Chapter 7: Quantum States of Light

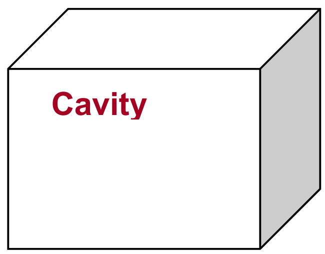
7.1 Cavity Fields

The operators for the fields inside a cavity are,

$$\hat{\vec{A}}(\vec{r},t)=\sum_m \sqrt{\frac{\hbar}{2\omega_m\varepsilon_o\varepsilon_m}}\left(\hat{a}_m(t)+\hat{a}_m^+(t)\right)\vec{U}_m(\vec{r})$$

$$\hat{\vec{E}}(\vec{r},t)=\sum_m i\sqrt{\frac{\hbar\omega_n}{2\varepsilon_o\varepsilon_m}}\left(\hat{a}_m(t)-\hat{a}_m^+(t)\right)\vec{U}_m(\vec{r})$$

$$\hat{\vec{H}}(\vec{r},t)=\sum_m \frac{1}{\mu_o}\sqrt{\frac{\hbar}{2\omega_m\varepsilon_o\varepsilon_m}}\left(\hat{a}_m(t)+\hat{a}_m^+(t)\right)\nabla\times\vec{U}_m(\vec{r})$$



The Hamiltonian is,

$$\hat{H}=\sum_m \hbar\omega_m\left(\hat{a}_m^+\,\hat{a}_m+\frac{1}{2}\right)$$

The time dependence of the operators in the Heisenberg picture is,

$$\hat{a}_m(t)=\hat{a}_m\,e^{-i\omega_m t}$$

$$\hat{a}_m^+(t)=\hat{a}_m^+\,e^{i\omega_m t}$$

In this chapter we will mostly consider only a single mode of the field in a cavity and ignore the remaining modes. In order to keep the notation from getting too cumbersome, we will drop the mode number in the subscripts (e.g. "m" above) unless necessary, and write the Hamiltonian and the field operators as follows,

$$\hat{H}=\hbar\omega_o\left(\hat{a}^+\,\hat{a}+\frac{1}{2}\right)$$

$$\hat{\vec{A}}(\vec{r},t)=\sqrt{\frac{\hbar}{2\omega_o\varepsilon_o\varepsilon}}\left(\hat{a}(t)+\hat{a}^+(t)\right)\vec{U}(\vec{r})$$

$$\hat{\vec{E}}(\vec{r},t)=i\sqrt{\frac{\hbar\omega_o}{2\varepsilon_o\varepsilon}}\left(\hat{a}\;(t)-\hat{a}^+(t)\right)\vec{U}(\vec{r})$$

$$\hat{\vec{H}}(\vec{r},t)=\frac{1}{\mu_o}\sqrt{\frac{\hbar}{2\omega_o\varepsilon_o\varepsilon}}\left(\hat{a}(t)+\hat{a}^+(t)\right)\nabla\times\vec{U}(\vec{r})$$

7.2 Fock States or Photon Number States

Number states of a mode contain a definite number of photons. As discussed in an earlier Chapter, these are eigenstates of the photon number operator and are defined as,

$$|n\rangle = \frac{(a^{+})^{n}}{\sqrt{n!}}|0\rangle$$

$$\Rightarrow \hat{n}|n\rangle = \hat{a}^{+}\hat{a}|n\rangle = n|n\rangle$$

and,

$$\hat{H}|n\rangle = \hbar\omega_o\left(n+\frac{1}{2}\right)|n\rangle$$

Also,

$$\hat{a}|n\rangle = \sqrt{n-1}|n-1\rangle$$

$$\hat{a}^{+}|n\rangle = \sqrt{n}|n+1\rangle$$

It follows that,

$$\langle n|\hat{a}|n\rangle = \langle n|\hat{a}^{+}|n\rangle = 0$$

and,

$$\langle n|\hat{\vec{E}}(\vec{r},t)|n\rangle = \langle n|\hat{\vec{H}}(\vec{r},t)|n\rangle = 0\,.$$

So surely $|n\rangle$ cannot be the quantum state of radiation coming out of, say, antennas, where the field is expected to have a non-zero average value.

7.3 Coherent States

Coherent states of light are the closest approximation to the classical radiation emitted from oscillating currents. We define an operator $\hat{D}(\alpha)$ (called the displacement operator) as,

$$\hat{D}(\alpha) = e^{\alpha\hat{a}^{+} - \alpha^{*}\hat{a}}$$

where α is a complex number,

$$\alpha = |\alpha|\, e^{i\theta}$$

A coherent state of a radiation mode, $|\alpha\rangle$, is defined as,

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle$$

Since,

$$e^{\hat{A}+\hat{B}} = e^{-\frac{[\hat{A},\hat{B}]}{2}}\, e^{\hat{A}}\, e^{\hat{B}} \qquad \left(\text{provided} \quad \left[\hat{A},\left[\hat{A},\hat{B}\right]\right]=0 \quad \left[\hat{B},\left[\hat{A},\hat{B}\right]\right]=0\right)$$

we have,

$$\hat{D}(\alpha) = e^{-\frac{|\alpha|^2}{2}}\, e^{\alpha\,\hat{a}^{+}}\, e^{-\alpha^{*}\hat{a}}$$

$$\hat{D}(\alpha) = e^{\frac{|\alpha|^2}{2}}\, e^{-\alpha^{*}\hat{a}}\, e^{\alpha\,\hat{a}^{+}}$$

7.3.1 Properties of $\hat{D}(\alpha)$

$\hat{D}(\alpha)$ has the following properties:

(i) $\hat{D}^{+}(\alpha) = e^{-\alpha\,\hat{a}^{+} + \alpha^{*}\hat{a}} = e^{-\frac{|\alpha|^2}{2}}\, e^{-\alpha\,\hat{a}^{+}}\, e^{\alpha^{*}\,\hat{a}}$

$$\Rightarrow \hat{D}^{+}(\alpha)\hat{D}(\alpha) = \hat{1}$$

$$\Rightarrow \hat{D}^{+}(\alpha) = \hat{D}^{-1}(\alpha)$$

(ii) $\hat{D}^{+}(\alpha)\hat{a} \;\; D(\alpha) = \hat{a} + \alpha$

Proof:

$$e^{-\alpha\hat{a}^{+} + \alpha^{*}\hat{a}} \;\; \hat{a} \; e^{\alpha\hat{a}^{+} - \alpha^{*}\hat{a}}$$

Recall that,

$$e^{-\alpha\hat{B}} \hat{A} e^{\alpha\hat{B}} = \hat{A} + \alpha \quad \text{ if } \left[\hat{A}, \hat{B}\right] = 1$$

$$\Rightarrow \hat{D}^{+}(\alpha) \; \hat{a} \; \hat{D}(\alpha) = \hat{a} + \alpha$$

(iii) $\hat{D}^{+}(\alpha) \; \hat{a}^{+} \;\; \hat{D}(\alpha) = \hat{a}^{+} + \alpha^{*}$

The above relation is obtained by taking the adjoint of the relation in (ii).

7.3.2 Properties of Coherent States

Important properties of coherent states are as follows:

(i) Coherent states are properly normalized,

$$\langle \alpha \mid \alpha \rangle = \left\langle 0 \middle| \hat{D}^{+}(\alpha)\hat{D}(\alpha) \middle| 0 \right\rangle = \langle 0 \mid 0 \rangle = 1$$

(ii) A coherent state is a linear superposition of photon number states,

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle = \exp\left(-\frac{|\alpha|^{2}}{2}\right)\exp(\alpha \; \hat{a}^{+})\exp(-\alpha^{*} \; \hat{a})|0\rangle$$

$$= \exp\left(-\frac{|\alpha|^{2}}{2}\right)\exp(\alpha \; \hat{a}^{+})|0\rangle$$

$$= \exp\left(-\frac{|\alpha|^{2}}{2}\right)\sum_{n=0}^{\infty}\frac{(\alpha \; \hat{a}^{+})^{n}}{n!}|0\rangle$$

$$= \exp\left(-\frac{|\alpha|^{2}}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^{n}}{\sqrt{n!}}\frac{(\hat{a}^{+})^{n}}{\sqrt{n!}}|0\rangle$$

$$|\alpha\rangle = \exp\left(-\frac{|\alpha|^{2}}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^{n}}{\sqrt{n!}}|n\rangle$$

(iii) If a photon number measurement is performed on a coherent state, the probability $P(n)$ of finding n photons in a coherent state $|\alpha\rangle$ is,

$$P(n) = |\langle n \mid \alpha \rangle|^{2} = \left|\exp\left(-\frac{|\alpha|^{2}}{2}\right)\frac{\alpha^{n}}{\sqrt{n!}}\right|^{2}$$

$$= \frac{\left(|\alpha|^{2}\right)^{n}}{n!}\exp(-|\alpha|^{2})$$

The photon number distribution in a coherent state looks like a Poisson distribution.

(iv) Coherent states are eigenstates of the destruction operator $\hat{a}$,

$$\hat{a}|\alpha\rangle = \hat{a}\,\hat{D}(\alpha)|0\rangle$$

Since,

$$\hat{D}^{+}(\alpha)\hat{a}\,\hat{D}(\alpha) = \hat{a} + \alpha$$

we get upon multipying both sides with $\hat{D}(\alpha)$,

$$\hat{a}\,\hat{D}(\alpha) = \hat{D}(\alpha)[\hat{a} + \alpha]$$

$$= \hat{D}(\alpha)\hat{a} + \hat{D}(\alpha)\alpha$$

Therefore,

$$\hat{a}|\alpha\rangle = \hat{a}\,\hat{D}(\alpha)|0\rangle = \hat{D}(\alpha)\,\hat{a}|0\rangle + \hat{D}(\alpha)\,\alpha|0\rangle$$

$$= \hat{D}(\alpha)\,\alpha|0\rangle = \alpha|\alpha\rangle$$

$$\Rightarrow \hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

The above equation also implies,

$$\langle\alpha|\hat{a}^{+} = \langle\alpha|\alpha^{*}$$

(v) Mean photon number and variance in the photon number for a coherent state are as follows,

$$\langle\hat{n}\rangle = \langle\alpha|\hat{n}|\alpha\rangle = \langle\alpha|\hat{a}^{+}a|\alpha\rangle = \langle\alpha|\alpha^{*}\alpha|\alpha\rangle = |\alpha|^{2}\langle\alpha|\alpha\rangle = |\alpha|^{2}$$

$$\left\langle\hat{n}^{2}\right\rangle = \langle\alpha|\hat{a}^{+}a\hat{a}^{+}a|\alpha\rangle = \langle\alpha|\hat{a}^{+}\left(\left[\hat{a},\hat{a}^{+}\right] + \hat{a}^{+}\hat{a}\right)\hat{a}|\alpha\rangle$$

$$= \langle\alpha|\hat{a}^{+}a + \hat{a}^{+}\hat{a}^{+}\hat{a}\hat{a}|\alpha\rangle$$

$$= \langle\alpha|\hat{a}^{+}a|\alpha\rangle + \langle\alpha|\hat{a}^{+}\hat{a}^{+}\hat{a}\hat{a}|\alpha\rangle$$

$$= |\alpha|^{2} + |\alpha|^{4}$$

$$= \langle\hat{n}\rangle + \langle\hat{n}\rangle^{2}$$

Therefore,

$$\left\langle\Delta\hat{n}^{2}\right\rangle = \left\langle\hat{n}^{2}\right\rangle - \langle\hat{n}\rangle^{2} = \langle\hat{n}\rangle$$

The variance in photon number is equal to the mean. This is not surprising since we saw earlier that the photon number distribution is Poissonian.

(vi) Different coherent states are not orthogonal. Suppose α and β are two different complex numbers and $|\alpha\rangle$ and $|\beta\rangle$ are the corresponding coherent states. We now find the value of the inner product $\langle\alpha\,|\,\beta\rangle$,

$$\langle\alpha\,|\,\beta\rangle = \langle 0|\hat{D}^{+}(\alpha)\,\hat{D}^{+}(\beta)|0\rangle$$

Since,

$$\hat{D}^{+}(\alpha)\hat{D}(\beta) = e^{-\alpha\hat{a}^{+} + \alpha^{*}\hat{a}}\,e^{\beta\hat{a}^{+} - \beta^{*}\hat{a}}$$

$$= e^{-\frac{|\alpha|^{2}}{2}}\,e^{-\alpha\hat{a}^{+}}\,e^{\alpha^{*}\hat{a}}\,e^{-\frac{|\beta|^{2}}{2}}\,e^{\beta\hat{a}^{+}}\,e^{-\beta^{*}\hat{a}}$$

$$= e^{-\frac{|\alpha|^{2}}{2} - \frac{|\beta|^{2}}{2}}\,e^{-\alpha\,\hat{a}^{+}}\,e^{\beta\,\hat{a}^{+}}\,e^{\alpha^{*}\hat{a}}\,e^{-\beta^{*}\hat{a}}\,e^{\alpha^{*}\beta}$$

Therefore,

$$\Rightarrow \left\langle 0 \middle| \hat{D}^{+}(\alpha)\ \hat{D}^{+}(\beta) \middle| 0 \right\rangle = e^{-\frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2} + \alpha^* \beta}$$

$$\Rightarrow \left\langle \alpha \mid \beta \right\rangle = e^{-\frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2} + \alpha^* \beta}$$

$$\Rightarrow \left| \left\langle \alpha \middle| \beta \right\rangle \right|^2 = e^{-|\alpha - \beta|^2}$$

(vii) Coherent states form a complete set. The completeness relation can be written as,

$$\frac{1}{\pi} \int_{-\infty}^{\infty} d\alpha_r \int_{-\infty}^{\infty} d\alpha_i \left| \alpha \right\rangle \left\langle \alpha \right| = \hat{1}$$

where $\alpha = \alpha_r + i\alpha_i$.

(viii) Mean values of field operators are non-zero for coherent states. Note that,

$$\left\langle \alpha \middle| \hat{a} \middle| \alpha \right\rangle = \alpha$$

$$\left\langle \alpha \middle| \hat{a}^{+} \middle| \alpha \right\rangle = \alpha^*$$

Therefore, if the field operator for a single field mode is written as,

$$\hat{\vec{A}}(r,t) = \sqrt{\frac{\hbar}{2\omega_o \varepsilon_o \varepsilon}} \left(\hat{a} e^{-i\omega_o t} + \hat{a}^{+} e^{i\omega_o t} \right) \vec{U}(\vec{r})$$

and we have a coherent state, i.e.,

$$\left| \alpha \right\rangle = e^{\alpha \hat{a}^{+} - \alpha^* \hat{a}} \left| 0 \right\rangle$$

then,

$$\left\langle \alpha \middle| \hat{\vec{A}}(r,t) \middle| \alpha \right\rangle = \sqrt{\frac{\hbar}{2\omega_o \varepsilon_o \varepsilon}} \left(\alpha\, e^{-i\omega_o t} + \alpha^*\, e^{i\omega_o t} \right) \vec{U}(\vec{r})$$

and,

$$\left\langle \alpha \middle| \hat{\vec{E}}(r,t) \middle| \alpha \right\rangle = i \sqrt{\frac{\hbar \omega_o}{2\varepsilon_o \varepsilon}} \left(\alpha\, e^{-i\omega_o t} - \alpha^*\, e^{i\omega_o t} \right) \vec{U}(\vec{r})$$

$$\left\langle \alpha \middle| \hat{\vec{H}}(r,t) \middle| \alpha \right\rangle = \frac{1}{\mu_o} \sqrt{\frac{\hbar}{2\omega_o \varepsilon_o \varepsilon}} \left(\alpha\, e^{-i\omega_o t} + \alpha^*\, e^{i\omega_o t} \right) \nabla \times \vec{U}(\vec{r})$$

Notice that if $\alpha = |\alpha| e^{i\phi}$ then the phase ϕ is also the phase of the average values of the field oscillations.

7.3.3 Quadrature Operators and Quadrature Fluctuations of Coherent States

Recall from Chapter 6 that all narrow band real signals $y(t)$ can be represented by phasors,

$$y(t) = \mathrm{Re}\left\{ x(t) e^{-i\omega_o t} \right\} = \frac{1}{2}\left[x(t) e^{-i\omega_o t} + x^*(t) e^{i\omega_o t} \right]$$

If,

$$x(t) = x_1(t) + i x_2(t)$$

then $x_1(t)$ and $x_2(t)$ are the quadratures of $y(t)$. With this in mind, and the fact that (for a single mode cavity),

$$\hat{\vec{A}}(r,t) \propto \frac{1}{2}\left[\hat{a}\, e^{-i\omega t} + \hat{a}^{+}\, e^{i\omega t} \right]$$

we define the (Schrodinger picture) quadrature operators, $\hat{x}_1$ and $\hat{x}_2$, for a mode as (mode subscript suppressed),

$$\hat{a} = \hat{x}_1 + i\hat{x}_2$$

Note that $\hat{x}_1$ and $\hat{x}_2$ are Hermitian operators and observables. It follows that,

$$\Rightarrow \hat{a}^+ = \hat{x}_1 - i\hat{x}_2$$

$$\Rightarrow \hat{x}_1 = \frac{\hat{a}+\hat{a}^+}{2} \qquad \hat{x}_2 = \frac{\hat{a}-\hat{a}^+}{2i}$$

$$\Rightarrow [\hat{x}_1, \hat{x}_2] = \frac{i}{2}$$

If $\Delta\hat{x}_1 = \hat{x}_1 - \langle\hat{x}_1\rangle$ and $\Delta\hat{x}_2 = \hat{x}_2 - \langle\hat{x}_2\rangle$, then the commutator result above implies the uncertainty relation,

$$\langle\Delta\hat{x}_1^2\rangle\langle\Delta\hat{x}_2^2\rangle \geq \frac{1}{16}$$

For a coherent state $|\alpha\rangle$, with $\alpha = |\alpha|e^{i\phi}$, we have,

$$\langle\alpha|\hat{x}_1|\alpha\rangle = \frac{\alpha+\alpha^*}{2} = \mathrm{Re}\{\alpha\} = |\alpha|\cos\phi$$

$$\langle\alpha|\hat{x}_2|\alpha\rangle = \frac{\alpha-\alpha^*}{2i} = \mathrm{Im}\{\alpha\} = |\alpha|\sin\phi$$

$$\langle\alpha|\hat{x}_1^2|\alpha\rangle = \frac{1}{4}\left\{\alpha^2+\alpha^{*2}+2\alpha^*\alpha+1\right\} = \frac{1}{4}\left\{(\alpha+\alpha^*)^2+1\right\}$$

$$\langle\alpha|\hat{x}_2^2|\alpha\rangle = -\frac{1}{4}\left\{\alpha^2+\alpha^{*2}-2\alpha^*\alpha-1\right\} = -\frac{1}{4}\left\{(\alpha-\alpha^*)^2-1\right\}$$

This implies,

$$\langle\alpha\,|\,\Delta\hat{x}_1^2|\alpha\rangle = \langle\alpha\,|\,\Delta\hat{x}_1^2|\alpha\rangle - \left(\langle\alpha\,|\,\Delta\hat{x}_1|\alpha\rangle\right)^2$$

$$\Rightarrow \langle\Delta\hat{x}_1^2\rangle = \frac{1}{4}$$

Also,

$$\langle\Delta\hat{x}_2^2\rangle = \frac{1}{4}$$

Thus, for coherent states $\langle\Delta\hat{x}_1^2\rangle\langle\Delta\hat{x}_2^2\rangle = \frac{1}{16}$. Coherent states satisfy the quadrature uncertainty relation with equality and are therefore called minimum uncertainty states.

7.3.4 Vacuum Quadrature Fluctuations

The vacuum state $|0\rangle$ may also be considered a coherent state with $\alpha = 0$. Therefore, the quadrature fluctuations of the vacuum are the same as for the state $|\alpha\rangle$,

$$\langle 0|\Delta\hat{x}_1^2|0\rangle = \frac{1}{4} \qquad \langle 0|\Delta\hat{x}_2^2|0\rangle = \frac{1}{4}$$

7.3.5 Generalized Quadratures and Generalized Quadrature Fluctuations for Coherent States

Recall that a narrowband real signal $y(t)$ can be written as,

$$y(t) == \text{Re}\left\{x(t)\, e^{-i\omega_o t}\right\}$$

where,

$$x(t) = x_1(t) + i\, x_2(t)$$

The quadratures are the real and imaginary components of $x(t)$. In the complex plane, this means $x_1(t)$ and $x_2(t)$ are components of $x(t)$ along x-axis (real axis) and y-axis (imaginary axis), respectively. We may also write $x(t)$ as,

$$x(t) = x_\theta(t) e^{i\theta} + x_{\theta+\pi/2}(t) e^{\,i(\theta+\pi/2)} = \left[x_\theta(t) + i\, x_{\theta+\pi/2}(t)\right] e^{i\theta}$$

where now we are looking at the components of $x(t)$ along the two perpendicular axis of a coordinate system that is rotated at an angle θ with respect to the $x-y$ coordinate system. With this as motivation, we define the generalized quadrature operators as,

$$\hat{a} = \hat{x}_\theta\, e^{i\theta} + \hat{x}_{\theta+\pi/2}\, e^{i(\theta+\pi/2)} = \left[\hat{x}_\theta + i\, \hat{x}_{\theta+\pi/2}\right] e^{i\theta}$$

$$\hat{a}^+ = \hat{x}_\theta\, e^{-i\theta} + \hat{x}_{\theta+\pi/2}\, e^{-i(\theta+\pi/2)} = \left[\hat{x}_\theta - i\, \hat{x}_{\theta+\pi/2}\right] e^{-i\theta}$$

$$\Rightarrow \hat{x}_\theta = \frac{\hat{a}e^{-i\theta} + \hat{a}^+ e^{i\theta}}{2}$$

$$\Rightarrow \hat{x}_{\theta+\pi/2} = \frac{\hat{a}e^{-i\theta} - \hat{a}^+ e^{i\theta}}{2i}$$

The generalized quadratures satisfy the commutation relation,

$$\left[\hat{x}_\theta, \hat{x}_{\theta+\pi/2}\right] = \frac{i}{2}$$

$$\Rightarrow \left\langle \Delta\hat{x}_\theta^2 \right\rangle \left\langle \Delta\hat{x}_{\theta+\pi/2}^2 \right\rangle \geq \frac{1}{16}$$

For a coherent state,

$$\left\langle \alpha \right| \Delta\hat{x}_\theta^2 \left| \alpha \right\rangle = \frac{1}{4}$$

$$\left\langle \alpha \right| \Delta\hat{x}_{\theta+\pi/2} \left| \alpha \right\rangle = \frac{1}{4}$$

For a coherent state, mean square quadrature fluctuations are the same no matter which "direction" you look. This can be graphically illustrated by drawing error diagrams.

7.3.6 Error Diagrams of Quantum States of Radiation

The fluctuations in quantum optical states are sometimes depicted graphically in a $x_1 - x_2$ plane. An arrow is drawn and the tip of the arrow is at the point where x_1 equals $\left\langle a + a^+ \right\rangle / 2$ and x_2 equals $\left\langle a - a^+ \right\rangle / 2i$. The dimensions of the shaded figure (or the error figure) drawn around the tip of the arrow along different directions indicate the magnitude (root mean square value) of the fluctuations around the average value along those directions.

As an example, consider a coherent state $\left|\alpha\right\rangle$. We have,

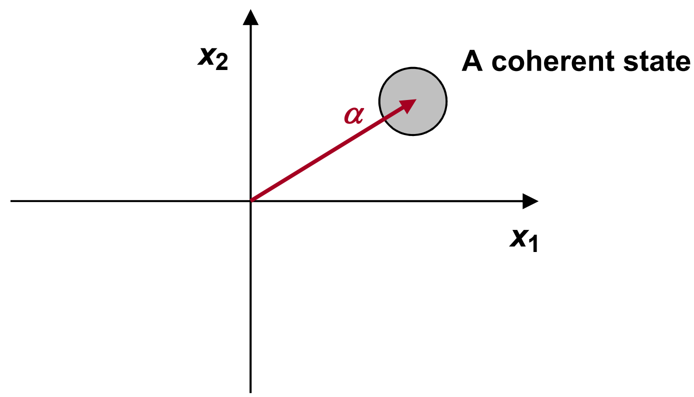
$$\langle\alpha|\hat{x}_1|\alpha\rangle = \frac{\alpha+\alpha^*}{2} = \text{Re}\{\alpha\}$$

$$\langle\alpha|\hat{x}_2|\alpha\rangle = \frac{\alpha-\alpha^*}{2i} = \text{Im}\{\alpha\}$$

So for a coherent state we draw an arrow in the complex $x_1 - x_2$ plane whose tip is at the coordinates $(\text{Re}\{\alpha\}, \text{Im}\{\alpha\})$. For a coherent state we know that,

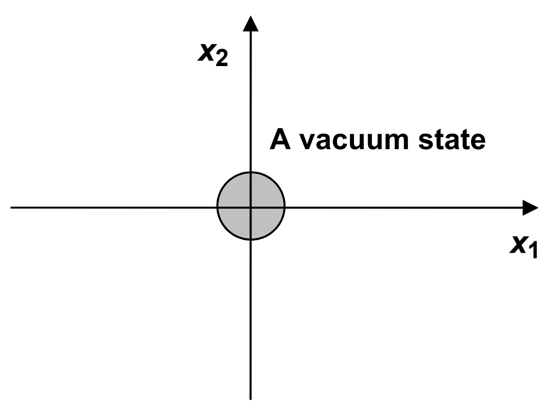
$$\langle\alpha|\Delta\hat{x}_1^2|\alpha\rangle = \frac{1}{4} \qquad \langle\alpha|\Delta\hat{x}_2^2|\alpha\rangle = \frac{1}{4}$$

In fact, for a coherent state, as discussed earlier, the fluctuations are the same along any direction, i.e., $\langle\Delta\hat{x}_\theta^2\rangle = 1/4$ for any value of θ. Therefore, the fluctuations in a coherent state are represented by drawing a circle of radius $1/2$, and area $\pi/4$, around the tip of the arrow, as shown below.



It should be noted here that unlike classical signals, where fluctuations represented the random motion in time of the tip of the phasor or the variations in an ensemble of signals, the fluctuations represented in the error diagram above are of quantum origin. The error diagram means that coherent states do not have a well defined value of, say, the x_1 quadrature. As will be shown later, a coherent state is in fact a superposition of quadrature eigenstates.

The vacuum state $|0\rangle$ is also a coherent state for which the average values of both the field quadratures are zero and the mean square fluctuations in each quadrature equal $1/4$. Therefore, a vacuum state is represented in the $x_1 - x_2$ plane as shown below.



7.3.7 Coherent States as Displaced Vacuum States

In this Section we will show that coherent states are "displaced" vacuum states, and clarify the meaning of the term "displaced." We saw earlier in Chapter 5 that the average values of fields $\left\langle \hat{\vec{A}}(\vec{r}) \right\rangle$, $\left\langle \hat{\vec{E}}(\vec{r}) \right\rangle$ and $\left\langle \hat{\vec{H}}(\vec{r}) \right\rangle$ in photon number states are zero. For example, if the field operator $\hat{\vec{A}}(\vec{r})$ for a single mode is,

$$\hat{\vec{A}}(\vec{r}) = \frac{\hat{q}}{\sqrt{\varepsilon_o \varepsilon}} \vec{U}(\vec{r})) = \sqrt{\frac{\hbar}{2\omega_o \varepsilon_o \varepsilon}} \left(\hat{a} + \hat{a}^+ \right) \vec{U}(\vec{r})$$

then for a state with n photons,

$$\left\langle \hat{\vec{A}}(\vec{r}) \right\rangle = \left\langle n \left| \hat{\vec{A}}(\vec{r}) \right| n \right\rangle = 0$$

We also showed that coherent states have non-zero average values of the field operators,

$$\left\langle \alpha \left| \hat{\vec{A}}(\vec{r}) \right| \alpha \right\rangle = \sqrt{\frac{\hbar}{2\omega_o \varepsilon_o \varepsilon}} \left(\alpha + \alpha^* \right) \vec{U}(\vec{r})$$

We now explore the reason behind the non-zero average values of the field operators for coherent states from a different angle. Let $\left| q \right\rangle$ be the eigenstates of the field amplitude operator $\hat{q}$, such that,

$$\hat{q} \left| q \right\rangle = q \left| q \right\rangle$$

and,

$$\int_{-\infty}^{\infty} dq \, \left| q \right\rangle \left\langle q \right| = \hat{1}$$

It follows that,

$$\begin{aligned}
\left\langle n \left| \hat{\vec{A}}(\vec{r}) \right| n \right\rangle &= \int_{-\infty}^{\infty} dq \left\langle n | q \right\rangle \left\langle q \left| \hat{\vec{A}}(\vec{r},t) \right| n \right\rangle \\
&= \int_{-\infty}^{\infty} dq \, \phi_n^*(q) \frac{q}{\sqrt{\varepsilon_o \varepsilon}} \vec{U}(\vec{r}) \phi_n(q) \\
&= \left[\int_{-\infty}^{\infty} dq \left| \phi_n(q) \right|^2 \, q \right] \frac{\vec{U}(\vec{r})}{\sqrt{\varepsilon_o \varepsilon}}
\end{aligned}$$

where $\phi_n(q)$ is the n-th Hermite-Gaussian. The above equation implies that the field operator will have a non-zero average value if q has a non-zero value when averaged with respect to the wavefunction $\phi_n(q)$. Recall that for the vacuum state the ground state wavefunction in q-space is Gaussian,

$$\left| \left\langle q | 0 \right\rangle \right|^2 = \left| \phi_0(q) \right|^2 = \frac{1}{\sqrt{\pi \hbar / \omega_o}} e^{-\frac{q^2}{\hbar/\omega_o}}$$

Since $\left| \phi_0(q) \right|^2$ is centered at $q = 0$, the average value for the $\hat{q}$ operator is zero. In fact, the Hermite-Gaussian wavefunctions $\left| \phi_n(q) \right|^2$ for all "n" are centered at zero, and so all photon number states have zero average field values.

What if we consider a "displaced" version of $\left| \phi_0(q) \right|^2$, say $\left| \phi'_0(q) \right|^2$, such that,

$$|\phi'_0(q)|^2 = \frac{1}{\sqrt{\pi\hbar/\omega_o}} e^{-\frac{(q-q_o)^2}{\hbar/\omega_o}}$$

Then, of course, with respect to $|\phi'_o(q)|^2$ the average value of the field operator is not zero, $$\left\langle \hat{\vec{A}}(r) \right\rangle = \left[\int dq\, |\phi'_0(q)|^2\, q \right] \frac{\vec{U}(\vec{r})}{\sqrt{\varepsilon_o \varepsilon}} = \frac{q_o\, \vec{U}(\vec{r})}{\sqrt{\varepsilon_o \varepsilon}}$$

The question then is, which quantum state corresponds to $|\phi'_0(q)|^2$? In other words, for what state $|\psi\rangle$ will $|\langle q\,|\,\psi\rangle|^2$ equal $|\phi'_0(q)|^2$? Recall that,

$$e^{-q_o\frac{\partial}{\partial q}} = 1 - q_o\frac{\partial}{\partial q} + \frac{(-q_o)^2}{2!}\frac{\partial^2}{q^2} + \frac{(-q_o)^3}{3!}\frac{\partial^3}{\partial q^3} + \cdots$$

Therefore,

$$\phi'_0(q) = \left(\frac{\omega_o}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{(q-q_o)^2}{2\hbar/\omega_o}}$$

$$= e^{-q_o\frac{\partial}{\partial q}}\, \phi_0(q)$$

Since the momentum operator $\hat{p}$ acts like a derivative on a q-space wavefunction,

$$\langle q|\hat{p}|0\rangle = \frac{\hbar}{i}\frac{\partial}{q}\phi_0(q)$$

we can write,

$$\langle q\,|\,\psi\rangle = \phi'_0(q) = e^{-q_o\frac{\partial}{\partial q}}\, \phi_0(q)$$

$$= \langle q|\, e^{-\frac{iq_o}{\hbar}\hat{p}}\, |0\rangle$$

This implies,

$$\Rightarrow |\psi\rangle = e^{-\frac{iq_o}{\hbar}\hat{p}}\, |0\rangle$$

But,

$$\hat{p} = \sqrt{\frac{\hbar\omega_o}{2}}\,\frac{\left(a - a^+\right)}{i}$$

Therefore,

$$|\psi\rangle = e^{-\frac{iq_o}{\hbar}\sqrt{\frac{\hbar\omega}{2}}\frac{\left(a-a^+\right)}{i}}\, |0\rangle$$

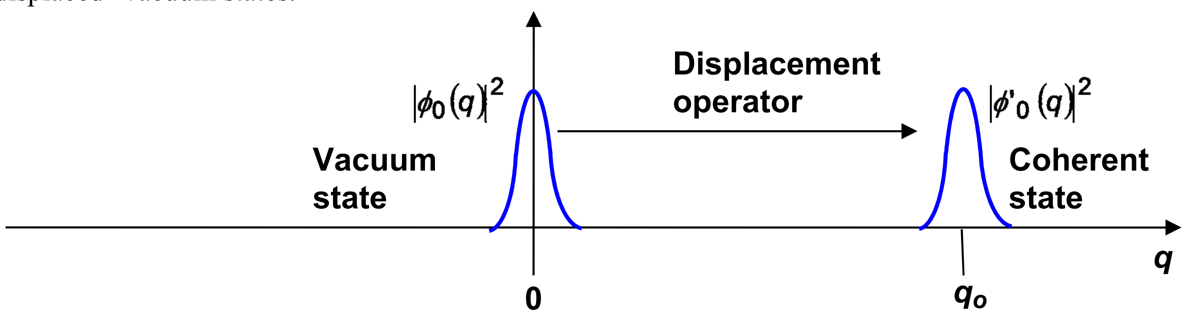
$$= e^{q_o\sqrt{\frac{\omega}{2\hbar}}a^+ - q_o\sqrt{\frac{\omega}{2\hbar}}a}\, |0\rangle$$

$$= e^{\alpha\, a^+ - \alpha^*\, a}\, |0\rangle = \hat{D}(\alpha)|0\rangle = |\alpha\rangle$$

where,

$$\alpha = \alpha^* = q_o \sqrt{\frac{\omega_o}{2\hbar}}$$

Therefore, the sought after state $|\psi\rangle$ is a coherent state! The Figure below shows the q-space probability distribution of a vacuum state and a coherent state and the action of the displacement operator $\hat{D}(\alpha)$. Since $\hat{D}(\alpha)$ displaces the vacuum state, it is called a displacement operator. Coherent states are thus "displaced" vacuum states.



7.3.8 Quadrature Eigenstates, Quadrature Fluctuations, and Error Figures

We had said earlier that a coherent state is a superposition of quadrature eigenstates. Here we quantify this notion. First note that both $\hat{x}_1$ and $\hat{x}_2$ quadratures cannot be measured simultaneously since they do not commute, $[\hat{x}_1(t), \hat{x}_2(t)] = i/2$. Recall that for a particle $[\hat{x}, \hat{p}] = i\hbar$, so we have a wavefunction in position $\psi(x,t) = \langle x \,|\, \psi(t)\rangle$, and we have a wavefunction in momentum $\psi(p,t) = \langle p \,|\, \psi(t)\rangle$, and the probabilities of finding a particular value of position or momentum (but not both) upon measurement are given by $|\psi(x,t)|^2$ and $|\psi(p,t)|^2$, respectively. Similarly, quantum states of radiation (e.g. coherent states) can be expanded in the eigenstates of the $\hat{x}_1$ quadrature or in the eigenstates of the $\hat{x}_2$ quadrature (but not both).

We start from (mode number subscript is suppressed),

$$\hat{a} = \frac{1}{\sqrt{2\hbar\omega_o}} \left\{ \omega_o \, \hat{q} + i \, \hat{p} \right\}$$

$$\hat{a}^+ = \frac{1}{\sqrt{2\hbar\omega_o}} \left\{ \omega_o \, \hat{q} - i \, \hat{p} \right\}$$

But we also have,

$$\hat{a} = \hat{x}_1 + i \, \hat{x}_2$$

$$\hat{a}^+ = x_1 - i \, \hat{x}_2$$

Therefore,

$$\hat{x}_1 = \sqrt{\frac{\omega_o}{2\hbar}} \, \hat{q}$$

$$\hat{x}_2 = \sqrt{\frac{1}{2\hbar\omega_o}} \, \hat{p}$$

Since the two quadrature operators are proportional to the operators $\hat{q}$ and $\hat{p}$, therefore, the eigenstates of $\hat{q}$ (i.e $|q\rangle$) are also eigenstates of $\hat{x}_1$ with eigenvalue $\sqrt{\frac{\omega_o}{2\hbar}}q$, and eigenstates of $\hat{p}$ (i.e. $|p\rangle$) are also eigenstates of $\hat{x}_2$ with eigenvalue $\sqrt{\frac{1}{2\hbar\omega_o}}p$. To avoid confusion below, I will write $|q\rangle$ as $|q\rangle_{\hat{q}}$ and $|p\rangle$ as $|p\rangle_{\hat{p}}$ where the subscripts indicate the operators of which the states are eigenstates. From the above discussion,

$$|q\rangle_{\hat{q}} \propto \left|\sqrt{\frac{\omega_o}{2\hbar}}\,q\right\rangle_{\hat{x}_1}$$

$$|p\rangle_{\hat{p}} \propto \left|\sqrt{\frac{1}{2\hbar\omega_o}}p\right\rangle_{\hat{x}_2}$$

Let,

$$|q\rangle_{\hat{q}} = b\left|\sqrt{\frac{\omega_o}{2\hbar}}\,q\right\rangle_{\hat{x}_1}$$

$$|p\rangle_{\hat{p}} = c\left|\frac{1}{2\hbar\omega_o}\,\rho\right\rangle_{\hat{x}_2}$$

where c and b will be determined to properly normalize the eigenstates of $\hat{x}_1$ and $\hat{x}_2$. We have,

$${}_{\hat{q}}\langle q'|q\rangle_{\hat{q}} = \delta(q'-q) \quad \Rightarrow \quad {}_{\hat{x}_1}\left\langle\sqrt{\frac{\omega}{2\hbar}}q'\middle|\sqrt{\frac{\omega}{2\hbar}}q\right\rangle_{\hat{x}_2} b^2 = \delta(q'-q)$$

Let,

$$x_1' = \sqrt{\frac{\omega_o}{2\hbar}}\,q' \qquad (1)$$

$$x_1 = \sqrt{\frac{\omega_o}{2\hbar}}\,q \qquad (2)$$

Then,

$${}_{\hat{x}_1}\langle x_1'|x_1\rangle_{\hat{x}_1} b^2 = \delta\left[\sqrt{\frac{2\hbar}{\omega}}(x_1'-x_1)\right] = \sqrt{\frac{\omega}{2\hbar}}\,\delta(x_1'-x_1)$$

We need,

$${}_{\hat{x}_1}\langle x_1'|x_1\rangle_{\hat{x}_1} = \delta(x_1'-x_1)$$

so b must equal,

$$b = \left(\frac{\omega_o}{2\hbar}\right)^{\frac{1}{4}}$$

Finally, we have,

$$|q\rangle_{\hat{q}} = \left(\frac{\omega_o}{2\hbar}\right)^{\frac{1}{4}}\left|\sqrt{\frac{\omega_o}{2\hbar}}\,q\right\rangle_{\hat{x}_1}$$

or, using (1) above,

$$\left|x_1\right\rangle_{\hat{x}_1} = \left(\frac{2\hbar}{\omega_o}\right)^{\frac{1}{4}} \left|\frac{2\hbar}{\omega_o} x_1\right\rangle_{\hat{q}}$$

Similarly, one can show that,

$$c = \left(\frac{1}{2\hbar\omega_o}\right)^{\frac{1}{4}}$$

and,

$$\left|x_2\right\rangle_{\hat{x}_2} = \left(2\hbar\omega_o\right)^{\frac{1}{4}} \left|\sqrt{2\hbar\omega_o}\, x_2\right\rangle_{\hat{p}}$$

$${}_{\hat{x}_2}\left\langle x_2' \middle| x_2\right\rangle_{\hat{x}_2} = \delta(x_2' - x_2)$$

Completeness: We have.

$$\int_{-\infty}^{\infty} dq\, \left|q\right\rangle_{\hat{q}}\; {}_{\hat{q}}\left\langle q\right| = \hat{1}$$

$$\Rightarrow \int dq \sqrt{\frac{\omega_o}{2\hbar}} \left|\sqrt{\frac{\omega_o}{2\hbar}} q\right\rangle_{\hat{x}_1} \; {}_{\hat{x}_1}\left\langle \sqrt{\frac{\omega_o}{2\hbar}}\, q\right| = \hat{1}$$

Define a change of variables,

$$x_1 = \sqrt{\frac{\omega_o}{2\hbar}}\, q$$

and obtain,

$$\int_{-\infty}^{\infty} dx_1\, \left|x_1\right\rangle_{\hat{x}_1}\; {}_{\hat{x}_1}\left\langle x_1\right| = \hat{1}$$

Similarly, one can obtain,

$$\int_{-\infty}^{\infty} dx_2 \left|x_2\right\rangle_{\hat{x}_2}\; {}_{\hat{x}_2}\left\langle x_2\right| = \hat{1}$$

from,

$$\int_{-\infty}^{\infty} dp \left|p\right\rangle_{\hat{p}}\; {}_{\hat{p}}\left\langle p\right| = \hat{1}$$

Now that we have the eigenstates of the quadrature operators, we will try to express coherent states in terms of these eigenstates. We can write,

$$\left|\alpha\right\rangle = \hat{1}\left|\alpha\right\rangle = \int_{-\infty}^{\infty} dx_1\; {}_{\hat{x}_1}\left\langle x_1 \middle| \alpha\right\rangle \quad \left|x_1\right\rangle_{\hat{x}_1}$$

$$\left|\alpha\right\rangle = \hat{1}\left|\alpha\right\rangle = \int_{-\infty}^{\infty} dx_2\; {}_{\hat{x}_2}\left\langle x_2 \middle| \alpha\right\rangle \quad \left|x_2\right\rangle_{\hat{x}_2}$$

We need the values of ${}_{\hat{x}_1}\left\langle x_1 \middle| \alpha\right\rangle$ and ${}_{\hat{x}_2}\left\langle x_2 \middle| \alpha\right\rangle$. We start from,

$$\left|\alpha\right\rangle = \hat{D}(\alpha)\left|0\right\rangle = e^{\alpha \hat{a}^{+} - \alpha^{*}\hat{a}} \left|0\right\rangle$$

Let,

$$\alpha = |\alpha| e^{i\phi}$$

We also know that,

$$\hat{a}^{+} = \frac{1}{\sqrt{2\hbar\omega_o}}\left(\omega_o\,\hat{q} - i\,\hat{p}\right)$$

$$\hat{a} = \frac{1}{\sqrt{2\hbar\omega_o}}\left(\omega_o\,\hat{q} + i\,\hat{p}\right)$$

Therefore,

$$|\alpha\rangle = \exp\left[\frac{|\alpha| e^{i\phi}}{\sqrt{2\hbar\omega_o}}\left(\omega_o\,\hat{q} - i\,\hat{p}\right) - \frac{|\alpha| e^{-i\phi}}{\sqrt{2\hbar\omega_o}}\left(\omega_o\,\hat{q} + i\hat{p}\right)\right]|0\rangle$$

$$|\alpha\rangle = \exp\left[i|\alpha|\sqrt{\frac{2\omega_o}{\hbar}}\sin\phi\,\hat{q} - i|\alpha|\sqrt{\frac{2}{\hbar\omega_o}}\cos\phi\,\hat{p}\right]|0\rangle$$

$$\Rightarrow |\alpha\rangle = \exp\left[-i\frac{|\alpha|^2}{2}\sin(2\phi)\right]\exp\left[i|\alpha|\sin\phi\sqrt{\frac{2\omega_o}{\hbar}}\,\hat{q}\right]\exp\left[-i|\alpha|\cos\phi\sqrt{\frac{2}{\hbar\omega_o}}\,\hat{p}\right]|0\rangle$$

The last line follows from using,

$$e^{\hat{A}+\hat{B}} = e^{-\frac{[\hat{A},\hat{B}]}{2}} e^{\hat{A}} e^{\hat{B}} \qquad \text{if } [\hat{A},\hat{B}] = \text{constant}$$

Take the inner product with the bra ${}_{\hat{q}}\langle q|$ on both sides to get,

$${}_{\hat{q}}\langle q|\alpha\rangle = e^{-i\frac{|\alpha|^2}{2}\sin(2\phi)}\, e^{i|\alpha|\sin\phi\sqrt{\frac{2\omega}{\hbar}}\,q}\, e^{-|\alpha|\cos\phi\sqrt{\frac{2\hbar}{\omega}}\frac{\partial}{\partial q}}\,\phi_0(q)$$

$$= e^{-i\frac{|\alpha|^2}{2}\sin(2\phi)}\, e^{i|\alpha|\sin\phi\sqrt{\frac{2\omega}{\hbar}}\,q}\,\phi_0(q - q_o)$$

where $q_o = |\alpha|\cos\phi\sqrt{2\hbar/\omega_o}$. Similarly, one can show that,

$${}_{\hat{p}}\langle p|\alpha\rangle = e^{i\frac{|\alpha|^2}{2}\sin(2\phi)}\, e^{-i|\alpha|\cos\phi\sqrt{\frac{2}{\hbar\omega}}\,p}\, e^{-|\alpha|\sin\phi\sqrt{2\hbar\omega}\frac{\partial}{\partial p}}\,\phi_0(p)$$

$$= e^{i\frac{|\alpha|^2}{2}\sin(2\phi)}\, e^{-i|\alpha|\cos\phi\sqrt{\frac{2}{\hbar\omega}}\,p}\,\phi_0(p - p_o)$$

where $p_o = |\alpha|\sin\phi\sqrt{2\hbar\omega_o}$. Now recall that,

$$\phi_0(q) = \left(\frac{\omega_o}{\pi\,\hbar}\right)^{\frac{1}{4}} e^{-\frac{q^2}{2\hbar/\omega_o}}$$

$$\phi_0(p) = \int_{-\infty}^{\infty} dq\,\frac{e^{-i\,pq}}{\sqrt{2\pi\,\hbar}}\,\phi_0(q) = \left(\frac{1}{\pi\,\hbar\omega_o}\right)^{\frac{1}{4}} e^{-\frac{p^2}{2\hbar\omega_o}}$$

Finally,

$$
\begin{aligned}
{}_{\hat{x}_1}\langle x_1|\alpha\rangle &= \left(\frac{2\hbar}{\omega_o}\right)^{\frac{1}{4}} {}_{\hat{q}}\left\langle \sqrt{\frac{2\hbar}{\omega_o}}\,x_1 \middle| \alpha \right\rangle \\
&= \left(\frac{2}{\pi}\right)^{\frac{1}{4}} e^{-i\frac{|\alpha|^2}{2}\sin(2\phi)}\; e^{i2|\alpha|\sin\phi\, x_1}\; e^{-[x_1-|\alpha|\cos\phi]^2}
\end{aligned}
$$

And similarly,

$$
\begin{aligned}
{}_{\hat{x}_2}\langle x_2|\alpha\rangle &= (2\hbar\omega_o)^{\frac{1}{4}} {}_{\hat{p}}\left\langle \sqrt{2\hbar\omega_o}\,x_2 \middle| \alpha \right\rangle \\
&= \left(\frac{2}{\pi}\right)^{\frac{1}{4}} e^{i\frac{|\alpha|^2}{2}\sin(2\phi)}\; e^{-i2|\alpha|\cos\phi\, x_2}\; e^{-[x_2-|\alpha|\sin\phi]^2}
\end{aligned}
$$

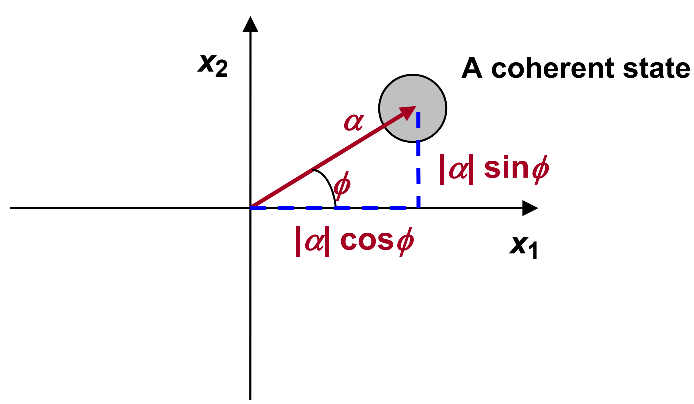
So we have,

$$
\left|{}_{\hat{x}_1}\langle x_1|\alpha\rangle\right|^2 = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} e^{-\frac{[x_1-|\alpha|\cos\phi]^2}{2(1/4)}}
$$

and,

$$
\left|{}_{\hat{x}_2}\langle x_2|\alpha\rangle\right|^2 = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} e^{-\frac{[x_2-|\alpha|\sin\phi]^2}{2(1/4)}}
$$

The above expressions show that a coherent state can be considered as a superposition of the eigenstates of the $\hat{x}_1$ operator. This superposition has a Gaussian probability distribution with a mean value centered at $|\alpha|\cos\phi$ and the variance of this distribution is $1/4$. Similarly, one may also consider a coherent state as a superposition of the eigenstates of the $\hat{x}_2$ operator. This superposition also has a Gaussian probability distribution with a mean value centered at $|\alpha|\sin\phi$ and the variance of this distribution is also $1/4$. These results justify the error diagram for the coherent states discussed earlier and show below.



Averages Using the Quadrature Distributions: Knowing the expansions of a coherent state in terms of the quadrature eigenstates, one can calculating averages of quantities of interest using these expansions. For example,

$$\langle \hat{x}_1 \rangle = \langle \alpha | \hat{x}_1 | \alpha \rangle = \langle \alpha | \hat{1}\hat{x}_1 | \alpha \rangle = \langle \alpha | \int_{-\infty}^{\infty} dx_1 | x_1 \rangle_{\hat{x}_1} \, {}_{\hat{x}_1}\langle x_1 | \hat{x}_1 | \alpha \rangle$$

$$= \int_{-\infty}^{\infty} dx_1 \, \left| {}_{\hat{x}_1}\langle x_1 | \alpha \rangle \right|^2 x_1 = |\alpha| \cos\phi$$

$$\Delta\hat{x}_1 = \hat{x}_1 - \langle \hat{x}_1 \rangle$$

$$\Rightarrow \left\langle \Delta\hat{x}_1^{\,2} \right\rangle = \langle \alpha | \Delta\hat{x}_1^{\,2} | \alpha \rangle$$

$$= \int_{-\infty}^{\infty} dx_1 \left| {}_{\hat{x}_1}\langle x_1 | \alpha \rangle \right|^2 \left[x_1^2 - |\alpha^2| \cos^2\phi \right]$$

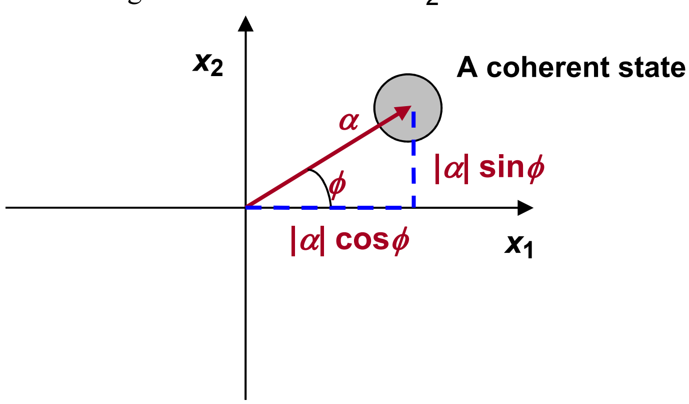
$$= \frac{1}{4}$$

And similarly,

$$\langle \hat{x}_2 \rangle = \langle \alpha | \hat{x}_2 | \alpha \rangle = \langle \alpha | \hat{1}\hat{x}_2 | \alpha \rangle = \int_{-\infty}^{\infty} dx_2 \left| {}_{\hat{x}_2}\langle x_2 | \alpha \rangle \right|^2 x_2 = |\alpha| \sin\phi$$

$$\left\langle \Delta\hat{x}_2^{\,2} \right\rangle = \langle \alpha | \Delta\hat{x}_2^2 | \alpha \rangle = \int dx_2 \left| {}_{\hat{x}_2}\langle x_2 | \alpha \rangle \right|^2 \left[x_2^{\,2} - |\alpha|^2 \sin^2\phi \right] = \frac{1}{4}$$

Note that the above results were obtained earlier in a different way without the knowledge of the probability distribution functions associated with the expansion of a coherent state in quadrature eigenstates. If one were to write $|\alpha\rangle$ as a linear super position of the eigenstates of $\hat{x}_1$, the resulting (root-mean-square) uncertainty would equal the extent of the figure in the direction of x_1 . Similarly, if one were to write $|\alpha\rangle$ as a linear super position of the eigenstates of $\hat{x}_2$, the resulting (root-mean-square) uncertainty would equal the extent of the figure in the direction of x_2.



One can define the eigenstates of the generalized quadrature operators, $\hat{x}_\theta$ and $\hat{x}_{\theta+\pi/2}$ (for any value of θ), and expand $|\alpha\rangle$ in terms of these eigenstates,

$$|\alpha\rangle = \int_{-\infty}^{\infty} dx_\theta \; {}_{\hat{x}_\theta}\langle x_\theta | \alpha \rangle \; | x_\theta \rangle_{\hat{x}_\theta}$$

or,

$$|\alpha\rangle = \int_{-\infty}^{\infty} dx_{\theta+\pi/2} \; {}_{\hat{x}_{\theta+\pi/2}}\langle x_{\theta+\pi/2} | \alpha \rangle \quad | x_{\theta+\pi/2} \rangle_{\hat{x}_{\theta+\pi/2}}$$

Since the error figure for a coherent state is a circle, it means that the fluctuations in all directions are equal and one may safely guess that,

$$\left|{}_{\hat{x}_\theta}\langle x_\theta|\alpha\rangle\right|^2 = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} e^{-\frac{\left[x_\theta - |\alpha|\cos(\phi-\theta)\right]^2}{2(1/4)}}$$

$$\left|{}_{\hat{x}_{\theta+\pi/2}}\langle x_{\theta+\pi/2}|\alpha\rangle\right|^2 = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} e^{-\frac{\left[x_{\theta+\pi/2} - |\alpha|\sin(\phi-\theta)\right]^2}{2(1/4)}}$$

7.3.9 Time Dependence of Coherent States

Suppose the quantum state of a single mode field at time $t = 0$ is a coherent state,

$$|\psi(t=0)\rangle = |\alpha\rangle = \hat{D}(\alpha)|0\rangle = e^{\alpha\hat{a}^+ - \alpha^*\hat{a}}\,|0\rangle$$

We need to find $|\psi(t)\rangle$. Suppose the Hamiltonian is,

$$\hat{H} = \hbar\omega_o\,\hat{a}^+ a$$

We have ignored the vacuum energy since it plays no interesting part in the discussion that follows. We have,

$$|\psi(t)\rangle = e^{-i\frac{\hat{H}t}{\hbar}}|\psi(t=0)\rangle = e^{-i\frac{\hat{H}t}{\hbar}}|\alpha\rangle$$

$$= e^{-i\frac{\hat{H}t}{\hbar}}\hat{D}(\alpha)|0\rangle$$

$$= e^{-i\frac{\hat{H}t}{\hbar}}\hat{D}(\alpha)e^{i\frac{\hat{H}t}{\hbar}}e^{-i\frac{\hat{H}t}{\hbar}}|0\rangle$$

$$= e^{\alpha\hat{a}^+(-t) - \alpha^*\hat{a}(-t)}\,|0\rangle$$

but,

$$\hat{a}^+(-t) = \hat{a}^+\,e^{-i\omega_o t}$$

$$\hat{a}(-t) = \hat{a}\,e^{i\omega_o t}$$

Therefore,

$$|\psi(t)\rangle = e^{\alpha(t)\hat{a}^+ - \alpha^*(t)\hat{a}}\,|0\rangle$$

where,

$$\alpha(t) = \alpha\,e^{-i\omega_o t}$$

$$\alpha^*(t) = \alpha^* e^{i\omega_o t}$$

One can write,

$$|\psi(t)\rangle = |\alpha(t)\rangle$$

Note that all the time dependence goes into the definitions of the complex numbers α and α^*.

7.3.10 Time Dependence of the Quadrature Operators

For a real signal $y(t) = \text{Re}\left\{x(t)e^{-i\omega_o t}\right\}$ writing $x(t) = x_1(t) + i\,x_2(t)$ implied that the fast time dependence of $y(t)$ given by $e^{-i\omega_o t}$ was not included in the definitions of $x_1(t)$ and $x_2(t)$ but was explicitly factored out. For quantum fields we know that the Heisenberg operator $\hat{\vec{A}}(r,t)$ is,

$$\hat{\vec{A}}(r,t) \quad \propto \quad \frac{1}{2}\left\{\hat{a}(t) + \hat{a}^+(t)\right\}$$

We write this as,

$$\hat{\vec{A}}(r,t) \quad \propto \quad \frac{1}{2}\left\{ \left[\hat{a}(t)e^{i\omega_o t}\right]e^{-i\omega_o t} + \left[\hat{a}^+(t)e^{-i\omega_o t}\right]e^{i\omega_o t} \right\}$$

Therefore, we define time dependent quadrature operators $\hat{x}_1(t)$ and $\hat{x}_2(t)$ as,

$$\hat{a}(t)e^{i\omega_o t} = \hat{x}_1(t) + i\,\hat{x}_2(t)$$

$$\hat{a}^+(t)e^{-i\omega_o t} = \hat{x}_1(t) - i\,\hat{x}_2(t)$$

For free fields (i.e. fields whose time dependence is governed by the Hamiltonian $\hat{H} = \hbar\omega_o\,\hat{a}^+ a$),

$$\hat{a}(t) = \hat{a}\,e^{-i\omega_o t}$$

$$\hat{a}^+(t) = \hat{a}^+ e^{i\omega_o t}$$

and therefore $\hat{x}_1(t)$ and $\hat{x}_2(t)$ will be independent of time. We can also write,

$$\hat{x}_1(t) = \frac{\hat{a}(t)\,e^{i\omega_o t} + \hat{a}^+(t)\,e^{-i\omega_o t}}{2}$$

$$\hat{x}_2(t) = \frac{\hat{a}(t)\,e^{i\omega_o t} - \hat{a}^+(t)\,e^{-i\omega_o t}}{2i}$$

And for the generalized quadratures we get,

$$\hat{x}_\theta(t) = \frac{\hat{a}(t)\,e^{-i\theta}\,e^{i\omega_o t} + \hat{a}^+(t)\,e^{i\theta}\,e^{-i\omega_o t}}{2}$$

$$\hat{x}_{\theta+\pi/2}(t) = \frac{\hat{a}(t)\,e^{-i\theta}\,e^{i\omega_o t} - \hat{a}^+(t)e^{i\theta}\,e^{-i\omega_o t}}{2i}$$

Note of Caution: The time dependence of a quadrature operator is not governed by the Heisenberg equation,

$$i\hbar\frac{d\,\hat{x}_\theta(t)}{dt} \neq \left[\hat{x}_\theta(t),\hat{H}\right] \qquad\qquad \hat{x}_\theta(t) \neq e^{i\frac{\hat{H}}{\hbar}t}\,\hat{x}_\theta e^{-i\frac{\hat{H}}{\hbar}t}$$

The time dependence of a quadrature operator is defined only through the equation,

$$\hat{x}_\theta(t) = \frac{\hat{a}(t)\,e^{-i\theta}\,e^{i\omega_o t} + \hat{a}^+(t)\,e^{i\theta}\,e^{-i\omega_o t}}{2}$$

The explicit presence of time exponentials in the above relation implies that the time dependence of the quadrature operators is not given directly by the Heisenberg equation. For any quantum state,

$$\langle\psi(t=0)|\,\hat{x}_\theta(t)\,|\psi(t=0)\rangle \neq \langle\psi(t)|\,\hat{x}_\theta\,|\psi(t)\rangle$$

The question then is how does one compute averages related to the quadrature operators in the Schrodinger and Heisenberg pictures. Below we discuss how to calculate the average of $\hat{x}_\theta(t)$ in the two pictures.

Heisenber Picture: Suppose one has the quantum state $|\psi(t=0)\rangle$ at time $t=0$. One first computes the Heisenebrg operators $\hat{a}(t), \hat{a}^{+}(t)$ at time t, find the desired quadrature operator $\hat{x}_{\theta}(t)$, and then compute,

$$\langle\psi(t=0)|\,\hat{x}_{\theta}(t)|\psi(t=0)\rangle$$

Schrodinger Picture: Suppose one has already computed the quantum state $|\psi(t)\rangle$ at time t. The quantity $\langle\psi(t=0)|\,\hat{x}_{\theta}(t)|\psi(t=0)\rangle$ found above is then the same as,

$$\langle\psi(t)|\frac{\hat{a}\,e^{-i\theta}\,e^{i\omega_o t}+\hat{a}^{+}\,e^{i\theta}\,e^{-i\omega_o t}}{2}|\psi(t)\rangle$$

Note that in the expression above the creation and destruction operators are in the Schrodinger picture. However, the time dependent complex exponentials remain.

Coherent States: For a coherent state, one obtains,

$$\langle\alpha|\,\hat{x}_{\theta}(t)|\alpha\rangle=\langle\alpha|\frac{\hat{a}(t)e^{-i\theta}\,e^{i\omega t}+\hat{a}^{+}(t)e^{i\theta}\,e^{-i\omega t}}{2}|\alpha\rangle$$

$$=\frac{\alpha\,e^{-i\theta}+\alpha^{*}\,e^{i\theta}}{2}$$

$$\langle\alpha|\,\hat{x}_{\theta+\pi/2}(t)|\alpha\rangle=\frac{\alpha\,e^{-i\theta}-\alpha^{*}e^{i\theta}}{2i}$$

And,

$$\langle\alpha|\,\Delta\hat{x}_{\theta}^{2}(t)|\alpha\rangle=\langle\alpha|\,\Delta\hat{x}_{\partial+\pi/2}^{2}(t)|\alpha\rangle=\frac{1}{4}$$

Note that the quadrature averages and variances are independent of time. This is true only for non-interacting free fields.

7.4 Squeezed States of Light

7.4.1 Introduction

For coherent states,

$$\left\langle\alpha\right|\Delta\hat{x}_{\theta}^{2}\left|\alpha\right\rangle=\frac{1}{4}$$

$$\langle\alpha|\Delta\hat{x}_{\theta+\pi/2}|\alpha\rangle=\frac{1}{4}$$

Coherent states satisfy the uncertainty relation,

$$\left\langle\Delta\hat{x}_{\theta}^{2}\right\rangle\left\langle\Delta\hat{x}_{\theta+\pi/2}^{2}\right\rangle\geq\frac{1}{16}$$

with equality. Squeezed states also satisfy the uncertainty condition with equality but have different fluctuations in quadratures $\hat{x}_{\theta}$ and $\hat{x}_{\theta+\pi/2}$. Obviously, they do that by decreasing the fluctuations in one quadrature at the expense of increasing the fluctuations in the other quadrature.

7.4.2 The Squeezing Operator

Squeezed states $|\alpha,\varepsilon\rangle$ are obtained by first squeezing the vacuum state $|0\rangle$ by the squeezing operator $\hat{S}(\varepsilon)$, where,

$$\hat{S}(\varepsilon) = e^{\frac{\varepsilon^*}{2}\hat{a}^2 - \frac{\varepsilon}{2}(\hat{a}^+)^2}$$

and then displacing it with $\hat{D}(\alpha)$,

$$|\alpha,\varepsilon\rangle = \hat{D}(\alpha)\hat{S}(\varepsilon)|0\rangle$$

The squeezing parameter ε is a complex number,

$$\varepsilon = r\,e^{+i\,2\phi}$$

$\hat{S}(\varepsilon)$ has the followng properties:

(i) $\hat{S}^+(\varepsilon)\hat{S}(\varepsilon) = \hat{1} \quad \Rightarrow \quad \hat{S}^+(\varepsilon) = \hat{S}^{-1}(\varepsilon) = \hat{S}(-\varepsilon)$

(ii) If $\varepsilon = r\,e^{+i\,2\phi}$, then,

$$\hat{S}^+(\varepsilon)\hat{a}\,\hat{S}(\varepsilon) = \hat{a}\cosh r - \hat{a}^+ e^{+2i\phi}\sinh r$$

$$\hat{S}^+(\varepsilon)\hat{a}^+\,\hat{S}(\varepsilon) = \hat{a}^+\cosh r - \hat{a}\,e^{-2i\phi}\sinh r$$

The proof follows from application of the formula,

$$\exp(\hat{A})\hat{B}\exp(-\hat{A}) = \hat{B} + \left[\hat{A},\hat{B}\right] + \frac{1}{2!}\left[\hat{A},\left[\hat{A},\hat{B}\right]\right] + \cdots$$

and then summing up the resulting series.

7.4.3 Properties of Squeezed States

In what follows, we will assume that $\varepsilon = r\,e^{+i\,2\phi}$. Squeezed states have the following properties:

(i) Averages of creation and destruction operators in squeezed states are as follows,

$$\langle\alpha,\varepsilon|\hat{a}|\alpha,\varepsilon\rangle = \langle 0|\hat{S}^+(\varepsilon)\hat{D}^+(\alpha)\hat{a}\,\hat{D}(\alpha)\hat{S}(\varepsilon)|0\rangle$$

$$= \langle 0|\hat{S}^+(\varepsilon)\left[\hat{a}+\alpha\right]\hat{S}(\varepsilon)|0\rangle = \alpha$$

$$\Rightarrow \langle\alpha,\varepsilon|\hat{a}^+|\alpha,\varepsilon\rangle = \alpha^*$$

It follows that,

$$\langle 0,\varepsilon|\hat{a}|0,\varepsilon\rangle = \langle 0|\hat{S}^+(\varepsilon)\;\hat{a}\,\hat{S}(\varepsilon)|0\rangle = 0$$

$$\langle 0,\varepsilon|\hat{a}^+|0,\varepsilon\rangle = 0$$

Also,

$$\langle\alpha,\varepsilon|\hat{a}^2|\alpha,\varepsilon\rangle = \langle 0|\hat{S}^+(\varepsilon)\hat{D}^+(\alpha)\hat{a}^2\,\hat{D}(\alpha)\hat{S}(\varepsilon)|0\rangle$$

$$= \langle 0|\hat{S}^+(\varepsilon)\hat{D}^+(\alpha)\hat{a}\,\hat{D}(\alpha)\hat{D}^+(\alpha)\hat{a}\,\hat{D}(\alpha)\hat{S}(\varepsilon)|0\rangle$$

$$= \langle 0|\hat{S}^+(\varepsilon)(\hat{a}+\alpha)^2\,\hat{S}(\varepsilon)|0\rangle$$

$$= \langle 0|\hat{S}^+(\varepsilon)\left[\hat{a}^2 + 2\hat{a}\alpha + \alpha^2\right]\hat{S}(\varepsilon)|0\rangle$$

$$= \langle 0|\hat{S}^+(\varepsilon)\hat{a}\,\hat{S}(\varepsilon)\hat{S}^+(\varepsilon)\hat{a}\,\hat{S}(\varepsilon)|0\rangle + \alpha^2$$

$$= \alpha^2 - e^{+2i\,\phi}\cosh r\sinh r$$

And,

$$\langle\alpha,\varepsilon|(\hat{a}^+)^2|\alpha,\varepsilon\rangle = \alpha^{*2} - e^{-2i\phi}\cosh r\sinh r$$

(ii) The number operator average is,

$$\begin{aligned}\langle \alpha,\varepsilon|\hat{a}^{+}\hat{a}|\alpha,\varepsilon\rangle &= \langle 0|\hat{S}^{+}\hat{D}^{+}\hat{a}^{+}\hat{D}\hat{D}^{+}\hat{a}\hat{D}\hat{S}|0\rangle \\ &= \langle 0|\hat{S}^{+}(\hat{a}^{+}+\alpha^{*})(\hat{a}+\alpha)\hat{S}|0\rangle \\ &= \langle 0|\hat{S}^{+}\hat{a}^{+}\hat{S}\hat{S}^{+}\hat{a}\hat{S}|0\rangle + |\alpha|^{2} \\ &= \sinh^{2}(r) + |\alpha|^{2}\end{aligned}$$

(iv) The quadrature operator average is,

$$\langle \alpha,\varepsilon|\hat{x}_{\theta}|\alpha,\varepsilon\rangle = \left\langle \alpha,\varepsilon\left|\frac{\hat{a}e^{-i\theta}+\hat{a}^{+}e^{-i\theta}}{2}\right|\alpha,\varepsilon\right\rangle = \frac{\alpha e^{-i\theta}+\alpha^{*}e^{i\theta}}{2}$$

$$\Rightarrow \left\langle \alpha,\varepsilon\left|\hat{x}_{\theta+\frac{\pi}{2}}\right|\alpha,\varepsilon\right\rangle = \frac{\alpha e^{-i\theta}-\alpha^{*}e^{i\theta}}{2i}$$

(v) The most distinguishing property of squeezed states compared to coherent states is the average fluctuations in the quadrature operators,

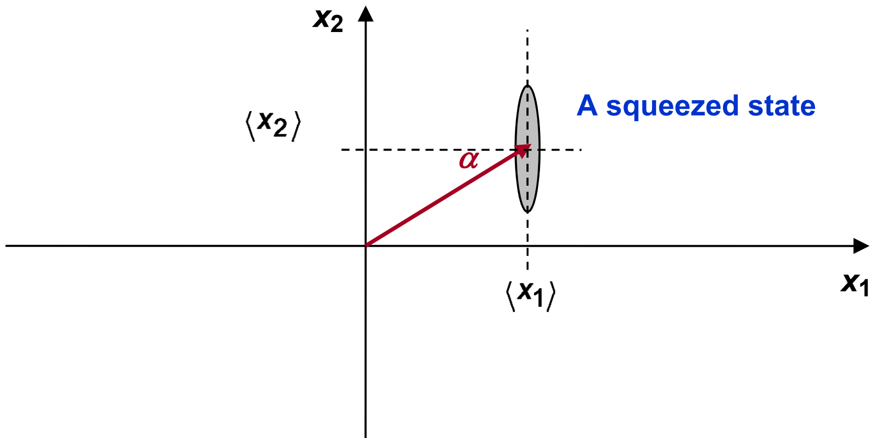
$$\begin{aligned}\langle \alpha,\varepsilon|\hat{x}_{\theta}^{2}|\alpha,\varepsilon\rangle &= \frac{1}{4}\left\langle \alpha,\varepsilon\left|\hat{a}^{2}e^{-i2\theta}+(\hat{a}^{+})^{2}e^{-i2\theta}+\hat{a}\hat{a}^{+}+\hat{a}^{+}a\right|\alpha,\varepsilon\right\rangle \\ &= \left(\alpha^{2}-e^{+2i\phi}\cosh r\sinh r\right)\frac{e^{-i2\theta}}{4} \\ &\quad + \left(\alpha^{2}-e^{-2i\phi}\cosh r\sinh r\right)\frac{e^{-i2\theta}}{4} + \left(|\alpha|^{2}+\cosh^{2}r\right)\frac{1}{4} \\ &\quad + \left(|\alpha|^{2}+\sinh^{2}r\right)\frac{1}{4}\end{aligned}$$

So if we let $\theta=\phi$ (i.e., look at the quadratures $\hat{x}_{\phi}$ and $\hat{x}_{\phi+\pi/2}$) where ϕ is the angle associated with the squeezing parameter ε, then,

$$\langle \alpha,\varepsilon|\Delta\hat{x}_{\phi}^{2}|\alpha,\varepsilon\rangle = \frac{1}{4}[\cosh r-\sinh r]^{2} = \frac{1}{4}e^{-2r}$$

$$\langle \alpha,\varepsilon|\Delta\hat{x}_{\phi+\pi/2}^{2}|\alpha,\varepsilon\rangle = \frac{1}{4}[\cosh r+\sinh r]^{2} = \frac{1}{4}e^{2r}$$

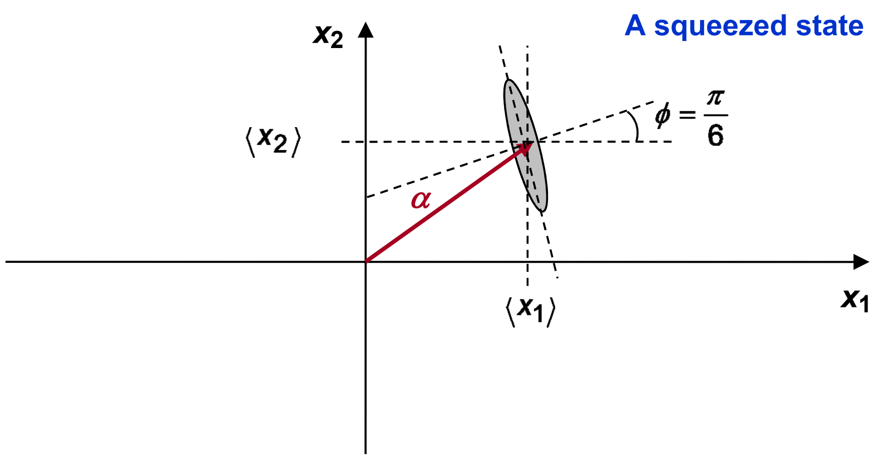
The squeezed state $|\alpha, re^{+2i\phi}\rangle$ has reduced fluctuations in the quadrature $\hat{x}_{\phi}$ compared to the quadrature $\hat{x}_{\phi+\pi/2}$. The error region in the x_1-x_2 plane for a squeezed state with $\phi=0$ is shown in the Figure below. The reduced fluctuations in one quadrature and the increased fluctuations in the other quadrature, make the error region look like an ellipse.



Also note that,

$$\left\langle \alpha, re^{+2i\phi} \middle| \Delta\hat{x}_{\phi}^2 \middle| \alpha, re^{+2i\phi} \right\rangle \left\langle \alpha, re^{+2i\phi} \middle| \Delta\hat{x}_{\phi+\pi/2}^2 \middle| \alpha, re^{+2i\phi} \right\rangle = \frac{1}{16}$$

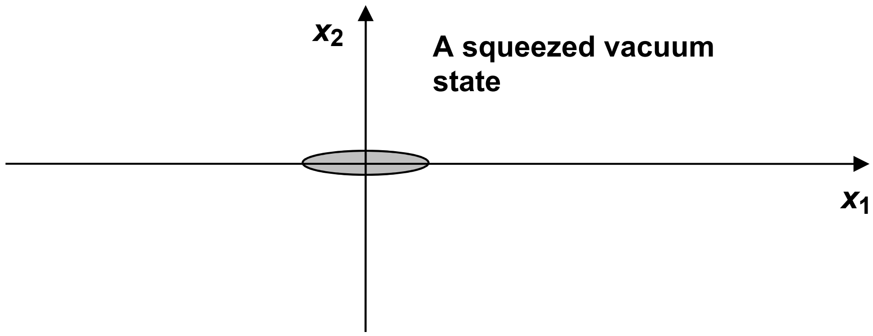
Squeezed states, like coherent states, are minimum uncertainty states since they satisfy the uncertainty relation with equality. The error region is an ellipse of semi-minor axis equal to $\sqrt{\frac{1}{4}e^{-2r}} = \frac{1}{2}e^{-r}$, and semi-major axis equal to $\sqrt{\frac{1}{4}e^{2r}} = \frac{1}{2}e^{r}$. The Figure below shows the error region for a squeezed state $\left| \alpha, re^{+2i\phi} \right\rangle$ for $\phi = \pi/6$. The squeezed state has reduced fluctuations in the quadrature $\hat{x}_{\phi}$ compared to the quadrature $\hat{x}_{\phi+\pi/2}$.



Squeezed Vacuum: A squeezed vacuum state $|0,\varepsilon\rangle$ is obtained by applying the squeezing operator to a vacuum state,

$$|0,\varepsilon\rangle = \hat{S}(\varepsilon)|0\rangle$$

For squeezed vacuum, the average values of the quadratures are zero and the quadrature fluctuations are dictated by the squeezing parameter $\varepsilon = r\,e^{+i\,2\phi}$. The figure below shows the squeezed vacuum state for $\phi = \pi/2$.



7.4.4 Squeezed States and Two-Photon Coherent States

Suppose we define two new operators $\hat{b}$ and $\hat{b}^+$ as a linear combination of the operators $\hat{a}$ and $\hat{a}^+$, as follows,

$$\hat{b} = \mu\,\hat{a} + \nu\,\hat{a}^+$$

$$\hat{b}^+ = \mu^*\,\hat{a}^+ + \nu^*\,\hat{a}$$

Here, μ and ν are complex numbers. If we also want to enforce the commutation relations for $\hat{b}$ and $\hat{b}^+$ (i.e. $\left[\hat{b},\hat{b}^+\right] = 1$), then we must have,

$$|\mu|^2 - |\nu|^2 = 1 .$$

Now suppose we define $\hat{b}$ and $\hat{b}^+$ as,

$$\hat{b} = \hat{S}^+(-\varepsilon)\,\hat{a}\,\hat{S}(-\varepsilon) = \hat{S}(\varepsilon)\hat{a}\hat{S}^+(\varepsilon)$$

$$\Rightarrow \hat{b} = \hat{a}\cosh r + \hat{a}^+\, e^{2i\phi}\sinh r$$

and,

$$\hat{b}^+ = \hat{S}(\varepsilon)\hat{a}^+\hat{S}^+(\varepsilon)$$

$$\Rightarrow \hat{b}^+ = \hat{a}\, e^{-2i\phi}\sinh r + \hat{a}^+\cosh r$$

You can verify that $\left[\hat{b},\hat{b}^+\right] = 1$. We now want to find eigenstates of the operator $\hat{b}$. We know that coherent states are eigenstates of the operator $\hat{a}$. $\left(\text{i.e. } \hat{a}|\alpha\rangle = \alpha|\alpha\rangle\right)$. We try the following state,

$$\hat{S}(\varepsilon)\hat{D}(\alpha)|0\rangle$$

In the above expression, the vacuum state is displaced first and then squeezed later. The resulting state is not exactly a squeezed state but it is closely related. Note that,

$$\begin{aligned}
\hat{b}\,\hat{S}(\varepsilon)\hat{D}(\alpha)|0\rangle &= \left[\hat{S}(\varepsilon)\hat{a}\hat{S}^+(\varepsilon)\right]\hat{S}(\varepsilon)\hat{D}(\alpha)|o\rangle \\
&= \hat{S}(\varepsilon)\hat{a}\,\hat{D}(\alpha)|0\rangle \\
&= \hat{S}(\varepsilon)\hat{a}\,|\alpha\rangle \\
&= \hat{S}(\varepsilon)\alpha\,|\alpha\rangle \\
&= \alpha\;\hat{S}(\varepsilon)\hat{D}(\alpha)|0\rangle
\end{aligned}$$

Therefore, the state $\hat{S}(\varepsilon)\hat{D}(\alpha)|0\rangle$ is an eigenstate of the operator $\hat{b}$ with eigenvalue α. These states are called two-photon coherent states. The $|0\rangle$ state is the vacuum (or ground state) of $\hat{a}$ in the sense that

$\hat{a}|0\rangle = 0$. The state $\hat{S}(\varepsilon)\hat{D}(\alpha=0)|0\rangle = \hat{S}(\varepsilon)|0\rangle = |0,\varepsilon\rangle$, or the squeezed vacuum state, is the ground state of the operator $\hat{b}$, i.e,

$$\hat{b}\,\hat{S}(\varepsilon)|0\rangle = 0$$

So we now have the following two sets of states:

Two-photon coherent states: $\hat{S}(\varepsilon)\hat{D}(\alpha)|0\rangle = |\alpha,\varepsilon\rangle_p$

Squeezed states: $\hat{D}(\alpha)\hat{S}(\varepsilon)|0\rangle = |\alpha,\varepsilon\rangle_s$ or just $|\alpha,\varepsilon\rangle$

The question that arises now is whether the above two sets represent the same or different states? We will answer this question next. One can also generate an eigenstate of $\hat{b}$ with eigenvalue α by operating with $\exp\left(\alpha\,\hat{b}^+ - \alpha^*\hat{b}\right)$on the ground state $\hat{S}(\varepsilon)|0\rangle$ of $\hat{b}$, i.e.,

$$e^{\alpha\,\hat{b}^+ - \alpha^*\hat{b}}\,\hat{S}(\varepsilon)|0\rangle$$

But,

$$\hat{b} = \hat{a}\cosh r + \hat{a}^+ e^{2i\phi}\sinh r$$

$$\hat{b}^+ = \hat{a}e^{-2i\phi}\sinh r + \hat{a}^+\cosh r$$

Therefore,

$$e^{\alpha\,\hat{b}^+ - \alpha^*\hat{b}}\,\hat{S}(\varepsilon)|0\rangle$$

$$= \exp\left[\left(\alpha\,\cosh r - \alpha^* e^{2i\phi}\sinh r\right)\hat{a}^+ - \left(-\alpha\,e^{-2i\phi}\sinh r + \alpha^*\cosh r\right)\hat{a}\right]\hat{S}(\varepsilon)|0\rangle$$

So,

$$\hat{S}(\varepsilon)\hat{D}(\alpha)|0\rangle = \hat{D}\left(\alpha\cosh(r) - \alpha^* e^{2i\phi}\,\mathrm{Sinh}\,(r)\right)\hat{S}(\varepsilon)|0\rangle$$

The above expression establishes the relationship between two-photon coherent states and squeezed states,

$$|\alpha,\varepsilon\rangle_p = \left|\alpha\,\cosh r - \alpha^* e^{2i\phi}\sinh r,\varepsilon\right\rangle_s$$

This implies that for two-photon coherent states with squeezing parameter $\varepsilon = r\,e^{+i\,2\phi}$,

$${}_p\langle\alpha,\varepsilon|\,\Delta\hat{x}_\phi^2\,|\alpha,\varepsilon\rangle_p = \frac{1}{4}e^{-2r}$$

$${}_p\langle\alpha,\varepsilon|\Delta\hat{x}_{\phi+\pi/2}|\alpha,\varepsilon\rangle_p = \frac{1}{4}e^{-2r}$$

7.4.5 Time Dependence of Squeezed States

Suppose the quantum state of a single mode field at time $t=0$ is a squeezed state,

$$|\psi(t=0)\rangle = |\alpha,\varepsilon\rangle = \hat{D}(\alpha)\hat{S}(\varepsilon)|0\rangle$$

We need to find $|\psi(t)\rangle$. Suppose the Hamiltonian is,

$$\hat{H} = \hbar\omega_o\,\hat{a}^+a$$

We have,

$$
\begin{aligned}
|\psi(t)\rangle &= e^{-i\frac{\hat{H}t}{\hbar}} |\psi(t=0)\rangle = e^{-i\frac{\hat{H}t}{\hbar}} |\alpha,\varepsilon\rangle \\
&= e^{-i\frac{\hat{H}t}{\hbar}} \hat{D}(\alpha)\hat{S}(\varepsilon)|0\rangle \\
&= e^{-i\frac{\hat{H}t}{\hbar}} \hat{D}(\alpha) e^{i\frac{\hat{H}t}{\hbar}} e^{-i\frac{\hat{H}t}{\hbar}} \hat{S}(\varepsilon) e^{i\frac{\hat{H}t}{\hbar}} e^{-i\frac{\hat{H}t}{\hbar}} |0\rangle \\
&= \hat{D}(\alpha(t))\hat{S}(\varepsilon(t))|0\rangle \\
&= |\alpha(t),\varepsilon(t)\rangle
\end{aligned}
$$

where,

$$\alpha(t) = \alpha\, e^{-i\omega_o t}$$

$$\varepsilon(t) = \varepsilon\, e^{-i2\omega_o t}$$

One can write,

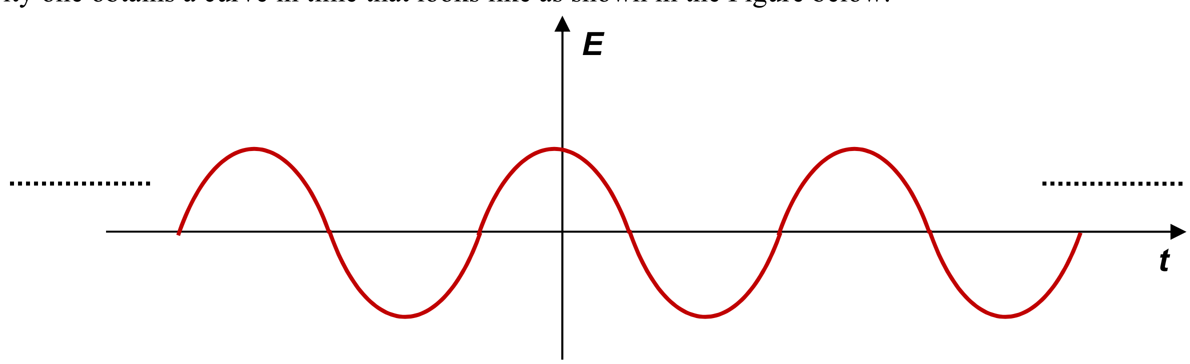
$$|\psi(t)\rangle = |\alpha(t),\varepsilon(t)\rangle$$

Note that all the time dependence goes into the definitions of the complex numbers α and ε .

7.5 Phase of Quantum States of Radiation

7.5.1 Introduction

We begin by asking the following question: What is the phase of a photon? And does this question even make sense? Let us start from what we already know about phase. We know that classical electric and magnetic fields inside a cavity oscillate and if one plots the strength of electric field at any location in the cavity one obtains a curve in time that looks like as shown in the Figure below.



This curve has a "phase" associated with it. So we do have a concept of a phase for a field. The phase of a "photon" is an ill-defined concept. The phase of the "field" is a more meaningful concept. We also know that since the electric field operator is,

$$\hat{\vec{E}}(\vec{r},t) = i\sqrt{\frac{\hbar\omega}{2\varepsilon_o\varepsilon}}\left(\hat{a}(t) - \hat{a}^{+}(t)\right)\vec{U}(\vec{r})$$

the electric field can have an average value that is non-zero only for states that are linear superpositions of photon number states. Photon number states $|n\rangle$, i.e. states with a definite number of photons, give an average value of zero for the electric field. For example, if $|\psi\rangle = |n\rangle$ then $\langle\psi|\hat{E}(\vec{r},t)|\psi\rangle = 0$, but if

$|\psi\rangle = \frac{1}{\sqrt{2}}\left[|n\rangle + |n-1\rangle\right]$ then $\langle\psi|\hat{E}(\vec{r},t)|\psi\rangle \neq 0$. Therefore, states for which the concept of "phase" should make sense must not be states of a definite photon number. These basic ingredients must be reflected in whatever operator we finally construct to describe the phase of the field.

Let us first look at a coherent state,

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle = e^{\frac{-|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

$$\left\langle\alpha\left|\hat{\vec{E}}(\vec{r},t)\right|\alpha\right\rangle = i\sqrt{\frac{\hbar\omega}{2\varepsilon_o\varepsilon}}\left[\alpha e^{-i\omega t} - \alpha^* e^{i\omega t}\right]\vec{U}(\vec{r})$$

If,

$$\alpha = |\alpha| e^{i\phi}$$

then,

$$\left\langle\alpha\left|\hat{\vec{E}}(\vec{r},t)\right|\alpha\right\rangle = 2\sqrt{\frac{\hbar\omega}{2\varepsilon_o\varepsilon}}|\alpha| \sin(\omega t - \phi)\, \vec{U}(\vec{r})$$

So the phase of the complex α parameter defines the phase of the average field. The average values of the quadrature operators,

$$\hat{x}_1(t) = \frac{\hat{a}(t)e^{i\omega t} + \hat{a}^+(t)e^{-i\omega t}}{2}$$

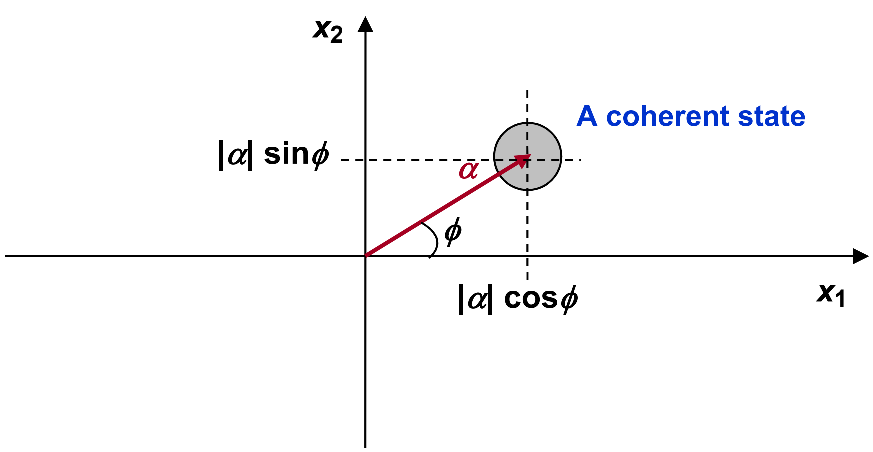
$$\hat{x}_2(t) = \frac{\hat{a}(t)e^{i\omega t} + \hat{a}^+(t)e^{-i\omega t}}{2i}$$

are,

$$\langle\alpha|\hat{x}_1(t)|\alpha\rangle = |\alpha|\cos\phi$$

$$\langle\alpha|\hat{x}_2(t)|\alpha\rangle = |\alpha|\sin\phi$$

In the $x_1 - x_2$ the state phasor is drawn as shown below.



One can see the problem in identifying the phase of the average field if $|\alpha| << 1$. In that case, the error circle would be close to the origin and the quadrature fluctuations would be larger than the mean quadrature values.

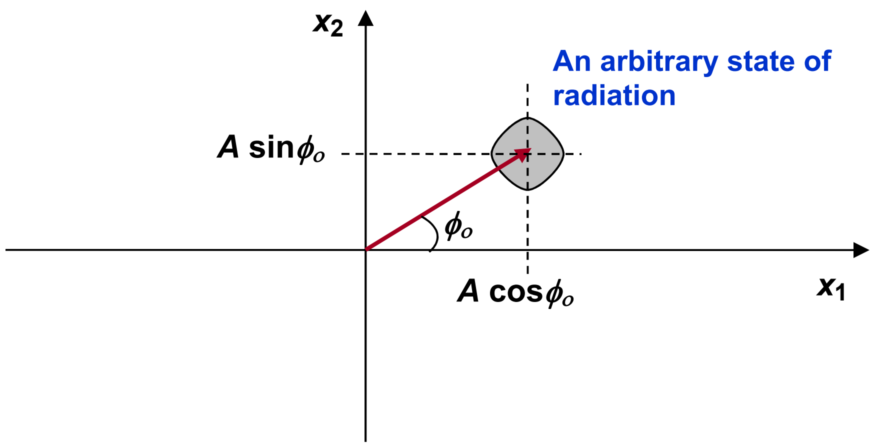
7.5.2 Phase Fluctuation Operator

A Hermitian phase operator for a field is not particularly easy to construct. This difficulty should not be a surprise since the absolute value of phase cannot be a measurable or an observable property. However, phase differences are measurable. In what follows, we will concentrate on constructing a phase fluctuation operator for quantum states.

Consider any arbitrary quantum state of radiation $|\psi\rangle$ for which the average values of the quadrature operators have the following non-zero values,

$$\langle\psi|\hat{a}(t)e^{i\omega t}|\psi\rangle = A\,e^{i\phi_o}$$
$$\langle\psi|\hat{x}_1(t)|\psi\rangle = A\cos\phi_o \qquad \{A \text{ is real}$$
$$\langle\psi|\hat{x}_2(t)|\psi\rangle = A\sin\phi_o$$

The average value ϕ_o of the phase for such a quantum state is well defined as long as $A >> 1$. If a measurement is made of any one quadrature, the relative uncertainty $\langle\Delta\hat{x}_\theta^2(t)\rangle / \langle\hat{x}_\theta(t)\rangle^2$ will be inversely proportional to A^2 and, therefore, small if, as assumed, $A >> 1$.



We can write,

$$\hat{a}(t)e^{i\omega t} = \hat{x}_1(t) + i\,\hat{x}_2(t) = \langle\hat{x}_1(t)\rangle + i\,\langle\hat{x}_2(t)\rangle + \Delta\hat{x}_1(t) + i\,\Delta\hat{x}_2(t)$$
$$= \langle\hat{a}(t)e^{i\omega t}\rangle + \Delta\hat{x}_1(t) + i\,\hat{x}_2(t)$$

where,

$$\Delta\hat{x}_1(t) = \hat{x}_1(t) - \langle\hat{x}_1(t)\rangle$$
$$\Delta\hat{x}_2(t) = \hat{x}_2(t) - \langle\hat{x}_2(t)\rangle$$

We can also write using generalized quadratures,

$$\hat{a}(t)e^{i\omega t} = \langle\hat{a}(t)e^{i\omega t}\rangle + \Delta\hat{x}_\theta(t)e^{i\theta} + i\,\Delta\hat{x}_{\theta+\pi/2}(t)e^{i\theta}$$

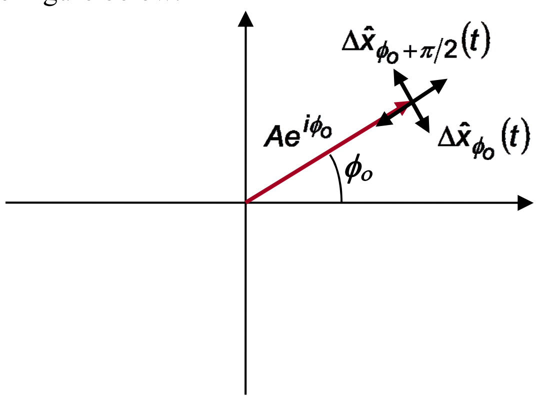
where,

$$\Delta\hat{x}_\theta(t) = \hat{x}_\theta(t) - \langle\hat{x}_\theta(t)\rangle$$
$$\Delta\hat{x}_{\theta+\pi/2}(t) = \hat{x}_{\theta+\pi/2}(t) - \langle\hat{x}_{\theta+\pi/2}(t)\rangle$$

If the average phase of the field is ϕ_o, i.e.,

$$\left\langle \psi \left| \hat{a}(t) e^{i\omega t} \right| \psi \right\rangle = A\, e^{i\phi_o}$$

then, as discussed earlier in the case of classical signals, the quadrature fluctuation operator $\Delta\hat{x}_{\phi_o}(t)$ in the direction ϕ_o will describe amplitude fluctuations and the quadrature fluctuation operator $\Delta\hat{x}_{\phi_o+\pi/2}(t)$ in the direction perpendicular to the direction ϕ_o will be proportional to the phase fluctuations. This is shown in the Figure below.



One can write,

$$\begin{aligned}\hat{a}(t) e^{i\omega t} &= \left\langle \hat{a}(t) e^{i\omega t} \right\rangle + \Delta\hat{x}_{\phi_o}(t) e^{i\phi_o} + i\,\Delta\hat{x}_{\phi_o+\pi/2}(t) e^{i\phi_o} \\ &= \left[A + \Delta\hat{x}_{\phi_o}(t) + i\,\Delta\hat{x}_{\phi_o+\pi/2}(t)\right] e^{i\phi_o}\end{aligned}$$

If the average value of the phase is ϕ_o, and if $A >> 1$, then a phase fluctuation operator, $\Delta\hat{\phi}(t)$, can be defined as follows,

$$\Delta\hat{\phi}(t) = \frac{\Delta\hat{x}_{\phi_o+\pi/2}(t)}{A}$$

The amplitude fluctuation operator is simply $\Delta\hat{x}_{\phi_o}(t)$. As an example, we calculate the phase fluctuations of a coherent state $|\alpha\rangle$. Suppose that,

$$\left\langle \alpha \left| \hat{a}(t) e^{i\omega t} \right| \alpha \right\rangle = |\alpha|\, e^{i\phi_o}$$

Then,

$$\langle \alpha | \Delta\hat{\phi}(t) | \alpha \rangle = \langle \alpha | \frac{\Delta\hat{x}_{\phi_o+\pi/2}(t)}{|\alpha|} | \alpha \rangle = 0$$

$$\langle \alpha | \Delta\hat{\phi}^2(t) | \alpha \rangle = \langle \alpha | \frac{\Delta\hat{x}^2{}_{\phi_o+\pi/2}(t)}{|\alpha|} | \alpha \rangle = \frac{1}{4\,|\alpha|}$$

The larger the magnitude $|\alpha|$ of a coherent state, the smaller the mean square phase fluctuations.

7.5.3 Photon Number Fluctuation Operator

Suppose for a quantum state the average photon number is n_o and the average phase is ϕ_o,

$$\left\langle \psi \left| \hat{a}(t) e^{i\omega t} \right| \psi \right\rangle = A\, e^{i\phi_o}$$

$$\left\langle \psi \left| \hat{n}(t) \right| \psi \right\rangle = \left\langle \psi \left| \hat{a}^{+}(t) \hat{a}(t) \right| \psi \right\rangle = n_o$$

We assume that $A >> 1$ and so the average phase ϕ_o is well defined. Then, as before, one can write,

$$\hat{a}(t)e^{i\omega t} = \left\langle \hat{a}(t)e^{i\omega t} \right\rangle + \Delta\hat{x}_{\phi_o}(t)e^{i\phi_o} + i\,\Delta\hat{x}_{\phi_o+\pi/2}(t)e^{i\phi_o}$$
$$= \left[A + \Delta\hat{x}_{\phi_o}(t) + i\,\Delta\hat{x}_{\phi_o+\pi/2}(t)\right] e^{i\phi_o}$$

The photon number operator becomes,

$$\hat{n}(t) = \langle \hat{n}(t) \rangle + \Delta\hat{n}(t)$$
$$= \hat{a}^{+}(t)\hat{a}(t) = \left[A + \Delta\hat{x}_{\phi_o}(t) + i\,\Delta\hat{x}_{\phi_o+\pi/2}(t)\right]\left[A + \Delta\hat{x}_{\phi_o}(t) - i\,\Delta\hat{x}_{\phi_o+\pi/2}(t)\right]$$
$$\approx A^2 + 2A\,\Delta\hat{x}_{\phi_o}(t)$$

We have ignored terms that contain squares of the quadrature fluctuation operators. The above relation shows that the average photon number n_o must approximately equal A^2 and the photon number fluctuation operator $\Delta\hat{n}(t)$ must approximately equal $2A\,\Delta\hat{x}_{\phi_o}(t)$. These approximations are excellent when $A = n_o >> 1$. So we define an approximate photon number fluctuation operator as,

$$\Delta\hat{n}(t) = 2A\,\Delta\hat{x}_{\phi_o}(t) = 2\sqrt{n_o}\;\Delta\hat{x}_{\phi_o}(t)$$

One can then write,

$$\left\langle \psi \middle| \hat{a}(t)e^{i\omega t} \middle| \psi \right\rangle = \sqrt{n_o}\, e^{i\phi_o}$$
$$\hat{a}(t)e^{i\omega t} = \left[\sqrt{n_o} + \Delta\hat{x}_{\phi_o}(t) + i\,\Delta\hat{x}_{\phi_o+\pi/2}(t)\right] e^{i\phi_o}$$
$$= \left[\sqrt{n_o} + \frac{\Delta\hat{n}(t)}{2\sqrt{n_o}} + i\sqrt{n_o}\,\Delta\hat{\phi}(t)\right] e^{i\phi_o}$$

7.5.4 Photon Number and Phase Uncertainty Relation

There is an interesting uncertainty relation between the photon number fluctuation operator and the phase fluctuation operator. The photon number fluctuation and the phase fluctuation operators for a quantum state with an average number of photons equal to n_o are,

$$\Delta\hat{n}(t) = 2\sqrt{n_o}\;\Delta\hat{x}_{\phi_o}(t)$$
$$\Delta\hat{\phi}(t) = \frac{\Delta\hat{x}_{\phi_o+\pi/2}(t)}{\sqrt{n_o}}$$

The commutation relation between the photon number fluctuation operator and the phase fluctuation operator follows from the commutation relation between the quadrature operators,

$$\left[\Delta\hat{n}(t)\,,\,\Delta\hat{\phi}(t)\right] = 2\left[\Delta\hat{x}_{\phi_o}(t)\,,\,\Delta\hat{x}_{\phi_o+\pi/2}(t)\right] = i$$
$$\Rightarrow \quad \left[\Delta\hat{n}(t), \Delta\hat{\phi}(t)\right] = i$$
$$\Rightarrow \quad \left\langle \Delta\hat{n}^2(t) \right\rangle \left\langle \Delta\hat{\phi}^2(t) \right\rangle \geq \frac{1}{4}$$

Therefore, the photon number and the phase (with respect to a reference) of a radiation field cannot be measured simultaneously with high accuracy. In other words, measurement of one will necessarily disturb the other. The other way to interpret the same result is that if a quantum state of radiation has a well defined value for the phase of the field then this quantum state cannot have a well defined number of photons and it must be a superposition of different photon number states. On the other hand, if a quantum state has a well defined number of photons then it cannot have a well defined value for the phase of the field.

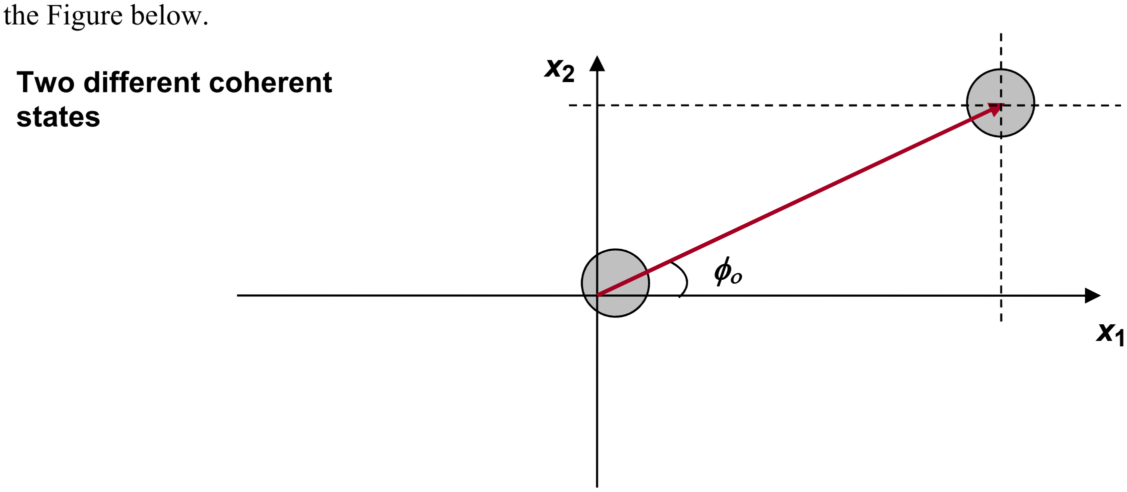
For a coherent state with average number of photons equal to n_o,

$$\langle \alpha | \Delta \hat{n}^2(t) | \alpha \rangle = |\alpha|^2 = n_o$$

$$\langle \alpha | \Delta \hat{\phi}^2(t) | \alpha \rangle = \frac{1}{4\,|\alpha|^2} = \frac{1}{4\,n_o}$$

The photon number and phase uncertainty relation is satisfied with equality. Note that larger the average photon number, the smaller the uncertainty in the phase.

The definitions for the phase fluctuation operator and the photon number fluctuation operator presented here work well only when the quantum state has a large average photon number n_o and a well defined average phase ϕ_o. As $\langle \hat{n}(t) \rangle = n_o \to 0$, the phase fluctuations described by the operator $\Delta\hat{\phi}(t) = \Delta\hat{x}_{\phi_o+\pi/2}(t) \big/ \sqrt{n_o}$ can become very large. The problem is that phase of a field is not a well defined concept when the photon number is very small. For example, consider two different coherent states, with the same phase, but for one $\langle \hat{n}(t) \rangle = |\alpha|^2 >> 1$ and for the other $\langle \hat{n}(t) \rangle = |\alpha|^2 << 1$, as shown in the Figure below.



When $|\alpha| << 1$, the quadrature fluctuations are comparatively large, and its difficult to categorize quadrature fluctuations as amplitude or phase fluctuations. For example, $\Delta\hat{x}_{\phi_o+\pi/2}(t)$ now also contributes to amplitude fluctuations. Although one can still talk about quadrature fluctuations, $\Delta\hat{x}_\theta(t), \Delta\hat{x}_{\theta+\pi/2}(t)$, but no quadrature fluctuation, for any value of θ, can be labled as an amplitude fluctuation and no quadrature fluctuation can be labled as a phase fluctuation.

Phase Noise in Squeezed States: Consider now a squeezed state $|\alpha, \varepsilon\rangle$ with a large average photon number,

$$\langle \hat{n}(t) \rangle = |\alpha|^2 + \sinh^2 r \approx |\alpha|^2 = n_o \qquad \left\{ \text{Assuming} \, |\alpha|^2 >> e^{2r} \right.$$

If,

$$\alpha = |\alpha| e^{i\theta_o} \qquad \varepsilon = r\, e^{2i\phi_o}$$

then,

$$\left\langle \alpha,\varepsilon \right| \hat{a}(t) e^{i\omega t} \left| \alpha,\varepsilon \right\rangle = \alpha$$

$$\left\langle \alpha,\varepsilon \right| \hat{\vec{E}}(\vec{r},t) \left| \alpha,\varepsilon \right\rangle = \sqrt{\frac{\hbar\omega}{2\varepsilon_o \varepsilon}} |\alpha| \sin(\omega t - \theta_o) \vec{U}(\vec{r})$$

The average phase of the field is θ_o. It is not necessary that the angles θ_o and ϕ_o be the same. First, we assume that $\phi_o = \theta_o$. We also assume that $|\alpha|^2$ is large and therefore n_o is large. The fluctuation operators are,

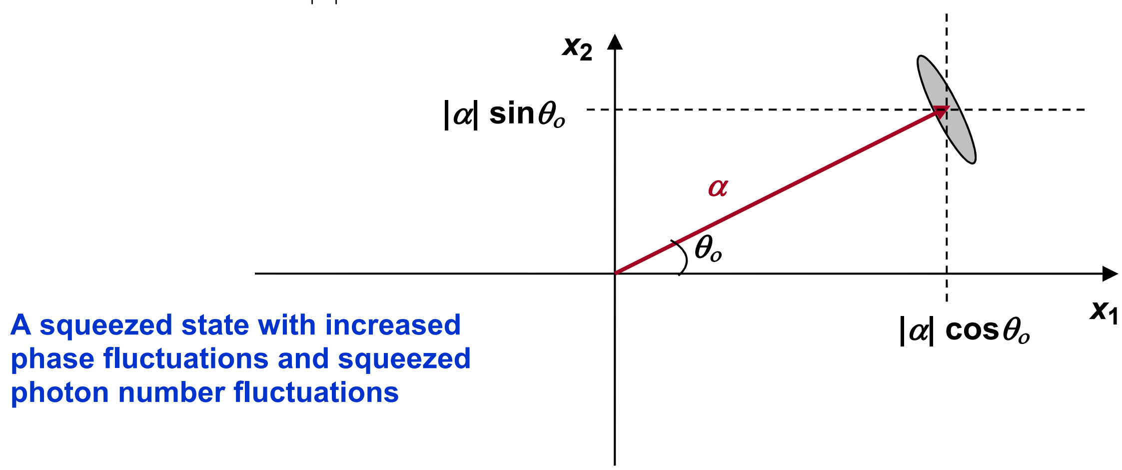
$$\Delta\hat{n}(t) = 2\sqrt{n_o}\ \Delta\hat{x}_{\theta_o}(t)$$

$$\Delta\hat{\phi}(t) = \frac{\Delta\hat{x}_{\theta_o+\pi/2}(t)}{\sqrt{n_o}}$$

We get,

$$\left\langle \alpha,\varepsilon \middle| \Delta\hat{n}^2(t) \middle| \alpha,\varepsilon \right\rangle = 4\ n_o \left\langle \alpha,\varepsilon \middle| \Delta\hat{x}^2_{\theta_o}(t) \middle| \alpha,\varepsilon \right\rangle = 4|\alpha|^2 \frac{1}{4} e^{-2r}$$

$$= |\alpha|^2 e^{-2r}$$



The above result agrees reasonably well with the exact result for $\left\langle \Delta\hat{n}^2(t) \right\rangle$ for a squeezed state,

$$\left\langle \alpha,\varepsilon \middle| \Delta\hat{n}^2(t) \middle| \alpha,\varepsilon \right\rangle = |\alpha|^2 e^{-2r} + 2\cosh^2 r\ \sinh^2 r$$

provided $|\alpha|^2$ is large. For phase fluctuations we get,

$$\left\langle \alpha,\varepsilon \middle| \Delta\hat{\phi}^2(t) \middle| \alpha,\varepsilon \right\rangle = \frac{1}{4}\frac{e^{2r}}{n_o} \approx \frac{1}{4}\frac{e^{2r}}{|\alpha|^2}$$

The number-phase uncertainty relation is satisfied with equality (approximately),

$$\left\langle \Delta\hat{n}^2(t) \right\rangle \left\langle \Delta\hat{\phi}^2(t) \right\rangle \approx \frac{1}{4}$$

Compared to a coherent state, the photon number fluctuations in this squeezed state are reduced (or squeezed) at the expense of increased phase fluctuations (see the Figure above). Now suppose, that the squeezing parameter's phase ϕ_o is $\theta_o + \pi/2$. As before,

$$\Delta\hat{n}(t) = 2\sqrt{n_o}\ \Delta\hat{x}_{\theta_o}(t)$$

$$\Delta\hat{\phi}(t) = \frac{\Delta\hat{x}_{\theta_o+\pi/2}(t)}{\sqrt{n_o}}$$

It follows that,

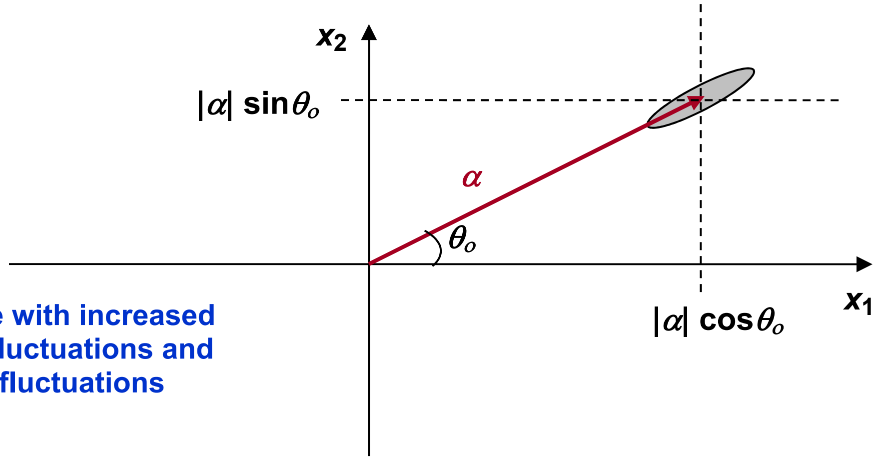
$$\langle\alpha,\varepsilon|\Delta\hat{n}^2(t)|\alpha,\varepsilon\rangle = 4\,n_o\langle\alpha,\varepsilon|\Delta\hat{x}^2_{\theta_o}(t)|\alpha,\varepsilon\rangle$$

$$\approx|\alpha|^2 e^{2r}$$

$$\langle\alpha,\varepsilon|\Delta\hat{\phi}^2(t)|\alpha,\varepsilon\rangle = \frac{\langle\alpha,\varepsilon|\Delta\hat{x}^2_{\theta_o+\pi/2}(t)|\alpha,\varepsilon\rangle}{n_o} \approx \frac{e^{-2r}}{4|\alpha|^2}$$

$$\Rightarrow \langle\Delta\hat{n}^2(t)\rangle\langle\Delta\hat{\phi}^2(t)\rangle = \frac{1}{4}$$

Now phase fluctuations are squeezed at the expense of increased photon number (or amplitude) fluctuations, as shown in the Figure below.



A squeezed state with increased photon number fluctuations and squeezed phase fluctuations

7.6 Thermal Radiation: A Statistical Mixture

So far we have looked at quantum states of radiation that could be described using a pure state. One can also have states of radiation that are statistical mixtures and are describable only by a density operator. One such state is the thermal state. Consider a single radiation mode inside a cavity. We know that at any temperature the radiation must be in thermal equilibrium with the walls of the cavity and the walls of the cavity must be continuously absorbing from and loosing photons to the radiation mode. Therefore, the radiation mode cannot have a fixed well defined number of photons. We assume that the density operator for the radiation mode can be written as,

$$\hat{\rho} = \sum_{n=0}^{\infty} P(n)\,|n\rangle\langle n|$$

In the photon number state basis, the density operator has no off-diagonal elements but only diagonal elements, $P(n)$, that give the probability of there being n photons in the radiation mode. From statistical physics we know that for a canonical ensemble the probability for a system at temperature T to have energy E is proportional to e^{-E/K_BT}. Therefore,

$$P(n) \propto e^{-\frac{\hbar\omega(n+1/2)}{K_BT}}$$

The constant of proportionality is obtained by requiring that,

$$\sum_{n=0}^{\infty} P(n) = 1$$

which gives,

$$P(n) = e^{-n\,\hbar\omega/K_B T}\left(1 - e^{-\hbar\omega/K_B T}\right)$$

The average number of photons in the mode equals,

$$\langle \hat{n} \rangle = \text{Trace}\{\hat{\rho}\,\hat{n}\} = \sum_{n=0}^{\infty} n\,P(n) = \frac{1}{e^{\hbar\omega/K_B T} - 1}$$

The expression for the average photon number is called the Bose-Einstein factor, and the thermal probability distribution,

$$P(n) = e^{-n\,\hbar\omega/K_B T}\left(1 - e^{-\hbar\omega/K_B T}\right)$$

is called the Bose-Einstein distribution. Bose-Einstein distribution is sometimes also written in terms of the average photon number as,

$$P(n) = e^{-n\,\hbar\omega/K_B T}\left(1 - e^{-\hbar\omega/K_B T}\right) = \frac{1}{1+\langle \hat{n} \rangle}\left(\frac{\langle \hat{n} \rangle}{1+\langle \hat{n} \rangle}\right)^{n}$$

The fluctuations in the photon number are,

$$\left\langle \Delta\hat{n}^2 \right\rangle = \left\langle \hat{n}^2 \right\rangle - \langle \hat{n} \rangle^2 = \langle \hat{n} \rangle \left(1 + \langle \hat{n} \rangle\right)$$

Note that for the same average number of photons, a thermal state has larger photon number fluctuations than a coherent state. Since, the thermal state is a statistical mixture of different photon number states (and not a linear superposition of different photon number states), it has no well defined phase.

The thermal or Bose-Einstein distribution played an important role in the development of the quantum theory. Max Plank studied the frequency distribution of radiation coming out of a blackbody cavity radiator and, in order to explain the experimental observations, postulated that radiation must be emitted and absorbed in discrete packets or “quanta” of energy. Max Plank was the first one to show that for a thermal state the energy in a radiation mode is distributed according to the Bose-Einstein distribution.