{b}^+(z-v_g t',0)}{2}\right]|\psi(t=0)\rangle$$

$$= \left[\frac{\alpha(z-v_g t)+\alpha^*(z-v_g t)}{2}\right]\left[\frac{\alpha(z-v_g t')+\alpha^*(z-v_g t')}{2}\right] + \frac{1}{4}\frac{\delta(t-t')}{v_g}$$

Therefore, the quadrature noise correlation function is,

$$\langle\psi(t=0)|\Delta\hat{x}_1(z,t)\Delta\hat{x}_1(z,t')|\psi(t=0)\rangle = \frac{1}{4}\frac{\delta(t-t')}{v_g}$$

Similarly, one can show that,

$$\langle \psi(t=0) | \Delta \hat{x}_2(z,t) \Delta \hat{x}_2(z,t') | \psi(t=0) \rangle = \frac{1}{4} \frac{\delta(t-t')}{v_g}$$

In general, for a coherent state,

$$\langle \psi(t=0) | \Delta \hat{x}_\theta(z,t) \Delta \hat{x}_\theta(z,t') | \psi(t=0) \rangle = \frac{1}{4} \frac{\delta(t-t')}{v_g}$$

Earlier, in the course we had seen that for coherent states inside a cavity, the mean square quadrature fluctuations equalled $1/4$ for every quadrature. For propagating coherent states, the quadrature noise correlations are delta-correlated with a weight proportional to $1/4$ for every quadrature.