

Chapter 6: Random Signals and Noise

6.1 Basic Concepts in Random Signals

6.1.1 Random Variables

A random variable x is characterized by a probability density function $P_x(x)$. The probability that x will have a value in an interval dx centered around x is equal to $P_x(x)dx$. Normalization requires,

$$\int_{-\infty}^{\infty} P_x(x)dx = 1$$

The mean and standard deviation for x are defined as,

$$\text{Mean of } x = \langle x \rangle = \int_{-\infty}^{\infty} x P_x(x)dx$$

$$\text{Standard deviation} = \langle x^2 \rangle - \langle x \rangle^2$$

The sample space of x is the set of all possible values of x (i.e. for which $P_x(x) \neq 0$).

6.1.2 Random Signals

A random signal $x(t)$ can be any signal from a set of signals $\{x_1(t), x_2(t), x_3(t), \dots\}$ which is the sample space for the random signal $x(t)$. The sample space need not be a discrete set. The probability that $x(t)$ will equal $x_n(t)$ is $P_{x(t)}[x_n(t)]$. The mean of $x(t)$ is,

$$m_x(t) = \langle x(t) \rangle = \sum_n x_n(t) P_{x(t)}[x_n(t)]$$

When the sample space is not discrete and is parameterized by a continuous variable y then,

$$m_x(t) = \int dy x_y(t) P_{x(t)}[x_y(t)]$$

The auto-correlation $R_{xx}(t_1, t_2)$ of $x(t)$ is defined as,

$$R_{xx}(t_1, t_2) = \langle x(t_1)x(t_2) \rangle = \sum_n x_n(t_1)x_n(t_2) P_{x(t)}[x_n(t)]$$

The cross-correlation between two random signals $x(t)$ and $y(t)$ is,

$$R_{xy}(t_1, t_2) = \langle x(t_1)y(t_2) \rangle = \sum_n \sum_m x_n(t_1)y_m(t_2) P_{x(t)y(t)}(x_n(t_1), y_m(t_2))$$

where $P_{x(t)y(t)}[x_n(t), y_m(t)]$ is the probability that $x(t) = x_n(t)$ and $y(t) = y_m(t)$.

6.1.3 Stationary and Ergodic Signals

Stationary random signals are those whose characteristics do not depend upon the time origin. This implies the following,

a) The mean values are independent of time, i.e. $m_x(t) = \langle x(t) \rangle = m_x$.

b) The auto- and cross-correlation functions are functions of the time difference only, i.e.,

$$R_{xx}(t_1, t_2) = \langle x(t_1)x(t_2) \rangle = R_{xx}(t_1 - t_2) \quad R_{xy}(t_1, t_2) = \langle x(t_1)y(t_2) \rangle = R_{xy}(t_1 - t_2)$$

Most (but not all) signals studied in this course will be stationary.

6.1.4 Ensemble Averages Vs Time Averages

Averages of signals can be done in two ways,

- a) **Ensemble averages:** Ensemble averages are averages with respect to a probability density function. For example,

$$\langle x(t) \rangle = \sum_n x_n(t) P_{x(t)}[x_n(t)]$$

- b) **Time averages:** One can time average a signal as follows,

$$\overline{x(t)} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t) dt$$

6.1.5 Ergodicity

When all ensemble averages equal the corresponding time averages the signal is called ergodic. Ergodicity implies stationarity but not the other way around. For ergodic signals,

$$\langle x(t) \rangle = \overline{x(t)} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t) dt = m_x$$

And

$$R_{xy}(t_1 - t_2) = \langle x(t_1)y(t_2) \rangle = \overline{x(t_1)y(t_2)} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t + t_1 - t_2)y(t) dt$$

Ergodicity implies that each signal in the sample set is representative of the whole set.

6.1.5 Fourier Transforms

The following definitions of Fourier transforms will be used,

$$x(\omega) = \int_{-\infty}^{\infty} e^{+i\omega t} x(t) dt$$

$$x(t) = \int_{-\infty}^{+\infty} e^{-i\omega t} x(\omega) \frac{d\omega}{2\pi}$$

Parseval's Identity: The total energy in a random signal can be written in time and frequency domains as,

$$\int_{-\infty}^{+\infty} x^2(t) dt = \int_{-\infty}^{\infty} |x(\omega)|^2 \frac{d\omega}{2\pi}$$

6.1.6 Power Spectral Density

The total energy of a real random signal $x(t)$, given by,

$$\int_{-\infty}^{\infty} x^2(t) dt = \int_{-\infty}^{\infty} |x(\omega)|^2 \frac{d\omega}{2\pi}$$

can be infinite. So in practice one does not work with total energies of signals but with the signal powers. Define a truncated signal $x_T(t)$ as,

$$x_T(t) = \begin{cases} x(t) & -\frac{T}{2} \leq t \leq \frac{T}{2} \\ 0 & \text{otherwise} \end{cases}$$

The power in the signal $x(t)$ is,

$$\begin{aligned}
 &= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t) dt = \frac{1}{T} \int_{-\infty}^{\infty} x_T^2(t) dt \\
 &= \frac{1}{T} \int_{-\infty}^{\infty} |x_T(\omega)|^2 \frac{d\omega}{2\pi}
 \end{aligned}$$

The ensemble averaged power in the signal is,

$$= \frac{1}{T} \left\langle \int_{-\infty}^{\infty} |x_T(\omega)|^2 \frac{d\omega}{2\pi} \right\rangle = \int_{-\infty}^{\infty} \frac{1}{T} \langle |x_T(\omega)|^2 \rangle \frac{d\omega}{2\pi}.$$

The integrand in the integral in the limit $T \rightarrow \infty$ can be interpreted as the power spectral density $S_{xx}(\omega)$ of the signal $x(t)$,

$$S_{xx}(\omega) = \lim_{T \rightarrow \infty} \frac{1}{T} \langle |x_T(\omega)|^2 \rangle$$

The power spectral density, as the name implies, is the power in the signal per unit frequency bandwidth.

6.1.7 Wiener-Kinchine Theorem

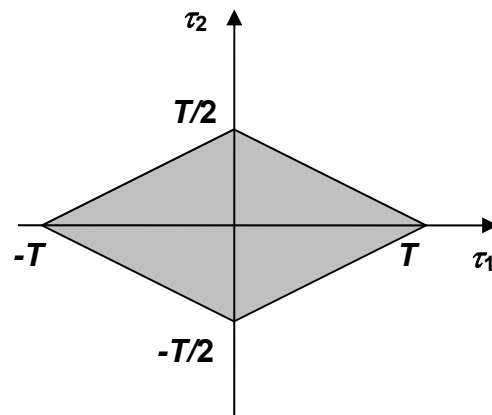
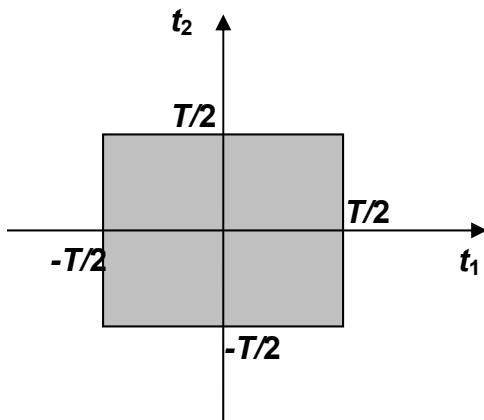
Weiner-Kinchine theorem relates the power spectral density of a stationary signal to the signal auto-correlation,

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} R_{xx}(t) dt$$

In other words, the power spectral density is the Fourier transform of the auto-correlation function.

Proof: We start from,

$$\begin{aligned}
 \langle |x_T(\omega)|^2 \rangle &= \left\langle \int_{-T/2}^{+T/2} dt_1 e^{i\omega t_1} x(t_1) \int_{-T/2}^{T/2} dt_2 e^{-i\omega t_2} x(t_2) \right\rangle \\
 &= \int_{-T/2}^{T/2} dt_1 \int_{-T/2}^{T/2} dt_2 \langle x(t_1)x(t_2) \rangle e^{i\omega(t_1-t_2)} \\
 &= \int_{-T/2}^{T/2} dt_1 \int_{-T/2}^{T/2} dt_2 R_{xx}(t_1-t_2) e^{i\omega(t_1-t_2)}
 \end{aligned}$$



Let,

$$\tau_1 = t_1 - t_2 \quad \tau_2 = \frac{t_1 + t_2}{2}$$

The Figure above shows how the integration region transform when going from the t_1 - t_2 plane to the τ_1 - τ_2 plane. It follows that,

$$\begin{aligned} \langle |x_T(\omega)|^2 \rangle &= \int_{-T}^0 d\tau_1 \int_{-\frac{T-\tau_1}{2}}^{\frac{T+\tau_1}{2}} d\tau_2 R_{xx}(\tau_1) e^{+i\omega\tau_1} + \int_0^T d\tau_1 \int_{-\frac{T-\tau_1}{2}}^{\frac{T-\tau_1}{2}} d\tau_2 R_{xx}(\tau_1) e^{+i\omega\tau_1} \\ &= \int_{-T}^T d\tau_1 R_{xx}(\tau_1) [T - |\tau_1|] e^{+i\omega\tau_1} \end{aligned}$$

and therefore,

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{1}{T} \langle |x_T(\omega)|^2 \rangle &= \lim_{T \rightarrow \infty} \int_{-T}^T d\tau_1 R_{xx}(\tau_1) \left[1 - \frac{|\tau_1|}{T} \right] e^{+i\omega\tau_1} \\ &= \int_{-\infty}^{\infty} d\tau_1 R_{xx}(\tau_1) e^{+i\omega\tau_1} \\ \Rightarrow S_{xx}(\omega) &= \int_{-\infty}^{\infty} d\tau_1 R_{xx}(\tau_1) e^{+i\omega\tau_1} \end{aligned}$$

Weiner-Kinchine theorem is a very important result and forms the basis of theoretical and experimental signal analysis.

Consider a stationary random signal $x(t)$. Then,

$$x(\omega) = \int_{-\infty}^{\infty} dt e^{+i\omega t} x(t)$$

It follows that,

$$\langle x^*(\omega') x(\omega) \rangle = 2\pi \delta(\omega - \omega') S_{xx}(\omega)$$

and,

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \langle x^*(\omega') x(\omega) \rangle$$

We will often use the above result.

6.2 Fluctuations and Noise: Concepts and Models

In quantum optics two types of noise are most commonly encountered:

- a) Shot noise
- b) Brownian or diffusion noise

In this chapter, we will look at these noise processes from a classical perspective.

6.2.1 Shot Noise

Suppose we are looking at a process that consists of a set of discrete events happening randomly in time. Suppose the j -th event happens at time t_j . At any time t , the rate $r(t)$ of the events (i.e. the number of events happening per unit time) is,

$$r(t) = \sum_j \delta(t - t_j)$$

The times t_j constitute the random part. We will assume the following:

- i) The times t_j are completely independent of each other
- ii) The probability that there is an event in a very short time interval dt is given by αdt where α is called the average rate of the process.

With these two assumptions we can compute the ensemble average of $r(t)$,

$$\langle r(t) \rangle = \left\langle \sum_j \delta(t - t_j) \right\rangle = \int_{-\infty}^{\infty} \alpha dt' \delta(t - t') = \alpha$$

The above result illustrates why α is the average rate. We want to find the spectral density of $r(t)$. We know that,

$$S_{rr}(\omega) = \int \frac{d\omega}{2\pi} e^{i\omega\tau} R_{rr}(\tau)$$

So we find the auto-correlation of the function $r(t)$,

$$\begin{aligned} R_{rr}(\tau) &= \langle r(t + \tau)r(t) \rangle \\ &= \left\langle \sum_j \sum_m \delta(t + \tau - t_j)\delta(t - t_m) \right\rangle \\ &= \left\langle \sum_j \sum_m \delta(t + \tau - t_j)\delta(t - t_m) [\delta_{jm} + 1 - \delta_{jm}] \right\rangle \\ R_{rr}(\tau) &= \left\langle \sum_j \delta(t + \tau - t_j)\delta(t - t_j) \right\rangle \quad \leftarrow \text{diagonal terms} \\ &\quad + \left\langle \sum_{j \neq m} \delta(t + \tau - t_j)\delta(t - t_m) \right\rangle \quad \leftarrow \text{off - diagonal terms} \\ &= \int_{-\infty}^{\infty} \alpha dt' \delta(t + \tau - t') \delta(t - t') + \left\langle \sum_{j \neq m} \delta(t + \tau - t_j)\delta(t - t_m) \right\rangle \end{aligned}$$

Noting that the times t_j are completely independent, the second term on the right hand side is approximately,

$$\left\langle \sum_j \delta(t + \tau - t_j) \right\rangle \left\langle \sum_m \delta(t - t_m) \right\rangle = \alpha^2$$

We therefore obtain,

$$R_{rr}(\tau) = \alpha \delta(\tau) + \alpha^2$$

The spectral density becomes,

$$S_{rr}(\omega) = \alpha + \alpha^2 2\pi\delta(\omega)$$

The noise in $r(t)$ is $n(t) = r(t) - \langle r(t) \rangle = r(t) - \alpha$. The auto-correlation and the spectral density of noise is,

$$\begin{aligned} R_{nn}(\tau) &= R_{rr}(\tau) - \alpha^2 = \alpha \delta(\tau) \\ \Rightarrow S_{nn}(\omega) &= S_{rr}(\omega) - \alpha^2 2\pi\delta(\omega) \\ S_{nn}(\omega) &= \alpha \end{aligned}$$

Noise $w(t)$ having the above characteristics is called shot noise. Shot noise implies a frequency independent spectral density with a magnitude equal to the average rate of the process. The noise is not present because the events are discrete. The noise is there because the events happen randomly in time (although the average rate is α). If a process consists of discrete events then just this does not mean that it will have shot noise. Shot noise implies that the timings of different events are random and completely independent of each other. If the times t_j get somehow correlated then the noise is not called shot noise. For example, consider the process $w(t)$ consisting of events happening at times t_j where $t_j = jT$. The events happen after fixed intervals of T . Each event is perfectly correlated in time with the other events. There is no randomness and therefore,

$$\langle w(t) \rangle = \left\langle \sum_j \delta(t - t_j) \right\rangle = \left\langle \sum_j \delta(t - jT) \right\rangle = \sum_j \delta(t - jT) = w(t)$$

The noise in $w(t)$ is zero.

6.2.2 Poisson Statistics and Shot Noise

For a process with shot noise, one may ask the question, "what is the probability of having n events in time T ?" Let $P(n, T)$ be the probability of having n events in time T . Then,

$$\begin{aligned} P(n, T + \Delta T) &= P(n-1, T) \alpha \Delta T + P(n, T)(1 - \alpha \Delta T) \\ \Rightarrow \frac{P(n, T + \Delta T) - P(n, T)}{\Delta T} &= P(n-1, T) \alpha - P(n, T) \alpha \\ \Rightarrow \frac{d}{dT} P(n, T) + \alpha P(n, T) &= \alpha P(n-1, T) \end{aligned}$$

Solution of the above differential equation with the boundary condition $P(n, T=0) = \delta_{n0}$ is,

$$P(n, T) = \frac{(\alpha T)^n}{n!} e^{-\alpha T}$$

When the statistics of events are according to the expression above, we call the process a Poisson process and the statistics are called Poisson statistics. The number of events in time T is,

$$N_T = \int_0^T r(t) dt = \int_0^T \sum_j \delta(t - t_j) dt$$

The ensemble average number of events in time T is,

$$\langle N_T \rangle = \int_0^T \langle r(t) \rangle dt = \int_0^T \left\langle \sum_j \delta(t - t_j) \right\rangle dt = \alpha T$$

This can also be confirmed from the probability distribution found above,

$$\begin{aligned} \langle N_T \rangle &= \sum_{n=0}^{\infty} n P(n, T) = \sum_{n=1}^{\infty} n \frac{(\alpha T)^n}{n!} e^{-\alpha T} \\ &= \sum_{n=1}^{\infty} \frac{(\alpha T)^{n-1}}{(n-1)!} e^{-\alpha T} (\alpha T) = \alpha T \end{aligned}$$

The standard deviation in the number of events in time T is,

$$\langle \Delta N_T^2 \rangle = \langle (N_T - \langle N_T \rangle)^2 \rangle = \langle N_T^2 \rangle - \langle N_T \rangle^2$$

The first term is,

$$\begin{aligned}\langle N_T^2 \rangle &= \sum_{n=0}^{\infty} n^2 \frac{(\alpha T)^n}{n!} e^{-\alpha T} = \sum_{n=1}^{\infty} n \frac{(\alpha T)^{n-1}}{(n-1)!} e^{-\alpha T} (\alpha T) \\ &= \sum_{n=1}^{\infty} (n-1+1) \frac{(\alpha T)^{n-1}}{(n-1)!} e^{-\alpha T} (\alpha T) = (\alpha T)^2 + (\alpha T)\end{aligned}$$

Therefore,

$$\langle \Delta N_T^2 \rangle = \langle N_T^2 \rangle - \langle N_T \rangle^2 = (\alpha T) = \langle N_T \rangle$$

For a process with shot noise, the standard deviation in the number of events during any time interval is equal to the mean. Of course, we can derive $\langle N_T \rangle$ and $\langle N_T^2 \rangle - \langle N_T \rangle^2$ directly from $r(t)$ as well. Since $r(t)$ is the rate of events, we can write,

$$N_T = \int_{t_0}^{t_0+T} r(t) dt$$

where t_0 is the time from where we start counting. The ensemble average value of N_T is,

$$\langle N_T \rangle = \left\langle \int_{t_0}^{t_0+T} r(t) dt \right\rangle = \int_{t_0}^{t_0+T} \alpha dt = \alpha T$$

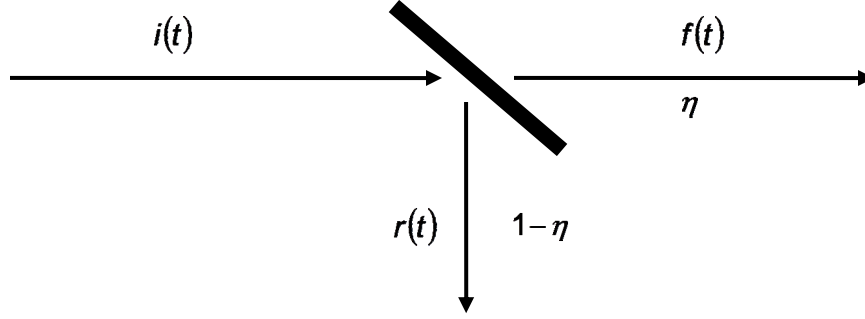
and,

$$\begin{aligned}\langle N_T^2 \rangle &= \left\langle \int_{t_0}^{t_0+T} r(t_1) dt_1 \int_{t_0}^{t_0+T} r(t_2) dt_2 \right\rangle \\ &= \int_{t_0}^{t_0+T} dt_1 \int_{t_0}^{t_0+T} dt_2 \langle r(t_1) r(t_2) \rangle \\ &= \int_{t_0}^{t_0+T} dt_1 \int_{t_0}^{t_0+T} dt_2 R_{rr}(t_1 - t_2) \\ &= \int_{t_0}^{t_0+T} dt_1 \int_{t_0}^{t_0+T} dt_2 \{ \alpha \delta(t_1 - t_2) + \alpha^2 \} \\ &= \alpha T + (\alpha T)^2\end{aligned}$$

And, therefore, $\langle N_T^2 \rangle - \langle N_T \rangle^2 = \alpha T = \langle N_T \rangle$ as before.

6.2.3 Partition Noise

Here we will consider noise introduced in particle splitting. Consider a stream of particles arriving at a splitter as shown below. The splitter decides whether each particle is transmitted in the forward direction or reflected in the downward direction. The decision is random but on the average the probability of forward transmission is η and the probability of reflection is $(1-\eta)$.



The incident particle rate is $i(t)$, the "reflected" particle rate is $r(t)$ and the rate of particles transmitted in the forward direction is $f(t)$. The average rates satisfy,

$$\begin{aligned}\langle i(t) \rangle &= \alpha \\ \langle f(t) \rangle &= \eta \langle i(t) \rangle = \eta \alpha \\ \langle r(t) \rangle &= (1 - \eta) \langle i(t) \rangle = (1 - \eta) \alpha.\end{aligned}$$

Let the noises be,

$$\begin{aligned}n^i(t) &= i(t) - \langle i(t) \rangle = i(t) - \alpha \\ n^f(t) &= f(t) - \eta \alpha \\ n^r(t) &= r(t) - (1 - \eta) \alpha\end{aligned}$$

If the splitter added no noise to the outgoing particles we would have had the following relations,

$$f(t) = \eta i(t) \quad (1)$$

$$r(t) = (1 - \eta) i(t) \quad (2)$$

and therefore,

$$\begin{aligned}S_{ff}(\omega) &= \eta^2 S_{ii}(\omega) \Rightarrow S_{n^f n^f}(\omega) = \eta^2 S_{n^i n^i}(\omega) \\ S_{rr}(\omega) &= (1 - \eta)^2 S_{ii}(\omega) \Rightarrow S_{n^r n^r}(\omega) = (1 - \eta)^2 S_{n^i n^i}(\omega).\end{aligned}$$

But (1) and (2) above are not correct since only the average of $f(t)$ equals η times the average of $i(t)$.

Let,

$$i(t) = \sum_j \delta(t - t_j)$$

and,

$$f(t) = \sum_j f_j \delta(t - t_j)$$

where f_j is a random variable that can take two values, 1 and 0, with probabilities η and $(1 - \eta)$, respectively. There are two sources of randomness in the above expression. The factors f_j are random and the times t_j of arrivals of the particles at the splitter are also random. The ensemble average of f_j is,

$$\langle f_j \rangle = \eta$$

It should also be noted that f_j and t_j are independent random variables. Therefore,

$$\langle f(t) \rangle = \left\langle \sum_j f_j \delta(t - t_j) \right\rangle = \eta \left\langle \sum_j \delta(t - t_j) \right\rangle = \eta \langle i(t) \rangle = \eta \alpha$$

In evaluating the above expression, we have first performed ensemble averaging with respect to the random variables f_j . The auto-correlation of $f(t)$ is,

$$\begin{aligned}
 R_{ff}(\tau) &= \langle f(t+\tau)f(t) \rangle \\
 &= \left\langle \sum_{j,m} \delta(t+\tau-t_j)\delta(t-t_m) f_j f_m \right\rangle \\
 R_{ff}(\tau) &= \left\langle \sum_j f_j^2 \delta(t+\tau-t_j)\delta(t-t_j) \right\rangle + \left\langle \sum_{j \neq m} f_j f_m \delta(t+\tau-t_j)\delta(t-t_m) \right\rangle
 \end{aligned}$$

We first perform ensemble averaging with respect to the factors f_j ,

$$\langle f_j \rangle = \eta \quad \langle f_j^2 \rangle = \eta \quad \langle f_j f_m \rangle_{j \neq m} = \langle f_j \rangle \langle f_m \rangle = \eta^2$$

It follows that,

$$\begin{aligned}
 R_{ff}(\tau) &= \eta \left\langle \sum_j \delta(t+\tau-t_j)\delta(t-t_j) \right\rangle + \eta^2 \left\langle \sum_{j \neq m} \delta(t+\tau-t_j)\delta(t-t_m) \right\rangle \\
 &= \eta(1-\eta) \left\langle \sum_j \delta(t+\tau-t_j)\delta(t-t_j) \right\rangle \\
 &\quad + \eta^2 \left\{ \left\langle \sum_j \delta(t+\tau-t_j)\delta(t-t_j) \right\rangle + \left\langle \sum_{j \neq m} \delta(t+\tau-t_j)\delta(t-t_m) \right\rangle \right\}
 \end{aligned}$$

The part in curly brackets is just $R_{ii}(\tau)$. Therefore,

$$R_{ff}(\tau) = \eta(1-\eta) \left\langle \sum_j \delta(t+\tau-t_j)\delta(t-t_j) \right\rangle + \eta^2 R_{ii}(\tau)$$

The first term on the right hand side is,

$$\begin{aligned}
 \left\langle \sum_j \delta(t+\tau-t_j)\delta(t-t_j) \right\rangle &= \int \alpha dt' \delta(t+\tau-t')\delta(t-t') \\
 &= \alpha \delta(\tau)
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 R_{ff}(\tau) &= \alpha \eta (1-\eta) \delta(\tau) + \eta^2 R_{ii}(\tau) \\
 \Rightarrow S_{ff}(\omega) &= \alpha \eta (1-\eta) + \eta^2 S_{ii}(\omega)
 \end{aligned}$$

Since,

$$n^f(t) = f(t) - \langle f(t) \rangle = f(t) - \eta \alpha$$

we have,

$$\begin{aligned}
 R_{n^f n^f}(\tau) &= R_{ff}(\tau) - \eta^2 \alpha^2 = \alpha \eta (1-\eta) \delta(\tau) + \eta^2 [R_{ii}(\tau) - \alpha^2] \\
 &= \alpha \eta (1-\eta) \delta(\tau) + \eta^2 R_{n^i n^i}(\tau)
 \end{aligned}$$

The noise spectral density in the transmitted particle stream is,

$$S_{n^f n^f}(\omega) = \alpha \eta (1-\eta) + \eta^2 S_{n^i n^i}(\omega)$$

The noise spectral density of the transmitted particles consist of two parts: i) the noise in the input particle stream multiplied by η^2 , ii) the noise added by the splitter which goes as $\alpha \eta (1-\eta)$. A similar analysis shows that,

$$R_{rr}(\tau) = \alpha\eta(1-\eta)\delta(\tau) + (1-\eta)^2 R_{ii}(\tau)$$

$$S_{rr}(\omega) = \alpha\eta(1-\eta) + (1-\eta)^2 S_{ii}(\omega)$$

$$S_{n^r n^r}(\omega) = \alpha\eta(1-\eta) + (1-\eta)^2 S_{n^i n^i}(\omega)$$

Let's look at,

$$S_{n^f n^f}(\omega) = \alpha\eta(1-\eta) + \eta^2 S_{n^i n^i}(\omega)$$

Case (i): Suppose η is very small such that $\eta^2 S_{n^i n^i}(\omega)$ can be neglected. Then,

$$S_{n^f n^f}(\omega) \cong \alpha\eta$$

But $\alpha\eta$ is the average rate at which particles go forward. So in the limit $\eta \rightarrow 0$, the output stream has shot noise irrespective of the noise in the input stream.

Case (ii): Suppose the incident stream has shot noise i.e. $S_{n^i n^i}(\omega)$ is equal to α , then,

$$S_{n^f n^f}(\omega) = \alpha\eta(1-\eta) + \eta^2 \alpha$$

$$S_{n^f n^f}(\omega) = \alpha\eta$$

If the input stream has shot noise then the output stream also has shot noise.

Case (iii): If $\eta \rightarrow 1$, then $S_{n^f n^f}(\omega) \approx S_{n^i n^i}(\omega)$ which makes sense since if no splitter is present, and no particle is reflected, then the noise in the incident and forward streams should be the same.

6.3 Introduction to Langevin Equations

6.3.1 Brownian Velocity

The best way to introduce Langevin equations is to look at a particle undergoing Brownian motion. Consider a particle of mass m in a gas (or a liquid) undergoing random Brownian motion. The particle is being kicked around randomly. The velocity of the particle is $v(t)$ which can be considered a random signal. We limit ourselves to one dimensional Brownian motion. The surrounding medium (other gas or liquid particles) exert a frictional force on the test particle so that any initial velocity given to the particle is quickly lost as a result of scattering with other particles. The equation for the particle velocity $v(t)$ can be written as,

$$\frac{dv(t)}{dt} = -\gamma v(t)$$

Here, γ describes the damping of the particle velocity due to collisions with other particles. The above equation cannot be a complete description of the particle motion since it gives,

$$v(t) = v(t=0) e^{-\gamma t}$$

This implies that,

$$v^2(t) = v^2(t=0) e^{-2\gamma t}$$

In the limit $t \rightarrow \infty$, the particle velocity and, therefore, energy both become zero. We know that the average particle energy is given by the equipartition theorem of statistical physics,

$$\left\langle \frac{1}{2} m v^2(t) \right\rangle = \frac{1}{2} K_B T \Rightarrow \langle v^2(t) \rangle = \frac{K_B T}{m}$$

One way to fix the equation for the motion of the particle is to add a force that describes the kicks the particle receives during collisions with other particles,

$$m \frac{dv(t)}{dt} = -\gamma m v(t) + F(t)$$

The force $F(t)$ is a random function. We require that the kicks at different times be completely uncorrelated and random in direction. In mathematical language, we require the ensemble average values of $F(t)$ and its auto-correlation to obey,

$$\begin{aligned} \langle F(t) \rangle &= 0 \\ \langle F(t_1)F(t_2) \rangle &= A\delta(t_1 - t_2) \end{aligned}$$

A-priori we don't know the value of A . We will determine A by enforcing the equipartition theorem. We start from the equation,

$$\frac{dv(t)}{dt} + \gamma v(t) = \frac{F(t)}{m}$$

The solution is,

$$\begin{aligned} v(t) &= v(t=0)e^{-\gamma t} + \frac{1}{m} \int_0^t F(t_1)e^{-\gamma(t-t_1)} dt_1 \\ \Rightarrow v^2(t) &= v^2(t=0)e^{-2\gamma t} + \frac{1}{m^2} \int_0^t dt_1 \int_0^t dt_2 F(t_1)F(t_2)e^{-\gamma(2t-t_1-t_2)} \\ &\quad + 2v(t=0) \frac{e^{-\gamma t}}{m} \int_0^t dt_1 F(t_1)e^{-\gamma(t-t_1)} \end{aligned}$$

We perform an ensemble average on both sides,

$$\begin{aligned} \langle v^2(t) \rangle &= \langle v^2(t=0) \rangle e^{-2\gamma t} + 2 \frac{e^{-\gamma t}}{m} \int_0^t dt_1 \langle v(t=0)F(t_1) \rangle e^{-\gamma(t-t_1)} \\ &\quad + \frac{1}{m^2} \int_0^t dt_1 \int_0^t dt_2 \langle F(t_1)F(t_2) \rangle e^{-\gamma(2t-t_1-t_2)} \end{aligned}$$

In steady state, when t can be assumed to be large, the only term that survives is the last term,

$$\begin{aligned} \lim_{t \rightarrow \infty} \langle v^2(t) \rangle &= \lim_{t \rightarrow \infty} \frac{1}{m^2} \int_0^t dt_1 \int_0^t dt_2 A\delta(t_1 - t_2) e^{-\gamma(2t-t_1-t_2)} \\ &= \lim_{t \rightarrow \infty} \frac{A}{m^2} \int_0^t dt_1 e^{-2\gamma(t-t_1)} \\ &= \lim_{t \rightarrow \infty} \frac{A}{m^2} \frac{(1 - e^{-2\gamma t})}{2\gamma} = \frac{A}{2\gamma m^2}. \end{aligned}$$

Since according to the equipartition theorem,

$$\langle v^2(t) \rangle = \frac{K_B T}{m}$$

we must have,

$$\frac{A}{2\gamma m^2} = \frac{K_B T}{m} \quad \Rightarrow \quad A = 2\gamma m K_B T$$

We therefore arrive at the following description of the velocity of a particle undergoing Brownian motion,

$$\frac{dv(t)}{dt} = -\gamma v(t) + \frac{F(t)}{m}$$

where the force $F(t)$ has a zero mean and the following auto-correlation function,

$$\langle F(t_1)F(t_2) \rangle = 2\gamma m K_B T \delta(t_1 - t_2)$$

The equation for the particle velocity is an example of a Langevin equation and $F(t)$ is a Langevin noise source. In most cases, the auto-correlation function of a Langevin noise source is a delta function in time and its magnitude is set by fundamental physical considerations irrespective of the microscopic details of the system. For example, we could have derived the factor $2\gamma m K_B T$ by a detailed microscopic calculation but it was easier to use a fundamental physical principle (i.e. equipartition theorem). The Langevin equation for the particle velocity is also an example of the "fluctuation dissipation" principle in statistical physics according to which every source of dissipation must also introduce fluctuations in a physical system.

Spectral Density: The spectral density of $v(t)$ can be derived in two ways:

- i) First find $R_{vv}(\tau)$ and then Fourier transform to obtain $S_{vv}(\omega)$.
- ii) The second method to obtain $S_{vv}(\omega)$ is to work directly in the frequency domain.

Both the above methods are illustrated below.

Start from,

$$v(t) = v(t=0)e^{-\gamma t} + \frac{1}{m} \int_0^t F(t_1) e^{-\gamma(t-t_1)} dt_1$$

The auto-correlation of the velocity is,

$$\begin{aligned} \langle v(t')v(t'') \rangle &= \left\langle v^2(0)e^{-\gamma(t'+t'')} \right\rangle + \frac{1}{m^2} \int_0^{t'} dt_1 \int_0^{t''} dt_2 \langle F(t_1)F(t_2) \rangle e^{-\gamma(t'+t'')} e^{\gamma(t_1+t_2)} \\ &\quad + \left\langle \frac{v(0)}{m} e^{-\gamma t'} \int_0^{t''} F(t_2) e^{-\gamma(t''-t_2)} dt_2 \right\rangle + \left\langle \frac{v(0)}{m} e^{-\gamma t''} \int_0^{t'} F(t_1) e^{-\gamma(t'-t_1)} dt_1 \right\rangle \end{aligned}$$

As both t' and t'' become large, the correlation function becomes,

$$\langle v(t')v(t'') \rangle = \lim_{\substack{t' \rightarrow \infty \\ t'' \rightarrow \infty}} \frac{2\gamma K_B T}{m} \int_0^{t'} dt_1 \int_0^{t''} dt_2 \delta(t_1 - t_2) e^{-\gamma(t'+t'')} e^{\gamma(t_1+t_2)}$$

Some care needs to be exercised in performing the double integral above:

- a) Suppose $t'' > t'$ then we can do the $\int_0^{t''} dt_2$ integral first and the result is,

$$\begin{aligned} &= \lim_{t', t'' \rightarrow \infty} \frac{2\gamma K_B T}{m} \int_0^{t'} dt_1 e^{-\gamma(t'+t'')} e^{2\gamma t_1} \\ &= \lim_{t', t'' \rightarrow \infty} \frac{K_B T}{m} \left(e^{-\gamma(t''-t')} - e^{-\gamma(t'+t'')} \right) \\ &= \frac{K_B T}{m} e^{-\gamma(t''-t')} \end{aligned}$$

- b) Suppose $t' > t''$ then we can do the $\int_0^{t'} dt_1$ integral first,

$$\begin{aligned}\langle v(t')v(t'') \rangle &= \lim_{\substack{t' \rightarrow \infty \\ t'' \rightarrow \infty}} \frac{2\gamma K_B T}{m} \int_0^{t''} dt_2 e^{-r(t'+t'')} e^{2rt_2} \\ &= \lim_{\substack{t' \rightarrow \infty \\ t'' \rightarrow \infty}} \frac{K_B T}{m} \left(e^{-r(t'-t'')} - e^{-r(t'+t'')} \right) \\ &= \frac{K_B T}{m} e^{-r(t'-t'')}\end{aligned}$$

Combining (a) and (b), one gets,

$$\langle v(t')v(t'') \rangle = \frac{K_B T}{m} e^{-\gamma|t'-t''|}$$

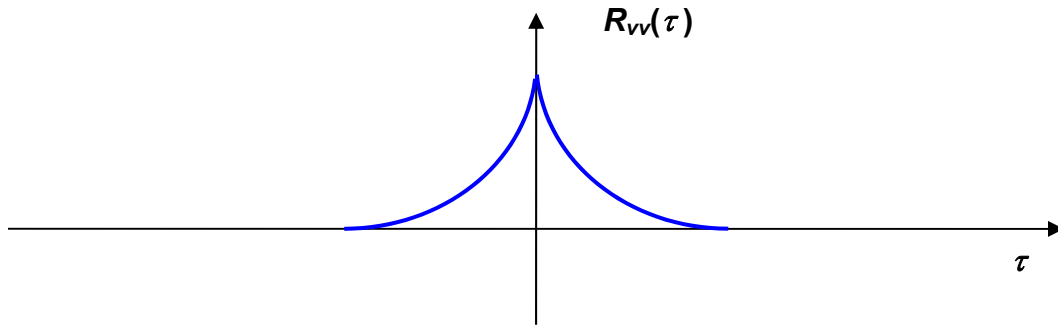
In the limit of large t' , t'' (when all initial correlations have died out) the random signal $v(t)$ is stationary (since $\langle v(t) \rangle = 0$ and $\langle v(t')v(t'') \rangle$ depends only on the time difference $(t'-t'')$),

$$R_{VV}(\tau) = \frac{K_B T}{m} e^{-\gamma|\tau|}$$

The spectral density of $v(t)$ is,

$$\begin{aligned}\Rightarrow S_{VV}(\omega) &= \frac{K_B T}{m} \int_{-\infty}^{\infty} d\tau e^{+i\omega\tau} e^{-\gamma|\tau|} \\ S_{VV}(\omega) &= \frac{2\gamma K_B T}{m(\omega^2 + \gamma^2)}\end{aligned}$$

The forces acting on the particle are delta-correlated (meaning their correlation function is proportional to a delta function in the time difference) but the particle velocity "remembers" the kicks it gets on a time scale of the order of $1/\gamma$. Therefore, the velocity-velocity correlation function is not delta-correlated but looks like,



and decays on a time scale $1/\gamma$.

The second method to obtain $S_{VV}(\omega)$ in steady state is to work directly in the frequency domain. Define,

$$F(\omega) = \int_{-\infty}^{\infty} dt e^{+i\omega t} F(t)$$

Then,

$$\langle F(\omega) \rangle = \int_{-\infty}^{\infty} dt e^{+i\omega t} \langle F(t) \rangle = 0$$

and,

$$\begin{aligned}
 \langle F^*(\omega_1)F(\omega_2) \rangle &= \int_{-\infty}^{\infty} dt_1 e^{-i\omega_1 t_1} \int_{-\infty}^{\infty} dt_2 e^{-i\omega_2 t_2} \langle F(t_1)F(t_2) \rangle \\
 &= 2\gamma m K_B T \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 e^{-i\omega_1 t_1} e^{+i\omega_2 t_2} \delta(t_1 - t_2) \\
 &= 2\gamma m K_B T \int_{-\infty}^{\infty} dt_1 e^{-i(\omega_1 - \omega_2)t_1} \\
 \langle F^*(\omega_1)F(\omega_2) \rangle &= 2\gamma m K_B T [2\pi \delta(\omega_1 - \omega_2)]
 \end{aligned}$$

Therefore, functions delta-correlated in time are also delta-correlated in frequency. Start from,

$$\frac{dv(t)}{dt} = -\gamma v(t) + \frac{F(t)}{m}$$

and take the Fourier transform of both sides,

$$-i\omega v(\omega) = -\gamma v(\omega) + F(\omega)$$

$$v(\omega) = \frac{F(\omega)/m}{-i[\omega + i\gamma]}$$

The spectral density $S_{VV}(\omega)$ is,

$$\begin{aligned}
 S_{VV}(\omega) &= \int_{-\infty}^{\infty} \langle v^*(\omega') v(\omega) \rangle \frac{d\omega'}{2\pi} \\
 &= \frac{1}{m^2} \int_{-\infty}^{\infty} \frac{\langle F^*(\omega')F(\omega) \rangle}{[\omega' - i\gamma][\omega + i\gamma]} \frac{d\omega'}{2\pi} \\
 &= \int_{-\infty}^{\infty} \frac{\frac{2\gamma K_B T}{m}}{[\omega' + i\gamma][\omega - i\gamma]} 2\pi \delta(\omega' - \omega) \frac{d\omega'}{2\pi} \\
 &= \frac{2\gamma K_B T}{\omega^2 + \gamma^2}
 \end{aligned}$$

After inverse Fourier transforming, we can find the correlation function as well,

$$R_{VV}(\tau) = \int_{-\infty}^{\infty} S_{VV}(\omega) e^{-i\omega\tau} \frac{d\omega}{2\pi} = \frac{K_B T}{m} e^{-\gamma|\tau|}$$

Working in frequency domain is often easier, and we don't have to deal with process of taking limits of large times in order to throw away the initial correlations.

6.3.2 Diffusive Motion (Random Walk)

The diffusion of a particle making random jumps is also described by a Langevin equation,

$$\frac{dx(t)}{dt} = W(t)$$

where,

$$\langle W(t) \rangle = 0$$

and,

$$\langle W(t_1)W(t_2) \rangle = 2D \delta(t_1 - t_2)$$

where D is the diffusion coefficient. The particle motion can also be described by a time-dependent probability density function $P(x,t)$ that gives the probability of finding the particle at position x at time t subject to some initial condition. $P(x,t)$ obeys the standard diffusion equation,

$$\frac{\partial P(x,t)}{\partial t} = D \frac{\partial^2 P(x,t)}{\partial x^2}$$

The Langevin equation and the diffusion equation are equivalent descriptions of the same process. Consider,

$$\frac{dx(t)}{dt} = W(t)$$

Since there is no dissipation, initial conditions cannot be ignored. The solution can be written as,

$$x(t) = x(0) + \int_0^t W(t) dt$$

$$\Rightarrow \langle x(t) \rangle = x(0)$$

The particle has an equal chance of diffusing in any direction, so the ensemble average of $x(t)$ equals $x(0)$. We first find the equal-time auto-correlation of the particle position as follows,

$$\langle x^2(t) \rangle = x^2(0) + 2x(0) \int_0^t \langle W(t) dt \rangle + \left\langle \int_0^t dt_1 \int_0^t dt_2 W(t_1) W(t_2) \right\rangle$$

$$\langle x^2(t) \rangle = x^2(0) + \int_0^t dt_1 \int_0^t dt_2 2D \delta(t_1 - t_2)$$

$$= x^2(0) + 2D \int_0^t dt_1$$

$$= x^2(0) + 2Dt$$

This gives,

$$\langle x^2(t) \rangle - x^2(0) = 2Dt$$

$$\Rightarrow \langle [x(t) - x(0)]^2 \rangle = 2Dt$$

The ensemble average of the square of the displacement of the particle from its initial location increases linearly with time. This property is a signature of diffusion. Next, we find the more general auto-correlation function. We have,

$$x(t_1) = x(0) + \int_0^{t_1} w(t) dt$$

$$x(t_2) = x(0) + \int_0^{t_2} w(t) dt$$

Therefore,

$$\begin{aligned}\langle x(t_1)x(t_2) \rangle &= x^2(0) + \int_0^{t_1} dt_1 \int_0^{t_2} dt_2 \langle w(t_1)w(t_2) \rangle \\ &= x^2(0) + \int_0^{t_1} dt_1 \int_0^{t_2} dt_2 2D\delta(t_1 - t_2) \\ \langle x(t_1)x(t_2) \rangle &= x^2(0) + 2D \min(t_1, t_2) \quad \rightarrow \quad \{\text{smaller of } t_1 \text{ and } t_2\} \\ \Rightarrow \langle [x(t_1) - x(t_2)]^2 \rangle &= 2D|t_1 - t_2|\end{aligned}$$

The process $x(t)$ is obviously not stationary.

We can get the same results as above from the diffusion equation,

$$\frac{\partial}{\partial t} P(x, t) = D \frac{\partial^2 P(x, t)}{\partial x^2}$$

The solution, subject to the initial condition, $P(x, t_0) = \delta(x - x(t_0))$, written as $P(x, t | x(t_0), t_0)$ for $t \geq t_0$ is,

$$P(x, t | x(t_0), t_0) = \frac{1}{\sqrt{2\pi 2D(t - t_0)}} \exp\left[-\frac{(x - x(t_0))^2}{2[2D(t - t_0)]}\right]$$

This means,

$$\langle x(t) - x(t_0) \rangle = 0$$

and,

$$\langle [x(t) - x(t_0)]^2 \rangle = 2D(t - t_0) \quad \{\text{for } t \geq t_0\}$$

Now consider the diffusion equation in the frequency domain,

$$-i\omega x(\omega) = W(\omega)$$

$$x(\omega) = \frac{W(\omega)}{-i\omega}$$

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \langle x^*(\omega')x(\omega) \rangle$$

$$= \int \frac{d\omega'}{2\pi} \frac{\langle W^*(\omega')W(\omega) \rangle}{\omega' \omega}$$

$$= \int \frac{d\omega'}{2\pi} \frac{2D 2\pi \delta(\omega' - \omega)}{\omega' \omega}$$

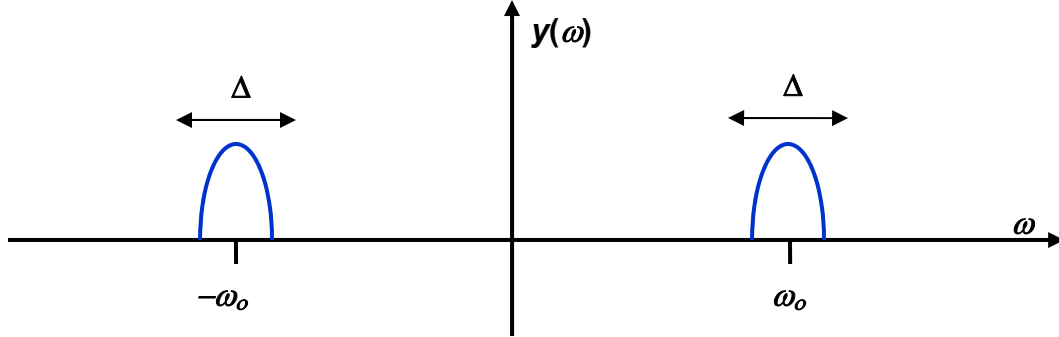
$$S_{xx}(\omega) = \frac{2D}{\omega^2}$$

Spectral density that goes as $1/\omega^2$ is a signature of a diffusion process. If you see the spectral density of a signal $y(t)$ going as $1/\omega^2$ then it means that $\langle [y(t_1) - y(t_2)]^2 \rangle \propto |t_1 - t_2|$. Since $x(t)$ is not stationary, and $\langle x(t_1)x(t_2) \rangle$ is not a function of $t_1 - t_2$, it makes little sense to take the inverse Fourier transform of $S_{xx}(\omega)$ for a diffusion process.

6.4 Phasor Representation of Signals and Signal Quadratures

6.4.1 Signal Quadratures and Phasors

Consider a narrowband real signal with spectrum centered near $\pm \omega_o$, as shown below.



Any real signal $y(t)$ with bandwidth centered at frequency ω_o can be written as,

$$y(t) = \int \frac{d\omega}{2\pi} y(\omega) e^{-i\omega t}$$

$$y(t) = \int_{\omega_o - \Delta/2}^{\omega_o + \Delta/2} \frac{d\omega}{2\pi} y(\omega) e^{-i\omega t} + \int_{-\omega_o - \Delta/2}^{-\omega_o + \Delta/2} \frac{d\omega}{2\pi} y(\omega) e^{-i\omega t}$$

Where Δ is large enough so that $y(\omega)$ is zero outside the bandwidth Δ centered at ω_o . One can remove the fast time dependence from inside the integral,

$$y(t) = e^{-i\omega_o t} \int_{\omega_o - \Delta/2}^{\omega_o + \Delta/2} \frac{d\omega}{2\pi} y(\omega) e^{-i(\omega - \omega_o)t} + e^{+i\omega_o t} \int_{-\omega_o - \Delta/2}^{-\omega_o + \Delta/2} y(\omega) e^{-i(\omega + \omega_o)t}$$

and write,

$$y(t) = e^{-i\omega_o t} \frac{x(t)}{2} + e^{i\omega_o t} \frac{x^*(t)}{2}$$

where,

$$x(t) = 2 \int_{\omega_o - \Delta/2}^{\omega_o + \Delta/2} \frac{d\omega}{2\pi} y(\omega) e^{-i(\omega - \omega_o)t}$$

The above relation expresses $y(t)$ as a product of a slowly time varying function $x(t)$ and a rapidly time varying complex exponential that has the center frequency of $y(t)$,

$$y(t) = \text{Re} \left\{ x(t) e^{-i\omega_o t} \right\}$$

Since $x(t)$ is complex, we can write,

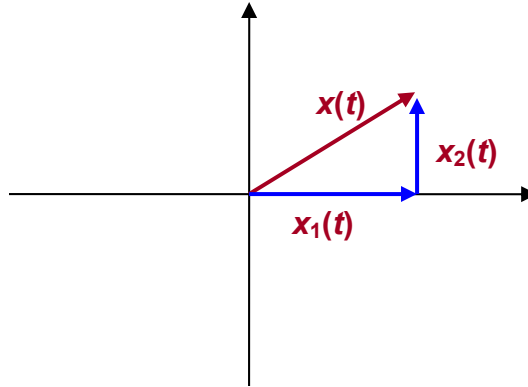
$$x(t) = x_1(t) + i x_2(t)$$

Then,

$$y(t) = \text{Re} \left\{ [x_1(t) + i x_2(t)] e^{-i\omega_o t} \right\}$$

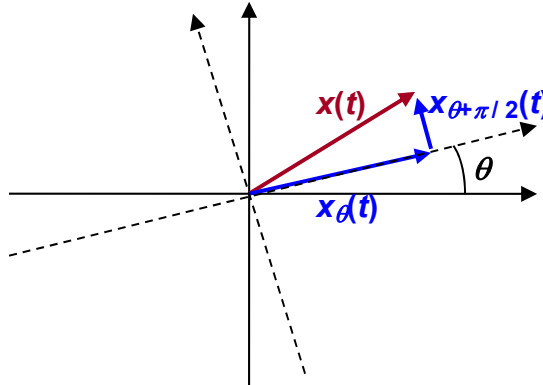
$$y(t) = x_1(t) \cos \omega_o t + x_2(t) \sin \omega_o t$$

$x_1(t)$ and $x_2(t)$ are called the quadratures of the signal $y(t)$. Usually, it is easier to deal with the slowly varying complex signal $x(t)$ than with the real signal $y(t)$. One can represent $x(t)$ as a vector (called phasor) in the complex plane, as shown below, and the quadratures are the orthogonal components of the phasor along the x-axis and the y-axis.



6.4.2 Generalized Quadratures

More often one is not interested in the components of the phasor along the x-axis or the y-axis but along axes that are rotated with respect to the x-y axes. Suppose we are interested in the components of the phasor that are along axes that are rotated at an angle θ with respect to the x-y axes, as shown below.



One can write $x(t)$ as,

$$x(t) = x_{\theta}(t)e^{i\theta} + x_{\theta+\pi/2}(t)e^{i(\theta+\pi/2)} = [x_{\theta}(t) + i x_{\theta+\pi/2}(t)]e^{i\theta}$$

$x_{\theta}(t)$ and $x_{\theta+\pi/2}(t)$ are the two orthogonal components of the phasor $x(t)$ along the rotated axes.

6.4.3 Examples

Case (i): Suppose,

$$x(t) = x_0 e^{i\phi}$$

Here, a is a complex constant. We can write the signal as,

$$x(t) = x_0 \cos \phi + i x_0 \sin \phi$$

It follows that,

$$\begin{aligned} y(t) &= \text{Re} \left\{ x(t) e^{-i\omega_0 t} \right\} \\ &= \text{Re} \left\{ x_0 e^{i\phi} e^{-i\omega_0 t} \right\} \\ y(t) &= x_0 \cos(\omega_0 t - \phi) + x_0 \sin(\omega_0 t - \phi) \end{aligned}$$

Case (ii): Suppose,

$$x(t) = x_0 + \eta_1(t)$$

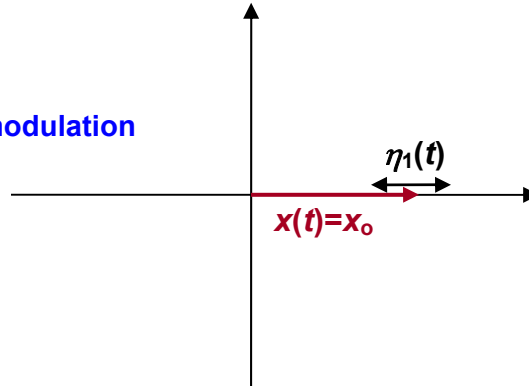
where x_0 and $\eta_1(t)$ are real and $\eta_1(t) \ll x_0$. Then,

$$y(t) = x(t) \cos \omega_0 t$$

$$= (x_0 + \eta_1(t)) \cos \omega_0 t$$

$\eta_1(t)$ thus represents amplitude modulation of the signal $y(t)$, as depicted below.

Case (ii): Amplitude modulation



Case (iii): Suppose,

$$x(t) = x_0 + i\eta_2(t)$$

where x_0 and $\eta_2(t)$ are real and $\eta_2(t) \ll x_0$. Then,

$$y(t) = x_0 \cos \omega_0 t + \eta_2(t) \sin \omega_0 t$$

The amplitude of the phasor is,

$$\sqrt{|x(t)|^2} = \sqrt{x_0^2 + \eta_2^2(t)} \approx \sqrt{x_0^2} = x_0$$

So a small imaginary part does not affect the amplitude of the phasor. We can also write,

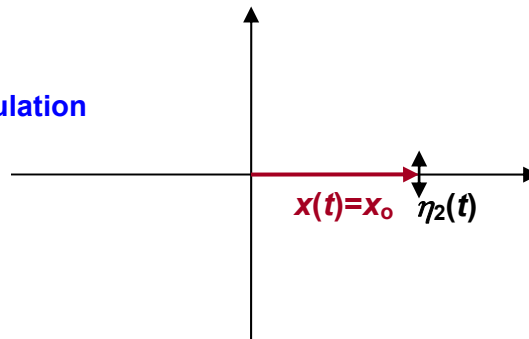
$$x(t) = [x_0 + i\eta_2(t)] = x_0 \left[1 + i \frac{\eta_2(t)}{x_0} \right]$$

$$\approx x_0 e^{i \left[\frac{\eta_2(t)}{x_0} \right]}$$

$$\Rightarrow y(t) \approx x_0 \cos \left[\omega_0 t - \frac{\eta_2(t)}{x_0} \right]$$

Therefore, $\eta_2(t)$ represents phase modulation of the signal $y(t)$, as depicted below.

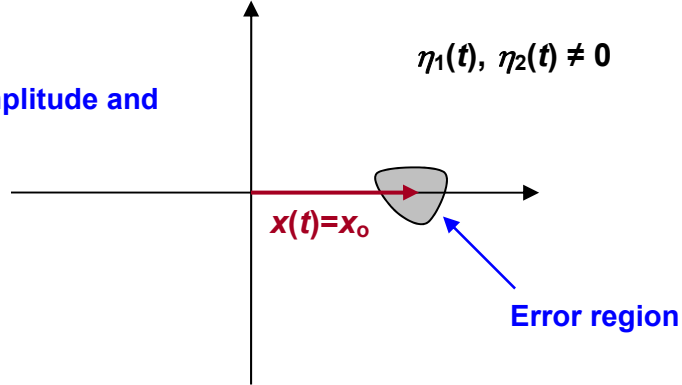
Case (iii): Phase modulation



Case (iv): If we have two zero-mean real random signals $\eta_1(t)$ and $\eta_2(t)$, and, $x(t) = x_0 + \eta_1(t) + i\eta_2(t)$, where x_0 is real and $\eta_1(t), \eta_2(t) \ll x_0$, then $\eta_1(t)$ describes the in-phase fluctuations or the amplitude fluctuations in the signal $y(t)$ and $\eta_2(t)$ describes the out-of-phase fluctuations or, simply, the phase fluctuations in the signal $y(t)$. The tip of the phasor will be randomly

moving in time (for example, the tip is shown to be moving in time in the shaded region in the Figure below). The shaded region is called the “error region.”

Case (iv): Random amplitude and phase fluctuations



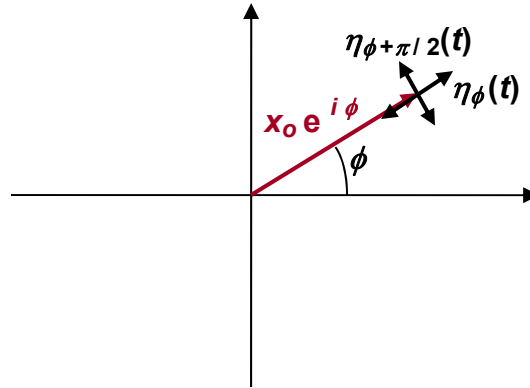
Case (v): Suppose,

$$x(t) = x_0 e^{i\phi} + \eta(t)$$

where x_0 is real and $\eta(t)$ is complex and $|\eta(t)| \ll x_0$. It is not obvious what effect does $\eta(t)$ have unless one writes the complex signal $\eta(t)$ using the generalized quadratures,

$$x(t) = x_0 e^{i\phi} + [\eta_\phi(t) + i\eta_{\phi+\pi/2}(t)] e^{i\phi}$$

Now it is clear that $\eta_\phi(t)$ and $\eta_{\phi+\pi/2}(t)$ represent amplitude and phase modulations, respectively, as shown below.



One can write,

$$\begin{aligned} x(t) &= x_0 e^{i\phi} + [\eta_\phi(t) + i\eta_{\phi+\pi/2}(t)] e^{i\phi} \approx [x_0 + \eta_\phi(t)] \left[1 + i \frac{\eta_{\phi+\pi/2}(t)}{x_0} \right] e^{i\phi} \\ &\approx [x_0 + \eta_\phi(t)] e^{i \left[\phi + \frac{\eta_{\phi+\pi/2}(t)}{x_0} \right]} \end{aligned}$$

The signal $y(t)$ is,

$$y(t) \approx [x_0 + \eta_\phi(t)] \cos \left[\omega_0 t - \phi - \frac{\eta_{\phi+\pi/2}(t)}{x_0} \right]$$

The amplitude and phase modulations in $y(t)$ are now obvious.