Chapter 5: Quantization of Radiation in Cavities and Free Space

5.1 Classical Electrodynamics

5.1.1 Classical Cavity Electrodynamics

Maxwell's equations for electromagnetism are,

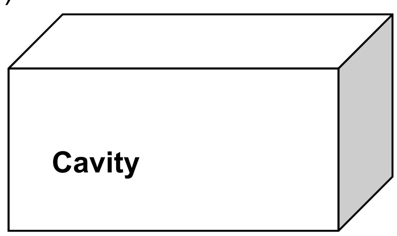
$$\nabla \cdot \vec{H}(\vec{r},t) = 0 \qquad (1)$$

$$\nabla \cdot \left(\varepsilon_o \varepsilon(\vec{r})\, \vec{E}(\vec{r},t)\right) = \rho(\vec{r},t) \qquad (2)$$

$$\nabla \times \vec{E}(\vec{r},t) = -\frac{\partial \mu_o \vec{H}(\vec{r},t)}{\partial t} \qquad (3)$$

$$\nabla \times \vec{H}(\vec{r},t) = \vec{J}(r,t) + \varepsilon_o \varepsilon(\vec{r}) \frac{\partial \vec{E}(\vec{r},t)}{\partial t} \qquad (4)$$

Here, $\varepsilon(\vec{r})$ is the relative dielectric constant. Consider fields inside a cavity. We need to find the eigenmodes of the radiation inside a cavity. The cavity is assumed to be absolutely lossless. We also assume that $\vec{J}(\vec{r},t) = 0$ and $\rho(\vec{r},t) = 0$.



From Maxwell's equations one can drive the wave equation,

$$(3) \Rightarrow \nabla \times \nabla \times \vec{E}(\vec{r},t) = -\mu_o \nabla \times \frac{\partial \vec{H}(\vec{r},t)}{\partial t}$$

$$\Rightarrow \nabla \times \nabla \times \vec{E}(\vec{r},t) = \frac{\varepsilon(\vec{r})}{c^2} \frac{\partial^2 \vec{E}(\vec{r},t)}{\partial t^2}$$

$$\Rightarrow \frac{1}{\varepsilon(\vec{r})} \nabla \times \nabla \times \vec{E}(\vec{r},t) = -\frac{1}{c^2} \frac{\partial^2 \vec{E}(\vec{r},t)}{\partial t^2}$$

The above equation can be solved to obtain the confined radiation modes, and their frequencies, inside the cavity. It is easier and better to work with the vector and scalar potentials, $\vec{A}(\vec{r},t)$ and $\phi(\vec{r},t)$, instead of $\vec{E}(\vec{r},t)$ and $\vec{H}(\vec{r},t)$ fields. These potentials are introduced below.

5.1.2 Vector and Scalar Potentials

We introduce vector and scalar potentials $\vec{A}(\vec{r},t)$ and $\phi(\vec{r},t)$ to describe $\vec{E}(\vec{r},t)$ and $\vec{H}(\vec{r},t)$ fields,

$$\vec{E}(\vec{r},t) = -\frac{\partial \vec{A}(\vec{r},t)}{\partial t} - \vec{\nabla}\phi(\vec{r},t)$$

$$\mu_o \vec{H}(r,t) = \nabla \times \vec{A}(r,t)$$

Since $\nabla \cdot \varepsilon_o \varepsilon(\vec{r})\, \vec{E}(\vec{r},t) = 0$ (no free charges),

$$\Rightarrow -\frac{\partial}{\partial t}\nabla\cdot\varepsilon_o\varepsilon(\vec{r})\,\vec{A}(\vec{r},t) - \nabla\cdot\varepsilon_o\varepsilon(\vec{r})\,\vec{\nabla}\phi(\vec{r},t) = 0$$

5.1.3 Gauge Transformations and Gauge Fixing

The fields $\vec{E}(\vec{r},t)$ and $\vec{H}(\vec{r},t)$ don't change if we make the following transformations,

$$\vec{A}(\vec{r},t) = \vec{A}'(\vec{r},t) + \vec{\nabla}F(\vec{r},t)$$

$$\phi(\vec{r},t) = \phi'(\vec{r},t) - \frac{\partial}{\partial t}F(\vec{r},t)$$

where $F(\vec{r},t)$ is any scalar function. So there is some degree of freedom in the choice of $\vec{A}(\vec{r},t)$ and $\phi(\vec{r},t)$. In other words, the choice of $\vec{A}(\vec{r},t)$ and $\phi(\vec{r},t)$ is not unique. One can impose an additional condition on the potentials to make them unique. In non-relativistic electrodynamics, one imposes the additional condition $\nabla\cdot\varepsilon_o\varepsilon(r)\,\vec{A}(\vec{r},t) = 0$. Introduction of a condition like this is called "gauge fixing", and this particular choice of gauge is called the coulomb gauge. In the coulomb gauge,

$$\nabla\cdot\varepsilon_o\varepsilon(\vec{r})\,\vec{E}(\vec{r},t) = -\frac{\partial}{\partial t}\nabla\cdot\varepsilon_0 o\varepsilon(\vec{r})\,\vec{A}(\vec{r},t) - \nabla\cdot\varepsilon_o\varepsilon(\vec{r})\,\vec{\nabla}\phi(\vec{r},t) = \rho(\vec{r},t)$$

$$\Rightarrow -\nabla\cdot\varepsilon_o\varepsilon(\vec{r})\,\vec{\nabla}\phi(\vec{r},t) = \rho(\vec{r},t)$$

The electric field can be divided into two parts,

$$\vec{E}(\vec{r},t) = \vec{E}_L(\vec{r},t) + \vec{E}_T(\vec{r},t)$$

In the coulomb gauge,,

$$\vec{E}_L(\vec{r},t) = -\vec{\nabla}\phi(\vec{r},t)$$

$$\vec{E}_T(\vec{r},t) = -\frac{\partial}{\partial t}\vec{A}(\vec{r},t)$$

It follows that,

$$\vec{\nabla}\cdot\varepsilon_o\varepsilon(\vec{r})\,\vec{E}_L(\vec{r},t) = \rho(\vec{r},t)$$

$$\nabla\cdot\varepsilon_o\varepsilon(\vec{r})\,\vec{E}_T(\vec{r},t) = 0$$

$\vec{E}_L(\vec{r},t)$ is called the longitudinal electric field and its source is the charge density $\rho(\vec{r},t)$ (whether static or time dependent). $\vec{E}_T(\vec{r},t)$ is called the transverse electric field and its source is the time-dependent current density $\vec{J}(\vec{r},t)$. The vector potential can also be divided into two parts,

$$\vec{A}(\vec{r},t) = \vec{A}_L(\vec{r},t) + \vec{A}_T(\vec{r},t)$$

In the coulomb gauge $\vec{A}_L(\vec{r},t) = 0$ and the vector potential is entirely transverse. In non-relativistic quantum electrodynamics, $\vec{E}_T(\vec{r},t)$ is associated with "photons". In this Chapter, since $\rho(\vec{r},t)$ will almost always be zero, $\vec{E}_L(\vec{r},t)$ will also be zero, and $\vec{E}(\vec{r},t)$ should be understood to be $\vec{E}_T(\vec{r},t)$. In the coulomb gauge (with $\rho(\vec{r},t) = 0$ and $\vec{J}(\vec{r},t) = 0$), we have,

$$\vec{E}(\vec{r},t) = -\frac{\partial\vec{A}(\vec{r},t)}{\partial t}$$

$$\mu_o\vec{H}(\vec{r},t) = \nabla\times\vec{A}(\vec{r},t)$$

Maxwell's equations give,

$$\frac{1}{\varepsilon(\vec{r})}\nabla\times\nabla\times\vec{A}(\vec{r},t) = -\frac{1}{c^2}\frac{\partial^2\vec{A}(\vec{r},t)}{\partial t^2}$$

Note that $\vec{E}(\vec{r},t)$ and $\vec{A}(\vec{r},t)$ satisfy the same wave equations.

5.1.4 Cavity Eigenmodes and Eigenvalues

To proceed further, we need to find the eigenmodes of the operator that appears in the wave equation,

$$\frac{1}{\varepsilon(\vec{r})}\nabla\times\nabla\times$$

Let these eigenmodes be $\vec{U}_n(\vec{r})$ and let the corresponding eigenvalues be ω_n^2/c^2 . Therefore,

$$\frac{1}{\varepsilon(\vec{r})}\nabla\times\nabla\times\vec{U}_n(\vec{r})=\frac{\omega_n^2}{c^2}\vec{U}_n(\vec{r})$$

These eigenfunctions must also satisfy the requirement that $\nabla\cdot\varepsilon_o\varepsilon(\vec{r})\vec{U}_n(\vec{r})=0$, which follows from the coulomb gauge condition, $\nabla\cdot\varepsilon_o\varepsilon(\vec{r})\vec{A}(\vec{r},t)=0$. We will assume that the eignemodes can be chosen to be completely real functions.

Orthogonality Relation for the Eigenmodes: Consider two different eigenmodes, $\vec{U}_n(\vec{r})$ and $\vec{U}_m(\vec{r})$. These satisfy,

$$\frac{1}{\varepsilon(\vec{r})}\nabla\times\nabla\times\vec{U}_n(\vec{r})=\frac{\omega_n^2}{c^2}\vec{U}_n(\vec{r})$$

$$\frac{1}{\varepsilon(\vec{r})}\nabla\times\nabla\times\vec{U}_m(\vec{r})=\frac{\omega_m^2}{c^2}\vec{U}_m(\vec{r})$$

Take the dot product of both sides of the first equation with $\vec{U}_m(\vec{r})$ to get,

$$\vec{U}_m(\vec{r})\cdot\left[\nabla\times\nabla\times\vec{U}_n(\vec{r})\right]=\frac{\omega_n^2}{c^2}\varepsilon(\vec{r})\vec{U}_m(\vec{r})\cdot\vec{U}_n(\vec{r})$$

Using the vector identity,

$$\vec{A}\cdot(\nabla\times\nabla\times\vec{C})=(\nabla\times\vec{A})\cdot(\nabla\times\vec{C})-\nabla\cdot\left(\vec{A}\times(\nabla\times\vec{C})\right)$$

and integrating over all space we get,

$$\int d^3\vec{r}\;\vec{U}_m(\vec{r})\cdot\left[\nabla\times\nabla\times\vec{U}_n(\vec{r})\right]=\frac{\omega_n^2}{c^2}\int d^3\vec{r}\;\varepsilon(\vec{r})\vec{U}_m(\vec{r})\cdot\vec{U}_n(\vec{r})$$

Integrate by parts twice on the left hand side to obtain,

$$\int d^3\vec{r}\,(\nabla\times\nabla\times\vec{U}_m(\vec{r}))\cdot\vec{U}_n(\vec{r})=\int d^3\vec{r}\,\frac{\omega_m^2}{c^2}\varepsilon(\vec{r})\vec{U}_m(\vec{r})\cdot\vec{U}_n(\vec{r})$$

$$\Rightarrow\left(\frac{\omega_m^2}{c^2}-\frac{\omega_n^2}{c^2}\right)\int d^3r\,\varepsilon(\vec{r})\vec{U}_m(\vec{r})\cdot\vec{U}_n(\vec{r})=0$$

Therefore, if $\omega_m\neq\omega_n$ then,

$$\int d^3\vec{r}\,\varepsilon(\vec{r})\vec{U}_m(\vec{r})\cdot\vec{U}_n(\vec{r})=0 \qquad (5)$$

Therefore, eigenvectors corresponding to different eigenvalues are orthogonal in the sense of (5) above. If $\vec{U}_n(\vec{r})$ is normalized such that,

$$\int\vec{U}_n(\vec{r})\cdot\vec{U}_n(\vec{r})d^3\vec{r}=1$$

then,

$$\int d^3\vec{r}\,\varepsilon(\vec{r})\vec{U}_m(\vec{r})\cdot\vec{U}_n(\vec{r})=\varepsilon_n\;\delta_{nm}$$

where ε_n is the average relative dielectric constant seen by the mode $\vec{U}_n(\vec{r})$.

5.1.5 Field Expansion using Eigenmodes

Since the eigenmodes form a complete basis set in the space of all transverse vector fields, one can always expand any transverse vector field, and in particular $\vec{A}(\vec{r},t)$, in terms of these eigenmodes,

$$\vec{A}(\vec{r},t) = \sum_n \frac{q_n(t)}{\sqrt{\varepsilon_o \varepsilon_n}} \vec{U}_n(\vec{r})$$

Substituting the above expansion in the wave equation we get,

$$\nabla \times \nabla \times \vec{A}(\vec{r},t) = -\frac{\varepsilon(\vec{r})}{c^2} \frac{\partial^2 A(\vec{r},t)}{\partial t^2}$$

$$\sum_n \frac{q_n(t)}{\sqrt{\varepsilon_o \varepsilon_n}} \nabla \times \nabla \times \vec{U}_n(\vec{r}) = -\sum_m \frac{\varepsilon(r)}{c^2} \frac{\partial^2 q_m(t)}{\partial t^2} \cdot \vec{U}_m(\vec{r})$$

$$\Rightarrow \sum_n \frac{q_n(t)}{\sqrt{\varepsilon_o \varepsilon_n}} \frac{\omega_n^2}{c^2} \varepsilon(\vec{r})\, \vec{U}_n(\vec{r}) = \sum_m \frac{\partial^2 q_m(t)}{\partial t^2} \frac{\varepsilon(\vec{r})}{c^2} \frac{\vec{U}_m(\vec{r})}{\sqrt{\varepsilon_o \varepsilon_m}}$$

Multiplying both sides by $\vec{U}_j(\vec{r})$, and integrate over all space, we get,

$$-q_j(t)\, \omega_j^2 = \frac{\partial^2 q_j(t)}{\partial t^2}$$

The above equation has the solution,

$$q_j(t) = q_j(t=0)\cos(\omega_j t) + \frac{\dot{q}_j(t=0)}{\omega_j} \sin(\omega_j t)$$

Or, equivalently, in complex time-harmonic notation,

$$q_j(t) = q_j\, e^{-i\,\omega_j\, t} + c.c.$$

5.1.6 Field Hamiltonian

The classical expression for the field energy is,

$$H = \int d^3\vec{r} \left[\frac{1}{2} \varepsilon_o \varepsilon(\vec{r}) \vec{E}(\vec{r},t) \cdot \vec{E}(\vec{r},t) + \frac{1}{2} \mu_o \vec{H}(\vec{r},t) \cdot \vec{H}(\vec{r},t) \right]$$

$$= \frac{1}{2} \sum_{nm} \int d^3\vec{r} \left[\frac{\varepsilon_o \varepsilon(\vec{r})}{\sqrt{\varepsilon_o \varepsilon_n}\sqrt{\varepsilon_o \varepsilon_m}} \dot{q}_n(t) \dot{q}_m(t)\, \vec{U}_n(\vec{r}) \cdot \vec{U}_m(\vec{r}) + \frac{1}{\mu_o} \frac{q_n(t)}{\sqrt{\varepsilon_o \varepsilon_n}} \frac{q_m(t)}{\sqrt{\varepsilon_0 \varepsilon_m}} \left(\nabla \times \vec{U}_n(\vec{r})\right) \cdot \left(\nabla \times \vec{U}_m(\vec{r})\right) \right]$$

Note that,

$$\int d^3 r \left(\nabla \times \vec{U}_n(\vec{r})\right) \cdot \left(\nabla \times \vec{U}_m(\vec{r})\right)$$

$$= \int \vec{U}_n(\vec{r}) \cdot \nabla \times \nabla \times \vec{U}_m(\vec{r}) d^3\vec{r} = \int \vec{U}_n(\vec{r}) \cdot \varepsilon(\vec{r}) \frac{\omega_m^2}{c^2} \vec{U}_m(\vec{r})$$

$$= \delta_{nm} \frac{\varepsilon_m\, \omega_m^2}{c^2}$$

Therefore,

$$H = \sum_m \left[\frac{\dot{q}_m^2(t)}{2} + \frac{\omega_m^2}{2} q_m^2(t) \right]$$

If one defines $p_m(t) = \dot{q}_m(t)$ then,

$$H = \sum_m \left[\frac{p_m^2(t)}{2} + \frac{\omega_m^2}{2} q_m^2(t) \right]$$

From Maxwell's equation we had found,

$$\frac{\partial^2 q_m(t)}{\partial t^2} = -\omega_m^2 \, q_m(t)$$

So the variables $p_m(t)$ and $q_m(t)$ satisfy the equations,

$$\begin{cases} \dfrac{d}{dt} p_m(t) = -\omega_m^2 \, q_m(t) \\ \dfrac{d}{dt} q_m(t) = p_m(t) \end{cases}$$

Comparing the above equations with those for a simple harmonic oscillator, we see that they are identical!

5.2 Cavity Quantum Electrodynamics

5.2.1 Quantization of Cavity Fields

For classical fields we derived the following expression for the energy,

$$\hat{H} = \sum_m \left(\frac{p_m^2(t)}{2} + \frac{\omega_m^2 \, q_m^2(t)}{2} \right)$$

And the dynamics are given by,

$$\begin{cases} \dfrac{d}{dt} p_m(t) = -\omega_m^2 \, q_m(t) \\ \dfrac{d}{dt} q_m(t) = p_m(t) \end{cases}$$

We need a fundamental relation (not derivable from any other theory) to quantize the field. Note that each field mode behaves like a simple harmonic oscillator (SHO). Not just that the expression for total energy matches that of a SHO, but the dynamics are also similar. So we can try quantizing by the following steps:

(a) Let the dynamic variables $q_m(t)$ and $\hat{p}_m(t)$ become operators $\hat{q}_m(t)$ and $\hat{p}_m(t)$

(b) Let the equal-time commutation relation between $\hat{q}_m(t)$ and $\hat{p}_m(t)$ be $\left[\hat{q}_m(t), \hat{p}_m(t)\right] = i\hbar$

(c) For different modes let the commutation relations be $\left[\hat{q}_m(t), \hat{p}_n(t)\right] = 0$

The Hamiltonian operator of the field becomes,

$$\hat{H} = \sum_m \frac{\hat{p}_m^2(t)}{2} + \frac{\omega_m^2 \, \hat{q}_m^2(t)}{2}$$

As in the case of SHO, define creation and destruction operators $\hat{a}_m(t)$ and $\hat{a}_m^+(t)$ for each field mode as,

$$\hat{a}_m(t) = \frac{1}{\sqrt{2\hbar\omega_m}} \left(\omega_m \hat{q}_m(t) - i \, \hat{p}_m(t) \right)$$

$$a_m^+(t) = \frac{1}{\sqrt{2\hbar\omega_m}} \left(\omega_m \hat{q}_m(t) + i \, \hat{p}_m(t) \right)$$

It follows that,

$$\left[\hat{a}_m(t),\hat{a}_n^+(t)\right] = \delta_{mn}$$

and,

$$\hat{H} = \sum_m \hbar\omega_m\left(\hat{a}_m^+(t)\hat{a}_m(t)+\frac{1}{2}\right)$$

The operator time dependence in the above expressions should be interpreted in the Heisenberg sense. So, for example, the Hamiltonian in the Schrodinger picture would be,

$$\hat{H} = \sum_m \hbar\omega_m\left(\hat{a}_m^+\hat{a}_m+\frac{1}{2}\right) = \sum_m \hbar\omega_m\left(\hat{n}_m+\frac{1}{2}\right)$$

where $\hat{n}_m$ is the number operator for the mode m.

5.2.2 Energy Eigenstates and Eigenenergies

Since all radiation eigenmodes behave independently, we consider one mode only. The Hamiltonian for a single mode is,

$$\hat{H} = \hbar\omega_m\left(\hat{a}_m^+ a_m+\frac{1}{2}\right)$$

Let the eigenstates of this Hamiltonian be $|n\rangle_m$. Following the discussion in Chapter 4, we have,

$$\hat{a}_m^+\hat{a}_m|n\rangle_m = n|n\rangle_m$$
$$\hat{a}_m^+|n\rangle_m = \sqrt{n+1}\,|n+1\rangle_m$$
$$\hat{a}_m|n\rangle_m = \sqrt{n}\,|n-1\rangle_m$$

and,

$$\hat{H}|n\rangle_m = \hbar\omega_m\left(n+\frac{1}{2}\right)|n\rangle_m$$

(i) $|n\rangle_m$ is an energy eigenstate with energy $\hbar\omega_m\left(n+\frac{1}{2}\right)$

(ii) The energies of different ergenstates are separated by multiples of $\hbar\omega_m$. The eigenenergies are,

$$\frac{1}{2}\hbar\omega_m,\hbar\omega_m+\frac{1}{2}\hbar\omega_m,2\hbar\omega_m+\frac{1}{2}\hbar\omega_m,3\hbar\omega_m+\frac{1}{2}\hbar\omega_m,.........$$

(iii) The ground state $|0\rangle_m$ has energy $\frac{1}{2}\hbar\omega_m$

5.2.3 The Photon

On quantizing the electromagnetic radiation we see that the energy of a single mode of the field with frequency ω_m can only take on values separated from each other by multiples of $\hbar\omega_m$ (i.e. the mode energy can only be increased or decreased in multiples of $\hbar\omega_m$). This is in contrast with classical electrodynamics where a radiation mode can take any energy value by increasing or decreasing the field amplitude in a continuous fashion. This minimum amount of energy (i.e. $\hbar\omega_m$) is associated with the term "photon". A state with a single photon is the smallest possible excitation above the ground state $|0\rangle_m$ of a field mode. A state with a single photon is $|1\rangle_m$ and has energy $\left(\hbar\omega_m+\frac{1}{2}\hbar\omega_m\right)$. A state with two photons is $\left|2\right\rangle_m$ and has energy $\left(2\hbar\omega_m + \frac{1}{2}\hbar\omega_m\right)$. A state with n photons is $\left|n\right\rangle_m$. The ground state $|0\rangle_m$ of a field mode has no photons but still has energy equal to $\frac{1}{2}\hbar\omega_m$.

The operator $\hat{n}_m = \hat{a}_m^+\hat{a}_m$ is called the number operator for the mode m because it gives the number of photons in a state,

$$\hat{n}_m\left|n\right\rangle_m = n\left|n\right\rangle_m$$

$\hat{a}_m^+$ and $\hat{a}_m$ are called photon creation and destruction operators because they increase or decrease the number of photons in a state,

$$\hat{a}_m^+\left|n\right\rangle_m = \sqrt{n+1}\left|n+1\right\rangle_m$$

$$\hat{a}_m\left|n\right\rangle_m = \sqrt{n}\left|n-1\right\rangle_m$$

The state $\left|n\right\rangle_m$ can be written as,

$$\left|n\right\rangle_m = \frac{(\hat{a}_m^+)^n}{\sqrt{n!}}\left|0\right\rangle_m$$

The states $\left|n\right\rangle_m$ are called photon number states since they are eigenstates of the number operator $\hat{n}_m = \hat{a}_m^+\hat{a}_m$.

A photon is not a particle and a photon is not a wave. A photon is just the smallest possible energy excitation of a field mode.

5.2.4 Multimode States

A multimode quantum state with 3 photons in mode m, 4 photons in mode n, 5 photons in mode p, is written as,

$$|\psi\rangle = |3\rangle_m \otimes |4\rangle_n \otimes |5\rangle_p$$

So,

$$\hat{n}_m\,|\psi\rangle = 3\,|\psi\rangle$$

$$\hat{n}_n\,|\psi\rangle = 4\,|\psi\rangle$$

$$\hat{n}_p\,|\psi\rangle = 5\,|\psi\rangle$$

Or sometimes $|\psi\rangle$ may be written as,

$$|\psi\rangle = |\,n_m = 3, n_n = 4, n_p = 5\rangle$$

The vacuum state $|0\rangle$ means the ground state of all modes,

$$|0\rangle = |0\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3 \otimes \ldots\ldots\ldots$$

If we have a state with 3 photons in mode 2 then,

$$|\psi\rangle = |0\rangle_1 \otimes |3\rangle_2 \otimes |0\rangle_3 \otimes |0\rangle_4 \otimes \ldots\ldots\ldots\ldots$$

The above notation is too cumbersome. Instead one typically writes,

$$|\psi\rangle = |3\rangle_2$$

One only writes down the states that have non-zero photon numbers or are of interest in some other way. Few examples are given below.

(1) A state with p photons in mode m,

$$|\psi\rangle = \frac{\left(\hat{a}_m^{+}\right)^p}{\sqrt{p!}}|0\rangle = |p\rangle_m$$

(2) A state with one photon in mode m and one in mode n,

$$|\psi\rangle = a_m^{+}a_n^{+}|0\rangle = |1\rangle_m \otimes |1\rangle_n = |n_m = 1, n_n = 1\rangle$$

(3) A state which is a linear superposition of one photon in mode m and me in mode n,

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left[a_m^{+}|0\rangle + a_n^{+}|0\rangle\right] = \frac{1}{\sqrt{2}}\left[|1\rangle_m \otimes |0\rangle_n + |0\rangle_m \otimes |1\rangle_n\right]$$

$$= \frac{1}{\sqrt{2}}\left[|n_m = 1, n_n = 0\rangle + |n_m = 0, n_n = 1\rangle\right] = \frac{1}{\sqrt{2}}\left[|n_m = 1\rangle + |n_n = 1\rangle\right]$$

Completeness Relation for the Photon Number States: A full blown completeness relation for photon states look like

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\ldots\ldots\left[|n_1, n_2, n_3, \ldots\ldots\rangle\langle n_1, n_2, n_3, \ldots\ldots|\right] = \hat{1}$$

But in any practical problem one is usually dealing with photons in one or a few modes. So for one mode (say the mode m) the completeness relation becomes,

$$\sum_{n=0}^{\infty}|n\rangle_m\, {}_m\langle n| = \hat{1} \quad \text{or} \quad \sum_{n_m=0}^{\infty}|n_m\rangle\langle n_m| = \hat{1}$$

Orthogonality Relation for the Photon Number States: We need to find the value of the inner product ${}_m\langle p|p'\rangle_n$. Since the corresponding mode spatial functions $\vec{U}_m(\vec{r})$ and $\vec{U}_n(\vec{r})$ are not orthogonal in the sense $\int \vec{U}_m(\vec{r}), \vec{U}_n(\vec{r})\, d^3\vec{r} = \delta_{nm}$, it is not immediately obvious what ${}_m\langle p|p'\rangle_n$ should be. But since $|p\rangle_m$ and $|p'\rangle_n$ are eigenstates of a Hermitian operator (i.e. $\hat{H}$), we have,

$${}_m\langle p|p'\rangle_n = \delta_{nm}\,\delta_{pp'}$$

5.2.5 Time Development of Creation and Destruction Operators

The time development of the creation and destruction operators follows from the Heisenberg equation,

$$i\hbar\frac{d\hat{a}_m(t)}{dt} = \left[\hat{a}_m(t), \hat{H}\right] = \hbar\omega_m \hat{a}_m(t)$$

$$\Rightarrow \hat{a}_m(t) = e^{-i\omega_m t}\,\hat{a}_m(t=0) = e^{-i\omega_m t}\,\hat{a}_m$$

and,

$$i\hbar\frac{d\hat{a}_m^{+}(t)}{dt} = \left[\hat{a}_m^{+}(t), \hat{H}\right] = -\hbar\omega_m \hat{a}_m(t)$$

$$\Rightarrow \hat{a}_m^{+}(t) = e^{i\omega_m t}\,\hat{a}_m^{+}(t=0) = e^{i\omega_m t}\,\hat{a}_m^{+}$$

5.2.6 Field Operators

The fields are physical observables and are therefore represented by operators. We can write $\hat{\vec{A}}(r,t)$ and $\hat{\vec{E}}(r,t)$ in Heisenberg picture using,

$$\begin{aligned}\hat{\vec{A}}(\vec{r},t) &= \sum_n \frac{\hat{q}_n(t)}{\sqrt{\varepsilon_o \varepsilon_n}} \vec{U}_n(\vec{r}) \\ &= \sum_m \sqrt{\frac{\hbar}{2\omega_m \varepsilon_o \, \varepsilon_m}} \left(\hat{a}_m(t) + \hat{a}_m^+(t) \right) \vec{U}_m(\vec{r}) \\ \hat{\vec{E}}(\vec{r},t) &= -\frac{\partial \hat{\vec{A}}(\vec{r},t)}{\partial t} \\ &= \sum_m i \sqrt{\frac{\hbar \omega_m}{2\varepsilon_o \varepsilon_m}} \left(\hat{a}_m(t) - \hat{a}_m^+(t) \right) \vec{U}_m(\vec{r}) \\ \hat{\vec{H}}(\vec{r},t) &= \frac{1}{\mu_o} \nabla \times \hat{\vec{A}}(\vec{r},t) \\ &= \sum_m \frac{1}{\mu_o} \sqrt{\frac{\hbar}{2\omega_m \varepsilon_0 \varepsilon_n}} \left(\hat{a}_m(t) + \hat{a}_m^+(t) \right) \nabla \times \vec{U}_m(\vec{r})\end{aligned}$$

Note that all field operators are Hermitian.

Now consider a quantum state with a billion photons in mode m, i.e $|\psi\rangle = |n_m = 10^9\rangle = |10^9\rangle_m$. Let's find the average electric field for this state,

$$\begin{aligned}\langle \psi | \hat{\vec{E}}(\vec{r},t) | \psi \rangle &= \sum_m i \sqrt{\frac{\hbar \omega_m}{2\varepsilon_0 \varepsilon_m}} \left[\langle \psi | \hat{a}_m(t) | \psi \rangle - \langle \psi | \hat{a}_m^+(t) | \psi \rangle \right] \vec{U}_m(\vec{r}) \\ &= 0\end{aligned}$$

There are a large number of photons in the state $|\psi\rangle = |10^9\rangle_m$ but it has a zero average value for the electric field. Therefore, the photon number state is likely not what emerges from, say, your cell phones.

5.2.7 Vacuum Fluctuations

The ground state of a mode $|0\rangle_m$ has energy $\hbar\omega_m/2$ even though there are no photons in this state. The energy is due to what are called vacuum fluctuations; in the ground state, the electric and magnetic fields have zero average values but averages of the squares of the fields are not zero. In fact, restricting ourselves to just one mode, a direct computation shows,

$$\int d^3\vec{r} \;\; {}_m\langle 0 | \frac{1}{2} \varepsilon_0 \varepsilon(\vec{r}) \hat{\vec{E}}(\vec{r},t) \cdot \hat{\vec{E}}(\vec{r},t) + \frac{1}{2} \mu_o \hat{\vec{H}}(\vec{r},t) \cdot \hat{\vec{H}}(\vec{r},t) | 0 \rangle_m = \frac{1}{2} \hbar \omega_m$$

The total vacuum energy, including contributions from all modes, is then $\sum_m \hbar\omega_m/2$, which is infinite. This infinity is not a problem since in experiments only the differences in energies are measured and not absolute energies.

We don't know whether the sum $\sum_m \hbar\omega_m/2$ is valid for values of ω_m all the way up to ∞. When m becomes very large, the modes $\vec{U}_m(\vec{r})$ vary very fast in space and over length scales comparable to or shorter than the Planck scale of 10^{-33} cm. Nobody knows if quantum electrodynamics is valid at such short spatial scales.

5.3 Electrodynamics in Free Space

5.3.1 Classical Electrodynamics in Free Space

Maxwell's equations in free space are,

$$\nabla.\mu_o\vec{H}(\vec{r},t) = 0$$

$$\nabla.\vec{E}(\vec{r},t) = 0$$

$$\nabla\times\vec{E}(\vec{r},t) = -\mu_o\frac{\partial\vec{H}(\vec{r},t)}{\partial t}$$

$$\nabla\times\vec{H}(\vec{r},t) = \varepsilon_o\frac{\partial\vec{E}(\vec{r},t)}{\partial t}$$

The above equations result in the wave equation,

$$\nabla\times\nabla\times\vec{E}(\vec{r},t) = -\frac{1}{c^2}\frac{\partial^2\vec{E}(\vec{r},t)}{\partial t^2}$$

Assuming, as before, coulomb gauge and all fields to be transverse (divergence free), let,

$$\vec{E}(\vec{r},t) = -\frac{\partial\vec{A}(\vec{r},t)}{\partial t}$$

$$\mu_o\vec{H}(\vec{r},t) = \nabla\times\vec{A}(\vec{r},t)$$

where,

$$\nabla\cdot\vec{E}(\vec{r},t) = \nabla\cdot\vec{A}(\vec{r},t) = 0$$

It follows that,

$$\nabla\times\nabla\times\vec{A}(\vec{r},t) = -\frac{1}{c^2}\frac{\partial^2\vec{A}(\vec{r},t)}{\partial t^2}$$

$$\nabla\times\nabla\times\vec{A}(\vec{r},t) = \nabla\left(\nabla\cdot\vec{A}(\vec{r},t)\right) - \nabla^2\vec{A}(\vec{r},t)$$

$$= -\nabla^2\vec{A}(\vec{r},t)$$

So the wave equation in free space becomes,

$$\nabla^2\vec{A}(\vec{r},t) = \frac{1}{c^2}\frac{\partial^2\vec{A}(\vec{r},t)}{\partial t^2}$$

5.3.2 Free-Space Eigenmodes and Eigenvalues

We need to find eigenmodes of the operator ∇^2. Let these be $\vec{U}_{\vec{k}}(\vec{r})$ with eigenvalues $-\omega_{\vec{k}}^2/c^2$, i.e.,

$$\nabla^2\vec{U}_{\vec{k}}(\vec{r},t) = -\frac{\omega_{\vec{k}}^2}{c^2}\vec{U}_{\vec{k}}(\vec{r})$$

The eigenvectors must also satisfy the coulomb gauge condition,

$$\nabla\cdot\vec{A}(\vec{r},t) = 0 \quad\Rightarrow\quad \nabla\cdot\vec{U}_{\vec{k}}(\vec{r}) = 0$$

The eigenvectors can be written in terms of plane waves,

$$\vec{U}_{\vec{k}}(\vec{r}) = \hat{\varepsilon}(\vec{k})\frac{e^{i\vec{k}\cdot\vec{r}}}{\sqrt{V}}$$

The eigenmodes are normalized in very large box of volume V. All physical results will turn out to be independent of V. The right hand side above represents a plane wave propagating in the direction of the vector $\vec{k}$ and with the vector potential (and the electric field) polarized in the direction of the unit vector $\hat{\varepsilon}(\vec{k})$. Note that,

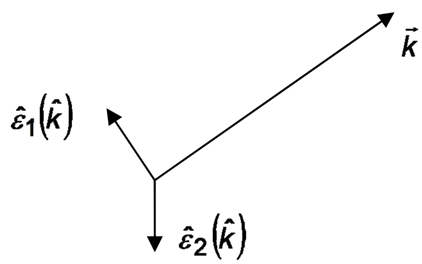
$$\nabla^2 \vec{U}_{\vec{k}}(\vec{r}) = \nabla^2 \hat{\varepsilon}(\vec{k}) \frac{e^{i\vec{k}\cdot\vec{r}}}{\sqrt{V}} = -k^2 \vec{U}_{\vec{k}}(\vec{r})$$

Therefore, the eigenvalue of the plane wave solution is,

$$\frac{\omega_{\vec{k}}^2}{c^2} = \vec{k}.\vec{k} = k^2$$

Since $\omega_{\vec{k}}$ does not depend on the direction of $\vec{k}$ but only on its magnitude, we will write it as ω_k. The constrain $\nabla \cdot \vec{U}_{\vec{k}}(\vec{r}) = 0$ implies $\vec{k}.\hat{\varepsilon}(\vec{k}) = 0$. The direction of field polarization is perpendicular to the direction of propagation of the plane wave given by the vector $\vec{k}$. For each direction $\vec{k}$, there are two independent and mutually orthogonal directions perpendicular to $\vec{k}$ that $\hat{\varepsilon}(\vec{k})$ can have. One can choose any two such directions for the polarization unit vectors and label them $\hat{\varepsilon}_1(\vec{k})$ and $\hat{\varepsilon}_2(\vec{k})$. For example, if $\vec{k} = k\hat{x}$ (i.e. pointing in the x-direction) then one can choose $\hat{\varepsilon}_1(\vec{k}) = \hat{y}$ and $\hat{\varepsilon}_2(\vec{k}) = \hat{z}$. Or, one may also choose,

$$\hat{\varepsilon}_1(\vec{k}) = \frac{1}{\sqrt{2}}(\hat{y} + \hat{z}) \qquad \hat{\varepsilon}_2(\vec{k}) = \frac{1}{\sqrt{2}}(-\hat{y} + \hat{z})$$



The plane wave eigenmodes form a complete set in the space of all transverse vector fields. Therefore, the field $\vec{A}(\vec{r},t)$ can be expanded in this complete set,

$$\vec{A}(\vec{r},t) = \sum_{\vec{k}} \sum_{j=1}^{2} \frac{q_j(\vec{k},t)}{\sqrt{\varepsilon_o}} \frac{e^{i\vec{k}.\vec{r}}}{\sqrt{V}} \hat{\varepsilon}_j(\vec{k}) \qquad (1)$$

Since $\vec{A}(\vec{r},t)$ is real (i.e., $\vec{A}(\vec{r},t) = A^*(\vec{r},t)$), one must have,

$$q_j(-\vec{k},t)\hat{\varepsilon}_j(-\vec{k}) = q_j^*(\vec{k},t)\hat{\varepsilon}_j(\vec{k})$$

It is convenient to choose,

$$q_j(-\vec{k},t) = q_j^*(\vec{k},t)$$
$$\hat{\varepsilon}_j(-\vec{k}) = \hat{\varepsilon}_j(\vec{k})$$

Keep in mind that since $\vec{k}$ can be pointing in any direction, the unit vectors $\hat{\varepsilon}_1(\vec{k})$ and $\hat{\varepsilon}_2(\vec{k})$ have no simple relationship with the Cartesian unit vectors $\hat{x}$, $\hat{y}$ and $\hat{z}$.

5.3.3 Periodic Boundary Conditions

Free space is infinite. For the purpose of calculations, it is useful to assume that free space is a large box of side L on each side, and volume V equal to L^3. We also assume that each facet of that box is connected with the opposite facet. Since the eigenmodes must be single-valued everywhere, the following periodic boundary conditions must hold,

$$e^{i\,\vec{k}.x\,\hat{x}} = e^{i\,\vec{k}.(x+L)\,\hat{x}} \quad \Rightarrow \quad k_x = \frac{2\pi\, n}{L} \qquad n = 0, \pm 1, \pm 2, \pm 3, \ldots\ldots$$

$$e^{i\,\vec{k}.y\,\hat{y}} = e^{i\,\vec{k}.(y+L)\,\hat{y}} \quad \Rightarrow \quad k_y = \frac{2\pi\, m}{L} \qquad m = 0, \pm 1, \pm 2, \pm 3, \ldots\ldots$$

$$e^{i\,\vec{k}.z\,\hat{z}} = e^{i\,\vec{k}.(z+L)\,\hat{z}} \quad \Rightarrow \quad k_z = \frac{2\pi\, p}{L} \qquad p = 0, \pm 1, \pm 2, \pm 3, \ldots\ldots$$

Thus, the vector $\vec{k}$ can only have certain discrete values. There is only one such allowed value in a cube of volume $(2\pi/L)^3$ in k-space. Therefore, noting that there are $V/(2\pi)^3$ different allowed $\vec{k}$ values per unit volume in k-space, the summation over $\vec{k}$ of the form $\sum_{\vec{k}}$, can be replaced by the integral,

$$\sum_{\vec{k}} \rightarrow \frac{V}{(2\pi)^3} \int d^3\vec{k}$$

The field $\vec{A}(\vec{r},t)$ can now be written as,

$$\vec{A}(\vec{r},t) = V \int \frac{d^3\vec{k}}{(2\pi)^3} \sum_{j=1}^{2} \frac{q_j(\vec{k},t)}{\sqrt{\varepsilon_o}} \frac{e^{i\,\vec{k}.\vec{r}}}{\sqrt{V}} \hat{\varepsilon}_j(\vec{k})$$

Orthogonality Relation for the Eigenmodes: The orthogonality relation for the eigenmodes is,

$$\int d^3\vec{r} \left\{ \frac{e^{-i\vec{k}'.\vec{r}}}{\sqrt{V}} \hat{\varepsilon}_r(\vec{k}') \right\} . \left\{ \frac{e^{i\vec{k}.\vec{r}}}{\sqrt{V}} \hat{\varepsilon}_s(\vec{k}) \right\} = \delta_{rs}\; \delta_{\vec{k}',\vec{k}}$$

Sometimes one is interested in the Cartesian components of $\vec{A}(\vec{r},t)$ (i.e. $A_x(\vec{r},t)$, $A_y(\vec{r},t)$, $A_z(\vec{r},t)$). These can be found as follows,

$$A_x(\vec{r},t) = \hat{x}.\vec{A}(\vec{r},t) = V \int \frac{d^3\vec{k}}{(2\pi)^3} \sum_{j=1}^{2} \frac{q_j(\vec{k},t)}{\sqrt{\varepsilon_0}} \frac{e^{i\,\vec{k}.\vec{r}}}{\sqrt{V}} \hat{x}.\hat{\varepsilon}_j(\vec{k})$$

$$\hat{\varepsilon}_j(\vec{k}) \,.\, \hat{\varepsilon}_j(\vec{k}) = 1$$
$$\Rightarrow \varepsilon_{jx}^2(\vec{k}) + \varepsilon_{jy}^2(\vec{k}) + \varepsilon_{jz}^2(\vec{k}) = 1$$

$$\left\{ \begin{array}{l} \varepsilon_{jx}(\vec{k}) = \hat{\varepsilon}_j(\vec{k}).\hat{x} \\ \varepsilon_{jy}(\vec{k}) = \hat{\varepsilon}_j(\vec{k}).\hat{y} \\ \varepsilon_{jz}(\vec{k}) = \hat{\varepsilon}_j(\vec{k}).\hat{z} \end{array} \right.$$

Finally, the electric and magnetic fields are,

$$\vec{E}(\vec{r},t) = -\frac{\partial \vec{A}(\vec{r},t)}{\partial t} = -V \int \frac{d^3\vec{k}}{(2\pi)^3} \sum_{j=1}^{2} \frac{\dot{q}_j(\vec{k},t)}{\sqrt{\varepsilon_o}} \frac{e^{i\,\vec{k}.\vec{r}}}{\sqrt{V}} \hat{\varepsilon}_j(\vec{k})$$

$$\vec{H}(\vec{r},t) = \frac{\nabla \times \vec{A}(\vec{r},t)}{\mu_o} = V \int \frac{d^3\vec{k}}{(2\pi)^3} \sum_{j=1}^{2} \frac{q_j(\vec{k},t)}{\mu_o \sqrt{\varepsilon_o}} \frac{e^{i\,\vec{k}.\vec{r}}}{\sqrt{V}} \left[i\vec{k} \times \hat{\varepsilon}_j(\vec{k}) \right]$$

5.3.4 Time Development

We can plug the expansion for $\vec{A}(\vec{r},t)$ in the equation,

$$\nabla^2 \vec{A}(\vec{r},t) = \frac{1}{c^2} \frac{\partial^2 \vec{A}(\vec{r},t)}{\partial t^2}$$

to get,

$$V\int \frac{d^3\vec{k}}{(2\pi)^3} \sum_{s=1}^{2} k^2 \frac{q_s(\vec{k},t)}{\sqrt{\varepsilon_o}} \frac{e^{i\vec{k}.\vec{r}}}{\sqrt{V}} \hat{\varepsilon}_s(\vec{k}) = V\int \frac{d^3\vec{k}}{(2\pi)^3} \sum_{s=1}^{2} \frac{-\ddot{q}_s(\vec{k},t)}{\sqrt{\varepsilon_o}} \frac{e^{i\vec{k}.\vec{r}}}{\sqrt{V}} \hat{\varepsilon}_s(\vec{k})$$

Multiplying by $e^{-i\vec{k}.\vec{r}}\hat{\varepsilon}_j(\vec{k})$ on both sides, integrating over all space, and using the orthogonality relation for the modes, we get,

$$\frac{d^2 q_j(\vec{k},t)}{dt_2} = -k^2c^2\, q_j(\vec{k},t)$$
$$= -\omega_k^2\, q_j(\vec{k},t)$$

5.3.5 Field Energy

The energy of the field can be found as follows,

$$H = \int d^3\vec{r}\, \frac{\varepsilon_o}{2}\, \vec{E}(\vec{r},t).\vec{E}(\vec{r},t) + \frac{\mu_o}{2}\vec{H}(\vec{r},t).\vec{H}(\vec{r},t)$$

$$= \int d^3\vec{r}\;\; V\int \frac{d^3\vec{k}}{(2\pi)^3}\sum_r V\int\frac{d^3\vec{k}'}{(2\pi)^3}\sum_s \left\{ \frac{1}{2}\dot{q}_r(\vec{k},t)\dot{q}_s(\vec{k}',t)\left[\hat{\varepsilon}_r(\vec{k}).\hat{\varepsilon}_s(\vec{k}')\right]\frac{e^{i(\vec{k}+\vec{k}').\vec{r}}}{V}\right.$$
$$\left. + \frac{c^2}{2}\left[\left(i\vec{k}\times\hat{\varepsilon}_r(\vec{k})\right).\left(i\vec{k}\times\hat{\varepsilon}_s(\vec{k}')\right)\right] q_r(\vec{k},t)q_s(\vec{k}',t)\frac{e^{i(\vec{k}+\vec{k}').\vec{r}}}{V}\right\}$$

First we carry out the integration over all space. From the exponentials one obtains a factor of $V\delta_{-\vec{k},\vec{k}'}$ that can be used to get rid of the integration/summation over $\vec{k}'$, and one can replace $\vec{k}'$ everywhere by $-\vec{k}$. Also, note the following relations,

$$q_j(-\vec{k},t) = q_j^*(\vec{k},t)$$
$$\hat{\varepsilon}_j(-\vec{k}) = \hat{\varepsilon}_j(\vec{k})$$
$$\sum_s \hat{\varepsilon}_r(\vec{k}).\hat{\varepsilon}_s(-\vec{k}) = 1$$
$$\sum_s \left[i\vec{k}\times\hat{\varepsilon}_r(\vec{k})\right]\cdot\left[-i\vec{k}\times\hat{\varepsilon}_s(-\vec{k})\right] = k^2$$

The final result is,

$$H = V\int\frac{d^3\vec{k}}{(2\pi)^3}\sum_j\left\{\frac{\dot{q}_j^*(\vec{k},t)\dot{q}_j(\vec{k},t)}{2} + \omega_k^2\,\frac{q_j^*(\vec{k},t)q_j(\vec{k},t)}{2}\right\}.$$

The expression for energy has complex quantities. So it is a little different from what you have seen in the past. If one defines,

$$p_j(\vec{k},t) = \dot{q}_j(\vec{k},t)$$

then,

$$\begin{cases} \dfrac{\partial p_j(\vec{k},t)}{\partial t} = -\omega_k^2\, q_j(\vec{k},t) \\ \dfrac{\partial q_j(\vec{k},t)}{\partial t} = p_j(\vec{k},t) \end{cases}$$

and the expression for the energy becomes,

$$H = V\int \frac{d^3\vec{k}}{(2\pi)^3} \sum_j \left\{ \frac{\left|p_j\left(\vec{k},t\right)\right|^2}{2} + \omega_k^2 \frac{\left|q_j\left(\vec{k},t\right)\right|^2}{2} \right\}.$$

5.3.6 Field Momentum

In classical electromagnetism the momentum of the electromagnetic field is,

$$\vec{P} = \varepsilon_o \mu_o \int d^3\vec{r} \;\; \vec{E}(\vec{r},t) \times \vec{H}(\vec{r},t)$$

Using the relation,

$$\sum_s - \hat{\varepsilon}_r\left(\vec{k}\right) \times \left(-i\,\vec{k} \times \hat{\varepsilon}_s\left(-\vec{k}\right)\right) = i\vec{k}$$

one obtains,

$$\vec{P} = V\int \frac{d^3\vec{k}}{(2\pi)^3} \sum_j \left\{ i\,\vec{k}\,\dot{q}_j\left(\vec{k},t\right) q_j^*\left(\vec{k},t\right) \right\}$$

5.3.7 Field Angular Momentum

Classical electromagnetic fields have a well-defined angular momentum. Recall that the angular momentum of a particle with momentum $\vec{p}(t)$ and position vector $\vec{r}(t)$ with respect to a point $\vec{r}_o$ is given as,

$$\vec{L}(t) = \left(\vec{r}(t) - \vec{r}_o\right) \times \vec{p}(t)$$

The classical expression for the momentum of the electromagnetic field is (from previous Section),

$$\vec{P} = \varepsilon_o \mu_o \int d^3\vec{r} \;\; \vec{E}(\vec{r},t) \times \vec{H}(\vec{r},t)$$

The classical expression for the angular momentum of the field is,

$$\vec{J} = \varepsilon_o \mu_o \int d^3\vec{r} \;\; \left(\vec{r} - \vec{r}_o\right) \times \left[\vec{E}(\vec{r},t) \times \vec{H}(\vec{r},t)\right]$$

Usually the point of reference is the origin (given the homogeneity of free space) and the above expression is written as,

$$\vec{J} = \varepsilon_o \mu_o \int d^3\vec{r} \;\; \vec{r} \times \vec{E}(\vec{r},t) \times \vec{H}(\vec{r},t)$$

5.4 Quantum Electrodynamics in Free Space

5.4.1 Quantlzation of Radiation in Free Space

The first step in the quantization of any system is the promotion of observables to operators and the imposition of commutation relations. In quantum electrodynamics the observables are the field operators, $\hat{\vec{E}}(\vec{r},t)$ and $\hat{\vec{H}}(\vec{r},t)$, and these operators must therefore be Hermitian. In classical electromagnetism,

$$q_j\left(-\vec{k},t\right) = q_j^*\left(\vec{k},t\right)$$

$$p_j\left(-\vec{k},t\right) = p_j^*\left(\vec{k},t\right)$$

The above conditions were necessary for the fields to be real. In quantum electrodynamics, the field operators will be Hermitian provided,

$$\hat{q}_j\left(-\vec{k},t\right) = \hat{q}_j^+\left(\vec{k},t\right)$$

$$\hat{p}_j\left(-\vec{k},t\right) = \hat{p}_j^+\left(\vec{k},t\right)$$

To quantize, we impose the following equal-time commutation relations,

$$\left[\hat{q}_r\left(\vec{k},t\right), p_s^+\left(\vec{k}',t\right)\right] = i\hbar\,\delta_{rs}\;\delta_{\vec{k},\vec{k}'}$$

It follows that,

$$\left[\hat{q}_r\left(\vec{k},t\right), p_s\left(\vec{k}',t\right)\right] = i\hbar\,\delta_{rs}\;\delta_{\vec{k},-\vec{k}'}$$

We define creation and destruction operators as before,

$$\hat{a}_j\left(\vec{k},t\right) = \frac{1}{\sqrt{2\hbar\omega_k}}\left[\omega_k\,\hat{q}_j\left(\vec{k},t\right) + i\,\hat{p}_j\left(\vec{k},t\right)\right]$$

$$\hat{a}_j^+\left(\vec{k},t\right) = \frac{1}{\sqrt{2\hbar\omega_k}}\left[\omega_k\,\hat{q}_j^+\left(\vec{k},t\right) - i\,\hat{p}_j^+\left(\vec{k},t\right)\right]$$

This gives,

$$\left[\hat{a}_r\left(\vec{k},t\right), \hat{a}_s^+\left(\vec{k}',t\right)\right] = \delta_{rs}\;\delta_{\vec{k},\vec{k}'}$$

$$\left[\hat{a}_r\left(\vec{k},t\right), \hat{a}_s\left(\vec{k}',t\right)\right] = 0$$

5.4.2 Field Hamiltonian

The field energy can be written as,

$$\hat{H} = V\int\frac{d^3\vec{k}}{(2\pi)^3}\sum_j\left\{\frac{\hat{p}_j^+\left(\vec{k},t\right)\hat{p}_j\left(\vec{k},t\right)}{2} + \omega_k^2\,\frac{\hat{q}_j^+\left(\vec{k},t\right)q_j\left(\vec{k},t\right)}{2}\right\}$$

and in terms of the creation and destruction operator it becomes,

$$\begin{aligned}
\hat{H} &= V\int\frac{d^3\vec{k}}{(2\pi)^3}\sum_j\left\{\frac{\hat{p}_j^+\left(\vec{k},t\right)\hat{p}_j\left(\vec{k},t\right)}{2} + \omega_k^2\,\frac{\hat{q}_j^+\left(\vec{k},t\right)q_j\left(\vec{k},t\right)}{2}\right\} \\
&= V\int\frac{d^3\vec{k}}{(2\pi)^3}\sum_j\frac{\hbar\omega_k}{4}\left\{\left(\hat{a}_j^+\left(\vec{k},t\right) - \hat{a}_j\left(-\vec{k},t\right)\right)\left(\hat{a}_j\left(\vec{k},t\right) - \hat{a}_j^+\left(-\vec{k},t\right)\right)\right. \\
&\qquad\qquad \left. + \left(\hat{a}_j^+\left(\vec{k},t\right) + \hat{a}_j\left(-\vec{k},t\right)\right)\left(\hat{a}_j\left(\vec{k},t\right) + \hat{a}_j^+\left(-\vec{k},t\right)\right)\right\} \\
&= V\int\frac{d^3\vec{k}}{(2\pi)^3}\sum_j\frac{\hbar\omega_k}{2}\left[\hat{a}_j^+\left(\vec{k},t\right)\hat{a}_j\left(\vec{k},t\right) + \hat{a}_j\left(\vec{k},t\right)\hat{a}_j^+\left(\vec{k},t\right)\right]
\end{aligned}$$

$$\hat{H} = V\int\frac{d^3\vec{k}}{(2\pi)^3}\sum_j\hbar\omega_k\left[\hat{a}_j^+\left(\vec{k},t\right)\hat{a}_j\left(\vec{k},t\right) + \frac{1}{2}\right]$$

The vacuum energy is,

$$V\int\frac{d^3\vec{k}}{(2\pi)^3}\sum_j\frac{\hbar\omega_k}{2} = V\int\frac{d^3\vec{k}}{(2\pi)^3}\;\hbar\omega_k$$

The vacuum energy density is therefore,

$$\int\frac{d^3\vec{k}}{(2\pi)^3}\hbar\omega_k$$

5.4.3 Time Development of Creation and Destruction Operators

The time development of the creation and destruction operators follows from the Heisenberg equation,

$$i\hbar\frac{d\hat{a}_r(\vec{k},t)}{dt} = \left[\hat{a}_r(\vec{k},t),\hat{H}\right] = \hbar\omega_k\hat{a}_r(\vec{k},t)$$
$$\Rightarrow \hat{a}_r(\vec{k},t) = e^{-i\omega_k t}\ \hat{a}_r(\vec{k},t=0) = e^{-i\omega_k t}\ \hat{a}_r(\vec{k})$$

and,

$$i\hbar\frac{d\hat{a}_r^+(\vec{k},t)}{dt} = \left[\hat{a}_r^+(\vec{k},t),\hat{H}\right] = -\hbar\omega_k\hat{a}_r^+(\vec{k},t)$$
$$\Rightarrow \hat{a}_r^+(\vec{k},t) = e^{i\omega_k t}\ \hat{a}_r^+(\vec{k},t=0) = e^{i\omega_k t}\ \hat{a}_r^+(\vec{k})$$

5.4.4 Field Operators

The field operator $\hat{\vec{A}}(\vec{r},t)$ is,

$$\hat{\vec{A}}(\vec{r},t) = V\int\frac{d^3\vec{k}}{(2\pi)^3}\sum_j\sqrt{\frac{\hbar}{2\omega_k\varepsilon_o}}\left[\hat{a}_j(\vec{k},t) + \hat{a}_j^+(-\vec{k},t)\right]\frac{e^{i\,\vec{k}.\vec{r}}}{\sqrt{V}}\hat{\varepsilon}_j(\vec{k})$$

and, of course, the field operator is Hermitian, $\hat{\vec{A}}^+(\vec{r},t) = \hat{\vec{A}}(\vec{r},t)$. The operators for the electric and magnetic fields are,

$$\hat{\vec{E}}(\vec{r},t) = -\frac{\partial\hat{\vec{A}}(\vec{r},t)}{\partial t} = V\int\frac{d^3\vec{k}}{(2\pi)^3}\sum_j i\sqrt{\frac{\hbar\omega_k}{2\varepsilon_o}}\left[\hat{a}_j(\vec{k},t) - \hat{a}_j^+(-\vec{k},t)\right]\frac{e^{i\,\vec{k}.\vec{r}}}{\sqrt{V}}\hat{\varepsilon}_j(\vec{k})$$
$$\hat{\vec{H}}(\vec{r},t) = \frac{\nabla\times\hat{\vec{A}}(\vec{r},t)}{\mu_o} = V\int\frac{d^3\vec{k}}{(2\pi)^3}\sum_j\frac{1}{\mu_o}\sqrt{\frac{\hbar}{2\omega_k\varepsilon_o}}\left[\hat{a}_j(\vec{k},t) + \hat{a}_j^+(-\vec{k},t)\right]\frac{e^{i\,\vec{k}.\vec{r}}}{\sqrt{V}}\left[i\vec{k}\times\hat{\varepsilon}_j(\vec{k})\right]$$

5.4.5 Energy Eigenstates and Photons

The energy eigenstate with m photons in mode $(\vec{k},j)$ is $|m\rangle_{\vec{k},j}$, so that,

$$\hat{H}\,|m\rangle_{\vec{k},j} = \hbar\omega_k\left(m+\frac{1}{2}\right)|m\rangle_{\vec{k},j}$$

5.4.6 Momentum of a Photon

The field momentum operator is,

$$\hat{\vec{P}} = \varepsilon_o\mu_o\int\ \hat{\vec{E}}(\vec{r},t)\times\hat{\vec{H}}(\vec{r},t)\,d^3\vec{r}$$

Using the eigenmode expansions for the field operators one gets,

$$\hat{\vec{P}} = V\int\frac{d^3\vec{k}}{\left(2\pi^3\right)}\sum_j\frac{\hbar\vec{k}}{2}\left[\left[\hat{a}_j(\vec{k},t) - \hat{a}_j^{\,+}(-\vec{k},t)\right]\left[\hat{a}_j(-\vec{k},t) + \hat{a}_j^+(\vec{k},t)\right]\right]$$

The only non-zero terms are,

$$\hat{\vec{P}} = V\int \frac{d^3\vec{k}}{(2\pi^3)} \sum_j \frac{\hbar\vec{k}}{2}\left[\hat{a}_j(\vec{k},t)\hat{a}_j^+(\vec{k},t) - \hat{a}_j^+(-\vec{k},t)\hat{a}_j(-\vec{k},t)\right]$$

$$= V\int \frac{d^3\vec{k}}{(2\pi^3)} \sum_j \frac{\hbar\vec{k}}{2}\left[\hat{a}_j(\vec{k},t)\hat{a}_j^+(\vec{k},t) + \hat{a}_j^+(\vec{k},t)\hat{a}_j(\vec{k},t)\right]$$

$$= V\int \frac{d^3\vec{k}}{(2\pi^3)} \sum_j \hbar\vec{k}\left[\hat{a}_j^+(\vec{k},t)\hat{a}_j(\vec{k},t) + \frac{1}{2}\right]$$

$$= V\int \frac{d^3\vec{k}}{(2\pi^3)} \sum_j \hbar\vec{k}\left[\hat{a}_j^+(\vec{k},t)\hat{a}_j(\vec{k},t)\right]$$

In the Schrodinger picture,

$$\hat{\vec{P}} = V\int \frac{d^3\vec{k}}{(2\pi^3)} \sum_j \hbar\vec{k}\left[\hat{a}_j^+(\vec{k})\hat{a}_j(\vec{k})\right]$$

Now we can check if the energy eigenstates (i.e. the photon number states) are also momentum eigenstates. We need to evaluate,

$$\hat{\vec{P}}\,|m\rangle_{\vec{k},j}$$

The result is,

$$\hat{\vec{P}}\,|m\rangle_{\vec{k},j} = m\,\hbar\vec{k}\,|m\rangle_{\vec{k},j}$$

Therefore, the momentum associated with a single photon in the radiation mode with wavevector $\vec{k}$ is $\hbar\vec{k}$.

5.4.7 Angular Momentum and Spin of a Photon

The classical expression for the angular momentum of electromagnetic field is,

$$\vec{J} = \varepsilon_o\mu_o \int d^3\vec{r}\;\vec{r}\times\left[\vec{E}(\vec{r},t)\times\vec{H}(\vec{r},t)\right]$$

The above expression consists of two contributions,
a) The orbital angular momentum
b) The intrinsic angular momentum or the spin angular momentum

These two parts are neither easily separable nor separately conserved. The reason is that spin angular momentum for a massive particle, say an electron, is clearly defined by looking at the angular momentum in the rest frame of the particle. Massless particles, like photons, have no rest frame and this procedure does not work. Nevertheless, if one looks at radiation states, like plane waves (with ideal infinite phase fronts), that are unlikely going to have any orbital component, then the angular momentum found using the expression above can tell us something about the intrinsic angular momentum of radiation. To see this more clearly we proceed as follows. We assume that the fields are transverse (i.e. divergence free). We start from the classical expression for the angular momentum above and perform a few manipulations,

$$\vec{J} = \varepsilon_o\mu_o \int d^3\vec{r}\;\vec{r}\times\left[\vec{E}(\vec{r},t)\times\vec{H}(\vec{r},t)\right]$$

$$= \varepsilon_o \int d^3\vec{r}\;\vec{r}\times\left[\vec{E}(\vec{r},t)\times\left(\nabla\times\vec{A}(\vec{r},t)\right)\right]$$

$$= \varepsilon_o \int d^3\vec{r}\;\;\vec{E}(\vec{r},t)\left[\vec{r}\,.\left(\nabla\times\vec{A}(\vec{r},t)\right)\right] - \varepsilon_o \int d^3\vec{r}\;\left[\vec{r}\,.\,\vec{E}(\vec{r},t)\right]\nabla\times\vec{A}(\vec{r},t)$$

$$= \varepsilon_o \int d^3\vec{r}\;\;\vec{E}(\vec{r},t)\left[\vec{r}\,.\left(\nabla\times\vec{A}(\vec{r},t)\right)\right] + \varepsilon_o \int d^3\vec{r}\;\nabla\left[\vec{r}\,.\,\vec{E}(\vec{r},t)\right]\times\vec{A}(\vec{r},t)$$

$$= \varepsilon_o \int d^3\vec{r}\;\;\vec{E}(\vec{r},t)\times\vec{A}(\vec{r},t) + \varepsilon_o \int d^3\vec{r}\;\sum_s E_s(\vec{r},t)\left[\vec{r}\times\nabla\right]A_s(\vec{r},t)$$

The first part is usually identified with the intrinsic or the spin angular momentum and the second part with the orbital angular momentum. It is tempting to write,

$$\vec{J} = \vec{S} + \vec{L}$$

However, as mentioned above, this decomposition runs into problems. For plane waves the second part can be shown to be zero. So for plane waves we will take the expression for the intrinsic angular momentum to be,

$$\vec{S} = \varepsilon_o \int d^3\vec{r} \;\; \vec{E}(\vec{r},t) \times \vec{A}(\vec{r},t)$$

Upon quantization, the fields become operators given by,

$$\hat{\vec{A}}(\vec{r},t) = V \int \frac{d^3\vec{k}}{(2\pi)^3} \sum_j \sqrt{\frac{\hbar}{2\omega_k \varepsilon_o}} \left[\hat{a}_j(\vec{k},t) + \hat{a}_j^+(-\vec{k},t) \right] \frac{e^{i\vec{k}.\vec{r}}}{\sqrt{V}} \hat{\varepsilon}_j(\vec{k})$$

$$\hat{\vec{E}}(\vec{r},t) = -\frac{\partial \hat{\vec{A}}(\vec{r},t)}{\partial t} = V \int \frac{d^3\vec{k}}{(2\pi)^3} \sum_j i \sqrt{\frac{\hbar\omega_k}{2\varepsilon_o}} \left[\hat{a}_j(\vec{k},t) - \hat{a}_j^+(-\vec{k},t) \right] \frac{e^{i\vec{k}.\vec{r}}}{\sqrt{V}} \hat{\varepsilon}_j(\vec{k})$$

Using the above expressions, the operator $\hat{\vec{S}}$ becomes,

$$\hat{\vec{S}} = V \int \frac{d^3\vec{k}}{(2\pi)^3} \sum_{j,\ell} \frac{i\hbar}{2} \hat{\varepsilon}_j(\vec{k}) \times \hat{\varepsilon}_\ell(-\vec{k}) \left[\hat{a}_j(\vec{k},t) - \hat{a}_j^+(-\vec{k},t) \right] \left[\hat{a}_\ell(-\vec{k},t) + \hat{a}_\ell^+(\vec{k},t) \right]$$

Using a convenient convention for the unit polarization vectors,

$$\hat{\varepsilon}_j(-\vec{k}) = \hat{\varepsilon}_j(\vec{k})$$

$$\hat{\varepsilon}_1(\vec{k}) \times \hat{\varepsilon}_2(\vec{k}) = \hat{k}$$

we obtain,

$$\hat{\vec{S}} = V \int \frac{d^3\vec{k}}{(2\pi)^3} \; i\hbar \hat{k} \left[\hat{a}_2^+(\vec{k},t)\hat{a}_1(\vec{k},t) - \hat{a}_1^+(\vec{k},t)\hat{a}_2(\vec{k},t) \right]$$

In the Schrodinger picture,

$$\hat{\vec{S}} = V \int \frac{d^3\vec{k}}{(2\pi)^3} \; i\hbar \hat{k} \left[\hat{a}_2^+(\vec{k})\hat{a}_1(\vec{k}) - \hat{a}_1^+(\vec{k})\hat{a}_2(\vec{k}) \right]$$

The operator $\hat{\vec{S}}$ is diagonal in the wavevector indices but not diagonal in the polarization indices. Recall that the operator $\hat{a}_1^+(\vec{k})$ creates a single photon with wavevector $\vec{k}$ and with linear polarization in the direction of the unit vector $\hat{\varepsilon}_1(\vec{k})$. Similarly, $\hat{a}_2^+(\vec{k})$ creates a single photon with wavevector $\vec{k}$ and with linear polarization in the direction of the unit vector $\hat{\varepsilon}_2(\vec{k})$. This means that a photon in a linearly polarized plane wave state is not in an eigenstates of $\hat{\vec{S}}$. We define two new destruction (and corresponding adjoint creation) operators as follows,

$$\hat{a}_R(\vec{k},t) = \frac{\hat{a}_1(\vec{k},t) - i\hat{a}_2(\vec{k},t)}{\sqrt{2}} \qquad \hat{a}_R^+(\vec{k},t) = \frac{\hat{a}_1^+(\vec{k},t) + i\hat{a}_2^+(\vec{k},t)}{\sqrt{2}}$$

$$\hat{a}_L(\vec{k},t) = \frac{\hat{a}_1(\vec{k},t) + i\hat{a}_2(\vec{k},t)}{\sqrt{2}} \qquad \hat{a}_L^+(\vec{k},t) = \frac{\hat{a}_1^+(\vec{k},t) - i\hat{a}_2^+(\vec{k},t)}{\sqrt{2}}$$

Note the following commutation relations,

$$\left[\hat{a}_R\left(\vec{k},t\right),\hat{a}_R^+\left(\vec{k}',t\right)\right]=\delta_{\vec{k},\vec{k}'}\qquad\left[\hat{a}_L\left(\vec{k},t\right),\hat{a}_L^+\left(\vec{k}',t\right)\right]=\delta_{\vec{k},\vec{k}'}$$

$$\left[\hat{a}_R\left(\vec{k},t\right),\hat{a}_L^+\left(\vec{k}',t\right)\right]=\left[\hat{a}_L\left(\vec{k},t\right),\hat{a}_R^+\left(\vec{k}',t\right)\right]=0$$

The operator $\hat{a}_R^+\left(\vec{k}\right)$ on the creates a single photon with wavevector $\vec{k}$ in a linear superposition of polarization vectors $\hat{\varepsilon}_1\left(\vec{k}\right)$ and $\hat{\varepsilon}_2\left(\vec{k}\right)$ with a 90-degrees phase difference between the two polarizations. In fact, $\hat{a}_R^+\left(\vec{k}\right)$ creates a single photon that is right-hand circularly polarized. Similarly, $\hat{a}_L^+\left(\vec{k}\right)$ creates a single photon that is left-hand circularly polarized. In terms of these new operators, the expression for $\hat{\vec{S}}$ becomes,

$$\hat{\vec{S}}=V\int\frac{d^3\vec{k}}{(2\pi)^3}\,\hbar\,\hat{k}\left[\hat{a}_R^+\left(\vec{k},t\right)\hat{a}_R\left(\vec{k},t\right)-\hat{a}_L^+\left(\vec{k},t\right)\hat{a}_L\left(\vec{k},t\right)\right]$$

In the Schrodinger picture,

$$\hat{\vec{S}}=V\int\frac{d^3\vec{k}}{(2\pi)^3}\,\hbar\,\hat{k}\left[\hat{a}_R^+\left(\vec{k}\right)\hat{a}_R\left(\vec{k}\right)-\hat{a}_L^+\left(\vec{k}\right)\hat{a}_L\left(\vec{k}\right)\right]$$

The operator $\hat{\vec{S}}$ is now diagonal in all indices. Consider a single photon state $|\psi\rangle=\hat{a}_R^+\left(\vec{k}\right)|0\rangle$. It is eigenstate of the operator $\hat{\vec{S}}$ with eigenvalue $+\hbar\hat{k}$,

$$\hat{\vec{S}}\,|\psi\rangle=+\hbar\hat{k}\,|\psi\rangle$$

We say that the angular momentum of a right-circularly polarized single photon plane wave state is $+\hbar\hat{k}$. Similarly, the state $|\psi\rangle=\hat{a}_L^+\left(\vec{k}\right)|0\rangle$ is an eigenstate of $\hat{\vec{S}}$ with eigenvalue $-\hbar\hat{k}$ and therefore the angular momentum of a left-circularly polarized single photon plane wave state is $-\hbar\hat{k}$. The direction of the angular momentum is always along the axis of propagation. The spin of a particle is related to its intrinsic angular momentum and is measured in units of $\hbar$. Therefore, the spin of a photon can have two values, +1 or -1, and these correspond to photons with right-circular or left-circular polarization states.

5.4.8 Position of a Photon

For photons position is not an observable, and there is no position operator for photons, and there are no position eigenstates. Any attempt to define position eigenstates for photons ends up violating Lorentz invariance. In fact, the same difficulty arises in the case of other massive particles, such as electrons, but for a massive particle one can always find a rest frame in which the particle is at rest (or moving very slowly compared to the speed of light) and then one can define approximate position eigenstates and the position operator in a non-relativistic setting. But photons, being massless, are always travelling at the speed of light and therefore no position operator can be rigorously defined. However, one may define the position of a photon in a measurement sense and ask the following question, "If there is a photon in an eigenmode $\vec{U}_m\left(\vec{r}\right)$ of the radiation, what is the probability of detecting the photon at location $\vec{r}$ at time t in a time interval Δt when a photo detector is placed at $\vec{r}$." We will discuss these questions in more detail when we discuss photon detection in later Chapters. Here we present a discussion of what goes wrong when trying to localize a photon in the detection sense.

Consider the single photon state,

$$|1\rangle_{\vec{k},1}=\hat{a}_1^+\left(\vec{k}\right)|0\rangle$$

The photon is created in a "plane wave state" and is therefore spread out in space. What if we create a photon in a superposition of plane wave states to make it more localized? With this as the motivation, consider the following state,

$$\psi_b^+(\vec{r}')|0\rangle = V\int \frac{d^3\vec{k}'}{(2\pi)^3} \sum_{j=1}^{2} \left[\hat{\varepsilon}_j(\vec{k}').\hat{e}_b\right] \frac{e^{-i\vec{k}'.\vec{r}'}}{\sqrt{V}}\, \hat{a}_j^+(\vec{k}')\,|0\rangle$$

The operator $\psi_b^+(\vec{r}')$ creates a single photon in a **maximal** superposition of plane wave states and with a polarization in the direction of the Cartesian unit vector $\hat{e}_b$. The question arises how localized is the photon at the location $\vec{r}'$. If it is really localized at $\vec{r}'$ then we could perhaps write a position eigenstate for the photon as,

$$|\vec{r}'\rangle_b = \psi_b^+(\vec{r}')|0\rangle$$

One way to answer the above question is to try destroying the photon at a different location $\vec{r}$ and in polarization along $\hat{e}_a$. Physically, this action corresponds to detecting a photon at location $\vec{r}$ and with polarization along $\hat{e}_a$, given the state $\psi_b^+(\vec{r}')|0\rangle$. So we evaluate the following matrix element,

$$\langle 0|\psi_a(\vec{r})\psi_b^+(\vec{r}')|0\rangle = {}_a\langle \vec{r}\,|\,\vec{r}'\rangle_b$$

If the photon created by $\psi_b^+(\vec{r}')$ is localized, and position eigenstates exist for photons, then we should expect the answer to look like,

$$\langle 0|\psi_a(\vec{r})\psi_b^+(\vec{r}')|0\rangle = {}_a\langle \vec{r}\,|\,\vec{r}'\rangle_b = \hat{e}_a.\hat{e}_b\;\delta^3(\vec{r}-\vec{r}') = \delta_{ab}\delta^3(\vec{r}-\vec{r}')$$

We evaluate this matrix element below,

$$
\begin{aligned}
&\langle 0|\psi_a(\vec{r})\psi_b^+(\vec{r}')|0\rangle \\
&= \sum_{\vec{k}}\sum_{\vec{k}'}\sum_{j=1}^{2}\sum_{\ell=1}^{2}\left[\hat{e}_a.\hat{\varepsilon}_j(\vec{k})\right]\frac{e^{i\vec{k}\cdot\vec{r}}}{\sqrt{V}}\left[\hat{\varepsilon}_\ell(\vec{k}').\hat{e}_b\right]\frac{e^{-i\vec{k}'\cdot\vec{r}'}}{\sqrt{V}}\langle 0|\hat{a}_j(\vec{k})\hat{a}_\ell^+(\vec{k}')|0\rangle \\
&= \sum_{\vec{k}}\sum_{\vec{k}'}\sum_{j=1}^{2}\sum_{\ell=1}^{2}\left[\hat{e}_a.\hat{\varepsilon}_j(\vec{k})\right]\frac{e^{i\vec{k}\cdot\vec{r}}}{\sqrt{V}}\left[\hat{\varepsilon}_\ell(\vec{k}').\hat{e}_b\right]\frac{e^{-i\vec{k}'\cdot\vec{r}'}}{\sqrt{V}}\langle 0|\left[\hat{a}_j(\vec{k}),\hat{a}_\ell^+(\vec{k}')\right]+\hat{a}_\ell^+(\vec{k}')\hat{a}_j(\vec{k})|0\rangle \\
&= \sum_{\vec{k}}\sum_{\vec{k}'}\sum_{j=1}^{2}\sum_{\ell=1}^{2}\left[\hat{e}_a.\hat{\varepsilon}_j(\vec{k})\right]\frac{e^{i\vec{k}\cdot\vec{r}}}{\sqrt{V}}\left[\hat{\varepsilon}_\ell(\vec{k}').\hat{e}_b\right]\frac{e^{-i\vec{k}'\cdot\vec{r}'}}{\sqrt{V}}\langle 0|\delta_{j\ell}\delta_{\vec{k},\vec{k}'}+\hat{a}_\ell^+(\vec{k}')\hat{a}_j(\vec{k})|0\rangle \\
&= \sum_{\vec{k}}\sum_{j=1}^{2}\left[\hat{e}_a.\hat{\varepsilon}_j(\vec{k})\right]\frac{e^{i\vec{k}\cdot\vec{r}}}{\sqrt{V}}\left[\hat{\varepsilon}_j(\vec{k}).\hat{e}_b\right]\frac{e^{-i\vec{k}\cdot\vec{r}'}}{\sqrt{V}} \\
&= \int\frac{d^3\vec{k}}{(2\pi)^3}\left(\delta_{ab}-\frac{k_a k_b}{k^2}\right)\frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}}{V} \\
&= \delta_{ab}^{\perp}(\vec{r}-\vec{r}')
\end{aligned}
$$

The answer $\delta_{ab}^{\perp}(\vec{r}-\vec{r}')$ is called the transverse delta function. It is not localized like a delta function but is spread out. One can write,

$$\delta_{ab}^{\perp}(\vec{r}-\vec{r}') = \delta_{ab}\delta^3(\vec{r}-\vec{r}') + \frac{1}{4\pi|\vec{r}-\vec{r}'|^3}\left[3\frac{(\vec{r}-\vec{r}).\hat{e}_a\;(\vec{r}-\vec{r}).\hat{e}_b}{|\vec{r}-\vec{r}'|^2}-\delta_{ab}\right]$$

The reason we did not just get $\delta_{ab}\delta^3(\vec{r}-\vec{r}')$, which would correspond to our expectation for position eigenkets, is that the eigenmodes are not scalars but vectors. One can superpose plane waves and produce a delta function but vector plane waves cannot be superposed to produce a delta function. This argument shows that photons cannot be localized.

5.4.9 Field Commutation Relations

The equal-time commutation relations among the fields express the possibility of simultaneous measurements. Consider first the electric and magnetic fields,

$$\hat{\vec{E}}(\vec{r},t) = -\frac{\partial \hat{\vec{A}}(\vec{r},t)}{\partial t} = V\int \frac{d^3\vec{k}}{(2\pi)^3} \sum_j i\sqrt{\frac{\hbar\omega_k}{2\varepsilon_o}}\left[\hat{a}_j(\vec{k},t) - \hat{a}_j^+(-\vec{k},t)\right]\frac{e^{i\vec{k}.\vec{r}}}{\sqrt{V}}\hat{\varepsilon}_j(\vec{k})$$

$$\hat{\vec{H}}(\vec{r},t) = \frac{\nabla\times\hat{\vec{A}}(\vec{r},t)}{\mu_o} = V\int \frac{d^3\vec{k}}{(2\pi)^3} \sum_j \frac{1}{\mu_o}\sqrt{\frac{\hbar}{2\omega_k\varepsilon_o}}\left[\hat{a}_j(\vec{k},t) + \hat{a}_j^+(-\vec{k},t)\right]\frac{e^{i\vec{k}.\vec{r}}}{\sqrt{V}}\left[i\vec{k}\times\hat{\varepsilon}_j(\vec{k})\right]$$

Let,

$$\hat{E}_a(\vec{r},t) = \hat{\vec{E}}(\vec{r},t).\hat{e}_a$$

$$\hat{H}_a(\vec{r},t) = \hat{\vec{H}}(\vec{r},t).\hat{e}_b$$

The equal-time commutation relations between these two components are,

$$\left[\hat{E}_a(\vec{r},t),\hat{H}_b(\vec{r}',t)\right] = i\hbar c^2\int\frac{d^3\vec{k}}{(2\pi)^3}e^{i\vec{k}.(\vec{r}-\vec{r}')}\sum_{j=1,2}\left[\hat{\varepsilon}_j(\vec{k}).\hat{e}_a\right]\left[\left(-i\vec{k}\times\hat{\varepsilon}_j(-\vec{k})\right).\hat{e}_b\right]$$

Noting that for any two vectors, the cross-product can be written as,

$$\left(\vec{A}\times\vec{B}\right)_a = \sum_{b,c}\varepsilon_{abc}A_bB_c$$

where ε_{abc} is the Levi-Civita symbol with the property that $\varepsilon_{123}=1$ and ε_{abc} picks a negative for any permutation of the indices (e.g. $\varepsilon_{321}=-1$ and $\varepsilon_{231}=+1$), one can write,

$$\sum_{j=1,2}\left[\hat{\varepsilon}_j(\vec{k}).\hat{e}_a\right]\left[\left(-i\vec{k}\times\hat{\varepsilon}_j(-\vec{k})\right).\hat{e}_b\right] = -i\sum_c\varepsilon_{abc}k_c$$

And,

$$\left[\hat{E}_a(\vec{r},t),\hat{H}_b(\vec{r}',t)\right] = -i\hbar c^2\sum_c\varepsilon_{abc}\partial_c\int\frac{d^3\vec{k}}{(2\pi)^3}e^{i\vec{k}.(\vec{r}-\vec{r}')}$$

$$= -i\hbar c^2\sum_c\varepsilon_{abc}\partial_c\,\delta^3(\vec{r}-\vec{r}')$$

The result is a derivative of a delta function. The symbol ∂_c stands for derivative with respect to the "c" Cartesian component ("c" could be x, y, or z). The result shows that same components of the electric and magnetic fields commute at all points but different components of the electric and magnetic fields do not commute.

Other interesting quantities are the field commutation relations at different locations and at different times,

$$\left[\hat{E}_a(\vec{r},t),\hat{E}_b(\vec{r}',t')\right]$$

Such commutation relation show whether accurate simultaneous measurements on fields at different locations and times are possible. Of course, one would expect in light of relativity that if field measurements are made at locations far enough such that no signal could travel in the time interval between the measurements then such measurements should not affect each other. In other words, all field operators must commute for space-like intervals (i.e. when $\left|\vec{r}-\vec{r}\right|^2 > c^2\left|t-t'\right|^2$). Using the expression for the electric field operator we get,

$$\left[\hat{E}_a(\vec{r},t),\hat{E}_b(\vec{r}',t')\right]$$
$$= \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}.(\vec{r}-\vec{r}')}\left[e^{-i\omega_k(t-t')} - e^{i\omega_k(t-t')}\right]\sum_{j=1,2}\left[\hat{\varepsilon}_j(\vec{k}).\hat{e}_a\right]\left[\hat{\varepsilon}_j(-\vec{k}).\hat{e}_b\right]$$
$$= \int \frac{d^3\vec{k}}{(2\pi)^3}\left(\frac{\hbar\omega_k}{2\varepsilon_o}\right)e^{i\vec{k}.(\vec{r}-\vec{r}')}\left[e^{-i\omega_k(t-t')} - e^{i\omega_k(t-t')}\right]\left(\delta_{ab} - \frac{k_a k_b}{k^2}\right)$$
$$= \int \frac{d^3\vec{k}}{(2\pi)^3}\left(\frac{\hbar\omega_k}{2\varepsilon_o}\right)\left[e^{i\vec{k}.(\vec{r}-\vec{r}')}e^{-i\omega_k(t-t')} - e^{-i\vec{k}.(\vec{r}-\vec{r}')}e^{i\omega_k(t-t')}\right]\left(\delta_{ab} - \frac{k_a k_b}{k^2}\right)$$
$$= \int \frac{d^3\vec{k}}{(2\pi)^3}\left(\frac{\hbar}{2\varepsilon_o\omega_k}\right)\left(\omega_k^2\delta_{ab} - c^2 k_a k_b\right)\left[e^{i\vec{k}.(\vec{r}-\vec{r}')}e^{-i\omega_k(t-t')} - e^{-i\vec{k}.(\vec{r}-\vec{r}')}e^{i\omega_k(t-t')}\right]$$
$$= \left(-\delta_{ab}\frac{\partial^2}{\partial t^2} + c^2\partial_a\partial_b\right)\int \frac{d^3\vec{k}}{(2\pi)^3}\left(\frac{\hbar}{2\varepsilon_o\omega_k}\right)\left[e^{i\vec{k}.(\vec{r}-\vec{r}')}e^{-i\omega_k(t-t')} - c.c\right]$$

To evaluate the above expression we need to evaluate the imaginary part of the following expression,

$$\int \frac{d^3\vec{k}}{(2\pi)^3}\left(\frac{\hbar}{2\varepsilon_o\omega_k}\right)e^{i\vec{k}.(\vec{r}-\vec{r}')}e^{-i\omega_k(t-t')}$$
$$= \frac{\hbar}{2\varepsilon_o c}\frac{1}{4\pi^2}\int_0^\infty k\,dk\int_0^\pi \sin\theta\, d\theta\, e^{ik|\vec{r}-\vec{r}'|\cos\theta}e^{-i\omega_k(t-t')}$$
$$= \frac{-i\hbar}{2\varepsilon_o c|\vec{r}-\vec{r}'|}\frac{1}{4\pi^2}\int_0^\infty dk\left(e^{ik|\vec{r}-\vec{r}'|} - e^{ik|\vec{r}-\vec{r}'|}\right)e^{-i\omega_k(t-t')}$$
$$= \frac{-i\hbar}{2\varepsilon_o c|\vec{r}-\vec{r}'|}\frac{1}{4\pi^2}\lim_{\eta\to 0}\int_0^\infty dk e^{-k\eta}\left(e^{ik|\vec{r}-\vec{r}'|} - e^{ik|\vec{r}-\vec{r}'|}\right)e^{-i\omega_k(t-t')}$$
$$= \frac{\hbar}{2\varepsilon_o c|\vec{r}-\vec{r}'|}\frac{1}{4\pi^2}\lim_{\eta\to 0}\left[\frac{1}{|\vec{r}-\vec{r}'| - c(t-t') + i\eta} + \frac{1}{|\vec{r}-\vec{r}'| - c(t-t') - i\eta}\right]$$

The imaginary part of the above expression is,

$$\frac{\hbar}{2\varepsilon_o c|\vec{r}-\vec{r}'|}\frac{1}{4\pi}\left[-\delta\left(|\vec{r}-\vec{r}'| - c(t-t')\right) + \delta\left(|\vec{r}-\vec{r}'| + c(t-t')\right)\right]$$

Finally, the field commutator becomes,

$$\left[\hat{E}_a(\vec{r},t),\hat{E}_b(\vec{r}',t')\right]$$
$$= \left(\delta_{ab}\frac{\partial^2}{\partial t^2} - c^2\partial_a\partial_b\right)\frac{i\hbar}{4\pi\varepsilon_o c|\vec{r}-\vec{r}'|}\left[\delta\left(|\vec{r}-\vec{r}'| - c(t-t')\right) - \delta\left(|\vec{r}-\vec{r}'| + c(t-t')\right)\right]$$

The above expression shows that the commutator is non-zero only on the light cone (i.e. when $|\vec{r}-\vec{r}| = c|t-t'|$) and therefore the fields commute for space-like intervals.