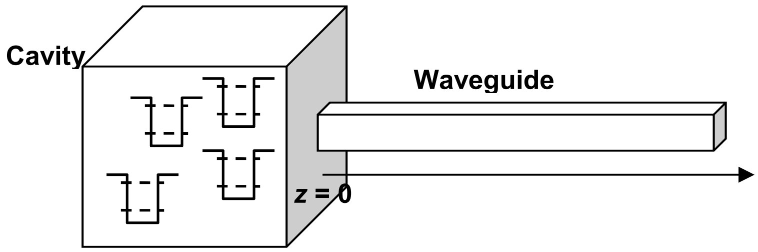
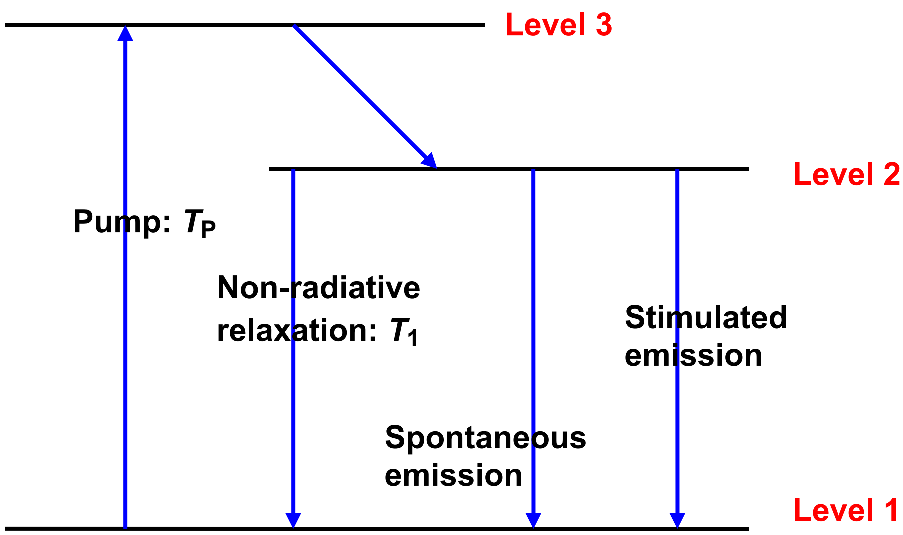
Chapter 13: Quantum Optics of Lasers

13.1 Introduction

Consider a N two-level system interacting with a single mode of radiation in a cavity, as shown below.



In this Chapter, we will discuss the operation of a laser composed of the above two-level systems interacting with a single mode of radiation in a cavity. We assume that the two-level systems are "pumped" into a population inverted state using a third level, as shown below.



All electrons pumped form the lowest level into the third level relax down very quickly into the second level. Including the effects of cavity photon loss, population relaxation, decoherence, and pumping one can write the following set of equations,

$$\frac{d\hat{N}_2(t)}{dt} = \frac{\hat{N}_1(t)}{T_p} - \frac{\hat{N}_2(t)}{T_1} - \frac{i}{\hbar}\left[k\hat{\sigma}_+(t)\hat{a}(t) - k^*\hat{a}^+(t)\hat{\sigma}_-(t)\right] - \hat{F}_N(t) + \hat{F}_P(t)$$

$$\frac{d\hat{N}_1(t)}{dt} = -\frac{\hat{N}_1(t)}{T_p} + \frac{\hat{N}_2(t)}{T_1} + \frac{i}{\hbar}\left[k\hat{\sigma}_+(t)\hat{a}(t) - k^*\hat{a}^+(t)\hat{\sigma}_-(t)\right] + \hat{F}_N(t) - \hat{F}_P(t)$$

$$\frac{d\hat{\sigma}_+(t)}{dt} = \left(i\frac{\Delta\varepsilon}{\hbar} - \frac{1}{T_2} \right)\hat{\sigma}_+(t) - \frac{i}{\hbar}k^*\,\hat{a}^+(t)\left[\hat{N}_2(t) - \hat{N}_1(t)\right] + \hat{F}_+(t)e^{i\omega_o t}$$

$$\frac{d\hat{\sigma}_-(t)}{dt} = \left(-i\frac{\Delta\varepsilon}{\hbar} - \frac{1}{T_2} \right)\hat{\sigma}_-(t) + \frac{i}{\hbar}k\left[\hat{N}_2(t) - \hat{N}_1(t)\right]\hat{a}(t) + \hat{F}_-(t)e^{-i\omega_o t}$$

$$\frac{d\hat{a}(t)}{dt} = \left(-i\omega_o - \frac{1}{2\tau_p} \right)\hat{a}(t) - \frac{i}{\hbar}k^*\hat{\sigma}_-(t) + \sqrt{\frac{1}{\tau_p}}\,\hat{S}_{in}(t)e^{-i\omega_o t}$$

$$\frac{d\hat{a}^+(t)}{dt} = \left(i\omega_o - \frac{1}{2\tau_p} \right)\hat{a}^+(t) + \frac{i}{\hbar}k\,\hat{\sigma}_+(t) + \sqrt{\frac{1}{\tau_p}}\,\hat{S}^+_{in}(t)e^{i\omega_o t}$$

Here, the input vacuum fluctuation operators are,

$$\hat{S}_{in}(t)e^{-i\omega_o t} = \sqrt{v_g}\,\hat{b}_L(z=0,t)e^{-i\omega_o t}$$

$$\hat{S}^+_{in}(t)e^{i\omega_o t} = \sqrt{v_g}\,\hat{b}^+_L(z=0,t)e^{i\omega_o t}$$

The photons coming out of the cavity are described by the equations,

$$\hat{S}_{out}(t)e^{-i\omega_o t} = \sqrt{v_g}\,\hat{b}_R(z=0,t)e^{-i\omega_o t}$$

$$= \sqrt{\frac{1}{\tau_p}}\,\hat{a}(t) - \sqrt{v_g}\,\hat{b}_L(z=0,t)e^{-i\omega_o t} = \sqrt{\frac{1}{\tau_p}}\,\hat{a}(t) - \hat{S}_{in}(t)e^{-i\omega_o t}$$

$$\hat{S}^+_{out}(t)e^{i\omega_o t} = \sqrt{v_g}\,\hat{b}^+_R(z=0,t)e^{i\omega_o t}$$

$$= \sqrt{\frac{1}{\tau_p}}\,\hat{a}^+(t) - \sqrt{v_g}\,\hat{b}^+_L(z=0,t)e^{i\omega_o t} = \sqrt{\frac{1}{\tau_p}}\,\hat{a}^+(t) - \hat{S}^+_{in}(t)e^{i\omega_o t}$$

As discussed in an earlier Chapter, in the limit of strong decoherence one can integrate out the equations for the operators $\hat{\sigma}_+(t)$ and $\hat{\sigma}_-(t)$ and obtain the following operator rate equations,

$$\frac{d\hat{N}_2(t)}{dt} = \frac{\hat{N}_1(t)}{T_p} - \frac{\hat{N}_2(t)}{T_1} - 2g_d\left[\hat{N}_2(t) - \hat{N}_1(t)\right]\hat{a}^+(t)\hat{a}(t) - \hat{F}_N(t) + \hat{F}_P(t)$$
$$- \left\{ e^{-i\omega_o t}\,\hat{a}^+(t)\hat{F}_{sp}(t) + e^{i\omega_o t}\,\hat{F}^+_{sp}(t)\hat{a}(t) \right\}$$

$$\frac{d\hat{N}_1(t)}{dt} = -\frac{\hat{N}_1(t)}{T_p} + \frac{\hat{N}_2(t)}{T_1} + 2g_d\left[\hat{N}_2(t) - \hat{N}_1(t)\right]\hat{a}^+(t)\hat{a}(t) + \hat{F}_N(t) - \hat{F}_P(t)$$
$$+ \left\{ e^{-i\omega_o t}\,\hat{a}^+(t)\hat{F}_{sp}(t) + e^{i\omega_o t}\,\hat{F}^+_{sp}(t)\hat{a}(t) \right\}$$

$$\frac{d\hat{a}(t)}{dt} = \left(-i\omega_o - \frac{1}{2\tau_p} \right)\hat{a}(t) + g_d\left[\hat{N}_2(t) - \hat{N}_1(t)\right]\hat{a}(t) + e^{-i\omega_o t}\,\hat{F}_{sp}(t) + \sqrt{\frac{1}{\tau_p}}\,\hat{S}_{in}(t)e^{-i\omega_o t}$$

$$\frac{d\hat{a}^+(t)}{dt} = \left(i\omega_o - \frac{1}{2\tau_p} \right)\hat{a}^+(t) + g_d\,\hat{a}^+(t)\left[\hat{N}_2(t) - \hat{N}_1(t)\right] + e^{i\omega_o t}\,\hat{F}^+_{sp}(t) + \sqrt{\frac{1}{\tau_p}}\,\hat{S}^+_{in}(t)e^{i\omega_o t}$$

$$\frac{d\hat{n}(t)}{dt} = -\frac{\hat{n}(t)}{\tau_p} + 2g_d\left[\hat{N}_2(t) - \hat{N}_1(t)\right]\hat{n}(t) + \left\{e^{-i\omega_o t}\,\hat{a}^+(t)\hat{F}_{sp}(t) + e^{i\omega_o t}\,\hat{F}_{sp}^+(t)\hat{a}(t)\right\}$$
$$+\sqrt{\frac{1}{\tau_p}}\left\{\hat{a}^+(t)\hat{S}_{in}(t)e^{-i\omega_o t} + e^{i\omega_o t}\hat{S}_{in}^+(t)\hat{a}(t)\right\}$$

where,

$$g_d = \text{differential gain} = \frac{|k|^2}{\hbar^2}\frac{1/T_2}{(\omega_o - \Delta\varepsilon/\hbar)^2 + (1/T_2)^2}$$

Suppose the average populations, photon number, and output photon flux are,

$$N_2(t) = \left\langle \hat{N}_2(t) \right\rangle$$
$$N_1(t) = \left\langle \hat{N}_1(t) \right\rangle$$
$$n(t) = \left\langle \hat{n}(t) \right\rangle$$
$$F_R(0,t) = \left\langle \hat{F}(0,t) \right\rangle = \left\langle \hat{S}_{out}^+(t)\hat{S}_{out}(t) \right\rangle = \frac{n(t)}{\tau_p}$$

The average populations and photon number obey the equations,

$$\frac{dN_2(t)}{dt} = \frac{N_1(t)}{T_p} - \frac{N_2(t)}{T_1} - 2g_d\,(N_2(t) - N_1(t))n(t) - 2g_g N_2(t) \tag{1}$$
$$\frac{dN_1(t)}{dt} = -\frac{N_1(t)}{T_p} + \frac{N_2(t)}{T_1} + 2g_d\,(N_2(t) - N_1(t))n(t) + 2g_d N_2(t) \tag{2}$$
$$\frac{dn(t)}{dt} = \left(2g_d\,(N_2(t) - N_1(t)) - \frac{1}{\tau_p}\right)n(t) + 2dN_2(t) \tag{3}$$

In (1) and (2), it is convenient to assume that the spontaneous emission term is absorbed in the definition of relaxation (i.e. $2g_d N_2$ is included in the term N_2/T_1). Define, $N_d(t) = N_2(t) - N_1(t)$, and since $N_2(t) + N_1(t) = N$, one can write,

$$N_2(t) = \frac{N + N_d(t)}{2} \qquad N_1(t) = \frac{N - N_d(t)}{2} \tag{4}$$

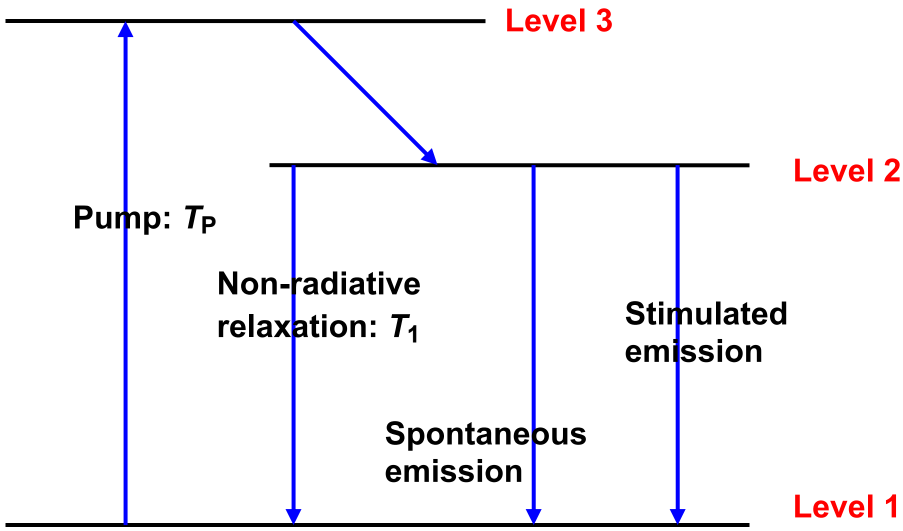
Subtract (2) from (1), to get,

$$\frac{dN_d(t)}{dt} = N\left(\frac{1}{T_p} - \frac{1}{T_1}\right) - N_d(t)\left(\frac{1}{T_p} + \frac{1}{T_1}\right) - 4g_d N_d(t)n(t) \tag{5}$$

so,

$$\frac{dn(t)}{dt} = \left(2g_d\,N_d(t) - \frac{1}{\tau_p}\right)n(t) + g_d\left[N + N_d(t)\right] \tag{6}$$

Equations (5) and (6) describe the operation of a laser.



In the steady state (5) and (6) give,

$$0 = N\left(\frac{1}{T_p} - \frac{1}{T_1}\right) - N_d\left(\frac{1}{T_p} + \frac{1}{T_1}\right) - 4g_d N_d n \qquad (5b)$$

$$0 = \left(2g_d N_d - \frac{1}{\tau_p}\right) n + g_d\left[N + N_d\right] \qquad (6b)$$

Of course, one can use a computer to solve (5b) and (6b), which form a system of coupled non-linear equations for the population difference and the photon number. To gain insight, we obtain approximate solutions in different regimes.

13.2 Regimes of Operation for a Laser

13.2.1 No Pumping ($1/T_p = 0$)

When the pumping rate is zero (i.e. $1/T_p = 0$), one obtains the steady state solution,

$$N_d = -N$$
$$\Rightarrow N_1 = N \qquad N_2 = 0$$
$$n = 0$$

When the pump is turned off, all electrons are sitting in the lower energy level and there are no photons inside the cavity.

13.2.2 Medium Offers Net Optical Loss ($0 < 1/T_p < 1/T_1$)

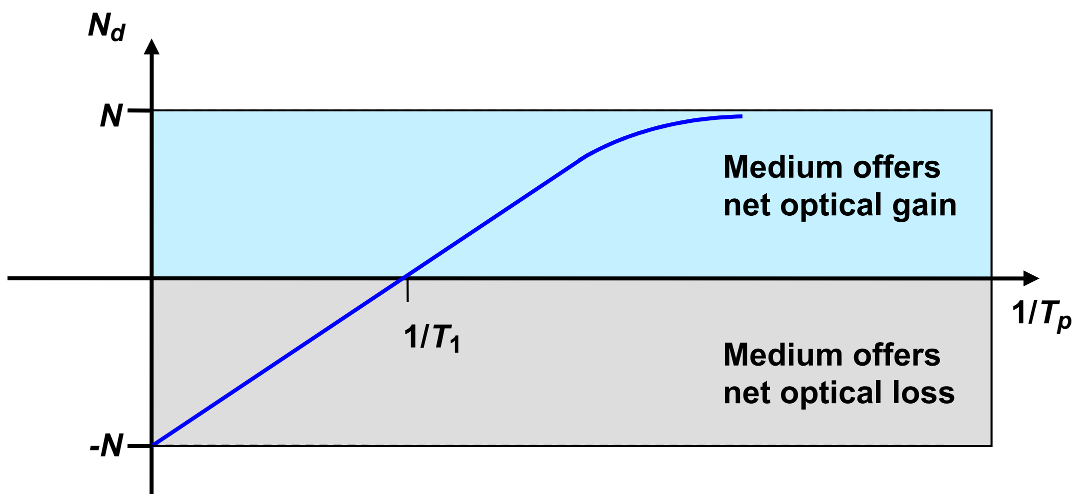
Suppose pumping rate is increased from zero. N_d will increase from $-N$ but will remain negative if $0 < 1/T_p < 1/T_1$. The photons in the cavity experience no net gain, but net loss (loss from stimulated absorption and from photons escaping from the cavity). The photon number in the steady state will be very small. Therefore, in this regime $(0 < 1/T_p < 1/T_1)$ we can solve (5b) and (6b) by first solving (5b) for N_d assuming $n \approx 0$, and then using the calculated value of N_d in (6b) to get n (and then verify that n is indeed small). In other words, the population behaves as if no photons are present. Since the photon number is small in this regime, stimulated emission and absorption rates are negligible compared to the pump rate and the relaxation rate and that is why it is justified to assume $n \approx 0$ in (5b). Equation (5b) gives,

$$N_d = N \frac{\left(\frac{1}{T_p} - \frac{1}{T_1}\right)}{\left(\frac{1}{T_p} + \frac{1}{T_1}\right) + 4g_d n} \approx N \frac{T_1 - T_p}{T_1 + T_p} \qquad (7)$$

Using the above result in (6b) gives,

$$n = \frac{g_d (N + N_d)}{\frac{1}{\tau_p} - 2g_d N_d} \quad \left\{ N_d \approx N \frac{T_1 - T_p}{T_1 + T_p} \right. \qquad (8)$$

N_d vs $1/T_p$ as obtained from Equation (7) is plotted below.



For all values of pumping rate $1/T_p$ less than the relxation rate $1/T_1$, N_d is negative. This means that the stimulated emission rate $(2g_d N_2 n)$ is less than the stimulated absorption rate $(2g_d N_1 n)$. The material offers net loss to the photons, and photons are lost from the cavity not just by escaping into the waveguide but also by stimulated absorption.

13.2.3 Medium is Transparent ($1/T_p = 1/T_1$)

When the pumping rate $1/T_P$ equals the relaxation rate $1/T_1$ we have,

$$N_d \approx N \frac{T_1 - T_p}{T_1 + T_p} \approx 0$$

which implies,

$$N_2 = N_1$$

Transparency is the condition when the stimulated emission rate $(2g_d N_2 n)$ is equal to the stimulated absorption rate $(2g_d N_1 n)$. A transparent medium offers no net optical gain and no net optical loss. However, the photons in the cavity will still see a net loss because photons can escape from the cavity. But at least in transparency the material is not contributing to the optical loss. The photon number n is still small, and so the approximation $n \approx 0$ in (7) remains valid.

13.2.4 Medium Offers Net Optical Gain ($1/T_p > 1/T_1$)

When the pumping rate $1/T_p$ exceeds the relaxation rate $1/T_1$, N_d becomes positive (population inversion). This means that the stimulated emission rate $(2g_d N_2 n)$ is greater than the stimulated absorption rate $(2g_d N_1 n)$, the material now provides net optical gain. Equation (6) shows that the net optical gain experienced by the photons in steady state equals,

$$\left(2g_d N_d - \frac{1}{\tau_p}\right)$$

As long as $2g_d N_d << 1/\tau_p$ the photons in the cavity experience net loss since the loss due to photons escaping from the cavity exceeds the gain from the population inverted medium. Consequently, the average number of photons n in the cavity is still small and therefore the approximation $n \approx 0$ in (7) remains valid.

As the pumping rate is increased further, N_d increases (as given by Equation (7)) and at some point N_d becomes so large that the optical gain from the medium approaches the optical loss due to photons escaping from the cavity, i.e.,

$$2g_d N_d \to \frac{1}{\tau_p}$$

Consequently, the number of photons in the cavity, as given earlier by Equation (8),

$$n = \frac{d(N + N_d)}{\frac{1}{\tau_p} - 2g_d N_d} \qquad (8)$$

increases dramatically since the denominator approaches zero. In physical terms, the net photon generation rate through stimulated processes (given by $2g_d N_d n$) is becoming equal to the photon loss rate due to photons escaping from the cavity (given by n/τ_p). Photons are multiplying at a rate nearly equal to the rate at which they are leaving the cavity. Consequently, a large population of photons builds up in the cavity in steady state. The relation,

$$n = \frac{d(N + N_d)}{\frac{1}{\tau_p} - 2g_d N_d}$$

suggests that the photon number will becomes ∞ if $2g_d N_d$ becomes equal to $1/\tau_p$, and even negative if $2g_d N_d$ were to exceed $1/\tau_p$. So it must be that $2g_d N_d$ should never exactly equal $1/\tau_p$ and must always be less than $1/\tau_p$. How is this guaranteed? We go back to Equation (7),

$$N_d = N \frac{\left(\frac{1}{T_p} - \frac{1}{T_1}\right)}{\left(\frac{1}{T_p} + \frac{1}{T_1}\right) + 4g_d n} \approx N \frac{T_1 - T_p}{T_1 + T_p} \qquad (7)$$

When the photon number in the cavity is large the second approximate equality in the above expression no longer remains valid. A large photon number will ensure in steady state the value of N_d, for a given pumping rate, is much smaller than the value predicted by the approximate relation,

$$N_d \approx N \frac{T_1 - T_p}{T_1 + T_p}$$

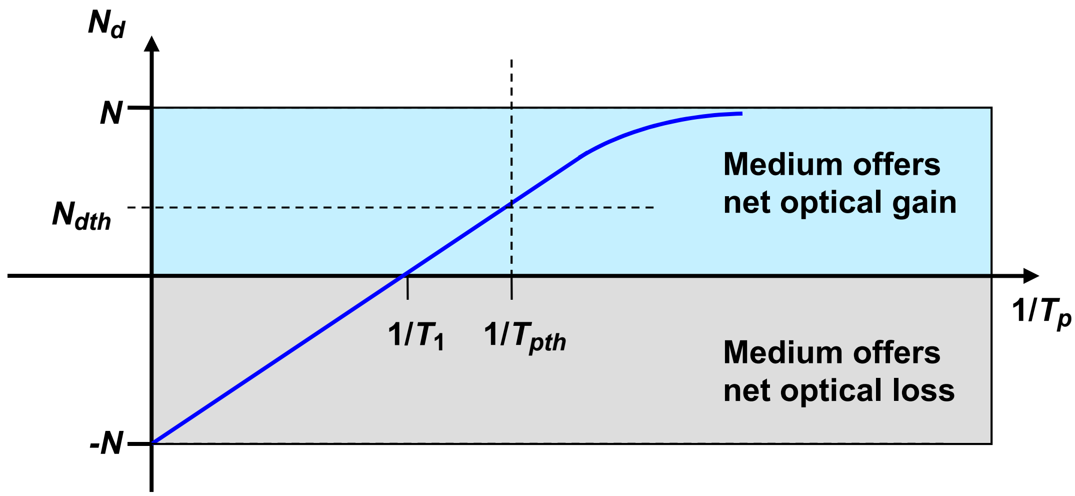
The value of the pumping rate for which the above approximate relation (falsely) predicts that the material gain $2g_d N_d$ will equal cavity photon loss $1/\tau_p$ is called the threshold pumping rate,

$$2g_d N_d \approx 2g_d N \frac{T_1 - T_{pth}}{T_1 + T_{pth}} = \frac{1}{\tau_p}$$

$$\Rightarrow \frac{1}{T_{pth}} = \frac{1}{T_1} \frac{(2g_d N\tau_p + 1)}{(2g_d N\tau_p - 1)}$$

The population difference value needed to make the material gain equal to the cavity loss is called the threshold population difference,

$$2g_d N_{dth} = \frac{1}{\tau_p}$$



The question then is what happens when the pumping rate $1/T_p$ approaches $1/T_{pth}$ and when $1/T_p$ exceeds $1/T_{pth}$. This we address next.

13.2.5 Threshold and Lasing ($1/T_p > 1/T_{pth}$)

We know that when $1/T_p$ approaches $1/T_{pth}$, $2g_d N_d$ approaches $1/\tau_p$, and consequently n becomes very large. A large photon number n means a large stimulated emission rate (given by $2g_d N_d n$). A large stimulated emission rate means that the population in the upper level is getting depleted at a large rate. Consequently, we cannot ignore the term that has n in the Equation,

$$N_d = N \frac{\left(\frac{1}{T_p} - \frac{1}{T_1}\right)}{\left(\frac{1}{T_p} + \frac{1}{T_1}\right) + 4g_d n}$$

Suppose $1/T_p$ is equal to or larger than $1/T_{pth}$, and the material gain $2g_d N_d n$ is close to $1/\tau_p$. If $1/T_p$ is now increased then the photon number increases so much that the accompanying increased stimulated emission rate keeps the population difference N_d from changing much at all. The net result is that when the pumping rate $1/T_p$ is increased beyond $1/T_{pth}$, the population difference does not change at all and only the photon number increases, and the increase in the photon number is such as to keep the value of population difference N_d from changing much. This pinning of the population difference N_d, and the material gain $2g_d N_d n$, when the pumping rate $1/T_p$ rate is equal to or larger than the threshold pumping rate $1/T_{pth}$, is called gain saturation. The rapid increase in the photon number (when the pumping rate $1/T_p$ is larger than the threshold pumping rate $1/T_{pth}$) is called lasing.

Approximate Model for Lasing: When the pumping rate $1/T_p$ exceeds the threshold pumping rate $1/T_{pth}$, the population difference does not change much and remains pinned at values close to the threshold value $N_{d\,th}$,

$$N_d \approx N_{dth} \qquad \left\{ 2g_d N_{dth} = \frac{1}{\tau_p} \right.$$

Of course, the above relation is not exactly true since the material gain $2g_d N_d n$ must always be less than the cavity loss $1/\tau_p$. We need to find the cavity photon number when $1/T_p$ exceeds the threshold pumping rate $1/T_{pth}$. We should not use Equation (8) (or Equation (6b)),

$$n = \frac{d(N + N_d)}{\frac{1}{\tau_p} - 2g_d N_d}$$

since our assumption of $N_d \approx N_{dth}$ will give an infinite value for n. Instead, we use Equation (5b) to determine n above threshold,

$$0 = N\left(\frac{1}{T_p} - \frac{1}{T_1}\right) - N_d\left(\frac{1}{T_p} + \frac{1}{T_1}\right) - 4g_d N_d n \qquad (5b)$$

We can replace N_d in the above equation by its approximate value N_{dth} (where $2g_d N_{d\,th} = 1/\tau_p$) to get,

$$0 = N\left(\frac{1}{T_p} - \frac{1}{T_1}\right) - \frac{1}{2g_d\tau_p}\left(\frac{1}{T_p} + \frac{1}{T_1}\right) - 2\frac{n}{\tau_p}$$

$$0 = N\left(\frac{1}{T_p} - \frac{1}{T_1}\right) - \frac{1}{2g_d\tau_p}\left(\frac{1}{T_p} + \frac{1}{T_1}\right) - 2\frac{n}{\tau_p}$$

$$\Rightarrow n = \frac{(2g_d N\tau_p - 1)}{4g_d}\left(\frac{1}{T_p} - \frac{1}{T_{pth}}\right)$$

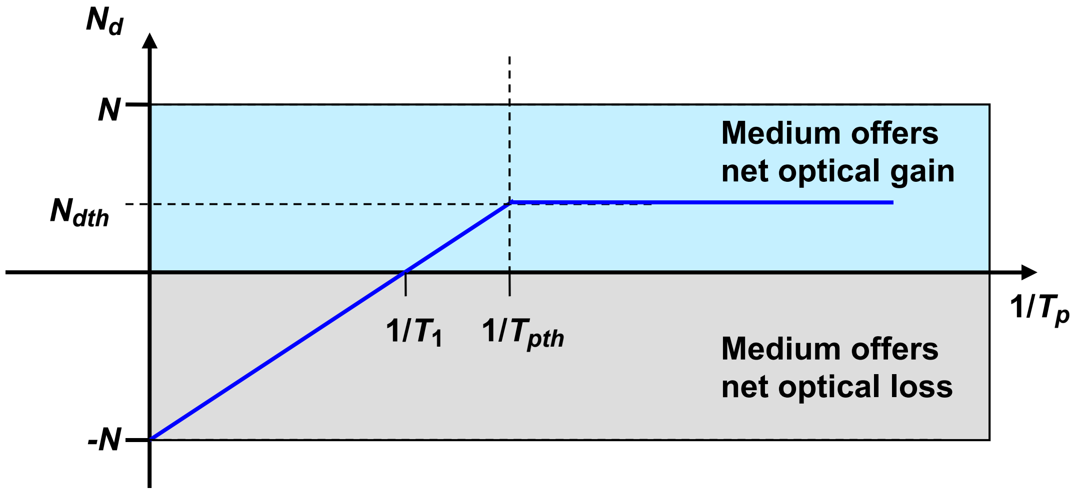
When $1/T_p$ exceeds the threshold pumping rate $1/T_{pth}$, n increases linearly with the pumping rate, and the population difference N_d remains fixed at a value close to N_{dth}.

13.2.6 Approximate model for $1/T_p < 1/T_{pth}$ (laser below threshold)

Below threshold, the population difference increases with the pumping rate. The photon number also increases with the pumping rate but remains relatively small.

$$N_d \approx N\frac{T_1 - T_p}{T_1 + T_p}$$

$$n = \frac{d(N+N_d)}{\frac{1}{\tau_p} - 2g_d N_d} \approx 0$$



13.2.7 Approximate model for $1/T_p > 1/T_{pth}$ (laser above threshold)

Above threshold the population difference is pinned at the threshold population difference value. The photon number increases linearly with the pumping rate.

$$N_d \approx N_{dth} \qquad \left\{ 2g_d N_{dth} = \frac{1}{\tau_p} \right.$$

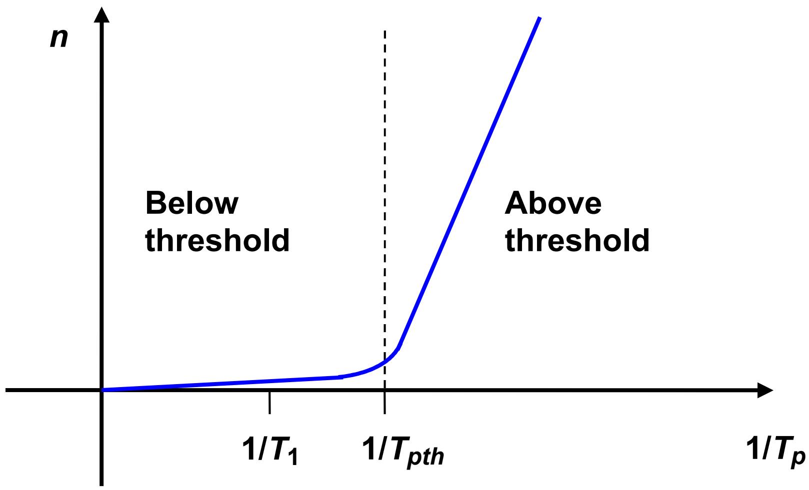
$$n = \frac{(2g_d N \tau_p - 1)}{4g_d}\left(\frac{1}{T_p} - \frac{1}{T_{pth}}\right)$$

Since,

$$N_d = N_2 - N_1$$
$$N = N_2 + N_1$$

above threshold the populations in both the upper and the lower level are also constant,

$$\Rightarrow \begin{cases} N_2 = \dfrac{N+N_{dth}}{2} = \dfrac{N}{2} + \dfrac{1}{4g_d\tau_p} \\ N_1 = \dfrac{N-N_{dth}}{2} = \dfrac{N}{2} - \dfrac{1}{4g_d\tau_p} \end{cases}$$



13.2.8 Output Photon Flux and Output Power

Below threshold, the output power is small and is due to amplified spontaneous emission and one can approximate the output power as being zero. Above threshold, the flux of photons leaving the cavity is,

$$\frac{n}{\tau_p} = \frac{(2g_d N \tau_p - 1)}{4 g_d \tau_p}\left(\frac{1}{T_p} - \frac{1}{T_{pth}}\right)$$

Therefore, the output power above threshold is.

$$P = \hbar\omega_o \frac{n}{\tau_p} = \hbar\omega_o \frac{(2g_d N \tau_p - 1)}{4 g_d \tau_p}\left(\frac{1}{T_p} - \frac{1}{T_{pth}}\right)$$

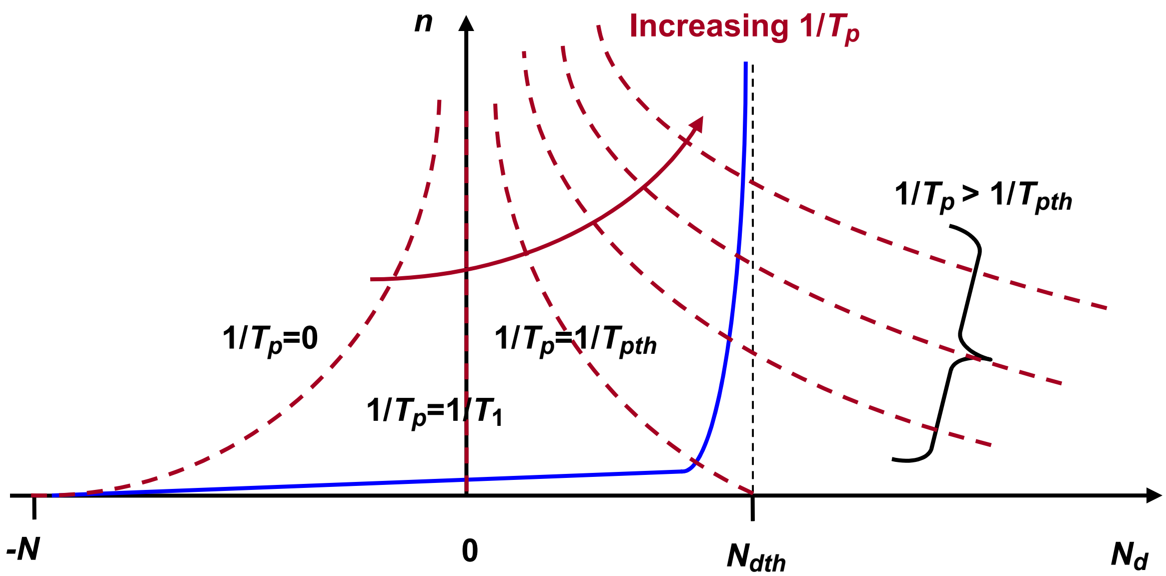
13.2.9 Exact Graphical Solution

The steady state Equations (5b) and (6b),

$$0 = N\left(\frac{1}{T_p} - \frac{1}{T_1}\right) - N_d\left(\frac{1}{T_p} + \frac{1}{T_1}\right) - 4 g_d N_d n \qquad (5b)$$

$$0 = \left(2 g_d N_d - \frac{1}{\tau_p}\right) n + d[N + N_d] \qquad (6b)$$

can be solved graphically to obtain solutions for both below and above threshold operation. The Figure below shows the graphical solution of Equations (5b) and (6b). Equation (5b) is plotted (dashed) for different values of the pump rate $1/T_p$. Equation (6b) is plotted (solid) as a single line. It can be seen that for pumping rates below the threshold pumping rate $1/T_{pth}$, the population difference N_d increases with the pumping rate and the photon number remains very small. When the pumping rate exceeds the threshold pumping rate, the population difference N_d remains fixed at values close to N_{dth} and the photon number increases rapidly with the pumping rate.



13.2.10 Laser Stability above Threshold and Relaxation Oscillations

The steady state solution of the laser rate equations above threshold is,

$$N_d \approx N_{dth} \qquad \left\{ 2dN_{dth} = \frac{1}{\tau_p} \right.$$

$$n = \frac{\left(2g_d N\tau_p - 1\right)}{4g_d}\left(\frac{1}{T_p} - \frac{1}{T_{pth}}\right)$$

The questions is how stable are the above steady state values? In other words, if the populations or the photon number are perturbed from their steady state values do they return to their steady state values? To study the stability of the laser above threshold expand the population difference and photon number around their steady state values,

$$N_d(t) = N_d + \Delta N_d(t)$$

$$n(t) = n + \Delta n(t)$$

Plug the above expansions in the laser rate equations,

$$\frac{dN_d(t)}{dt} = N\left(\frac{1}{T_p} - \frac{1}{T_1}\right) - N_d(t)\left(\frac{1}{T_p} + \frac{1}{T_1}\right) - 4g_d N_d(t) n(t)$$

$$\frac{dn(t)}{dt} = \left(2g_d\, N_d(t) - \frac{1}{\tau_p}\right) n(t) + g_d\left[N + N_d(t)\right]$$

Since $\Delta n(t)$ and $\Delta N_d(t)$ are small perturbations around the steady state, one can ignore terms that are of second order in the perturbations (i.e linearize the equations) to get,

$$\frac{d\Delta N_d(t)}{dt} = -\Delta N_d(t)\left[\frac{1}{T_p}+\frac{1}{T_1}+4g_d n\right] - \overbrace{4g_d N_d}^{\frac{2}{\tau_p}\text{ above threshold}} \Delta n(t)$$

$$\frac{d\Delta n(t)}{dt} = \Delta N_d(t)\left[\underbrace{2g_d\left(n+\frac{1}{2}\right)}_{\approx 2dn}\right] + \left[\underbrace{2g_d N_d - \frac{1}{\tau_p}}_{\approx 0}\right]\Delta n(t)$$

$$\Rightarrow \frac{d}{dt}\begin{bmatrix}\Delta N_d(t)\\ \Delta n(t)\end{bmatrix} = \begin{bmatrix} -\left(\frac{1}{T_p}+\frac{1}{T_1}+4g_d n\right) & -\frac{2}{\tau_p} \\ 2g_d n & 0\end{bmatrix}\begin{bmatrix}\Delta N_d(t)\\ \Delta n(t)\end{bmatrix}$$

Let,

$$\frac{1}{\tau_r} = \text{differential relaxation time } = \frac{1}{T_p}+\frac{1}{T_1}+4g_d n$$

$$\Rightarrow \frac{d}{dt}\begin{bmatrix}\Delta N_d(t)\\ \Delta n(t)\end{bmatrix} = \begin{bmatrix} -\frac{1}{\tau_r} & -\frac{2}{\tau_p} \\ 2g_d n & 0\end{bmatrix}\begin{bmatrix}\Delta N_d(t)\\ \Delta n(t)\end{bmatrix}$$

Solution: The above coupled differential equations give,

$$\left[\frac{d^2}{dt^2}+\frac{1}{\tau_r}\frac{d}{dt}+\omega_r^2\right]\Delta N_d(t) = 0$$

$$\left[\frac{d^2}{dt^2}+\frac{1}{\tau_r}\frac{d}{dt}+\omega_r^2\right]\Delta n(t) = 0$$

Where,

$$\omega_r^2 = \frac{4g_d n}{\tau_p} = \text{relaxation oscillation frequency}$$

The second order differential equations show that the perturbations $\Delta Nd(t)$ and/or $\Delta n(t)$ are damped and the laser is stable above threshold against small perturbations. The general solution can be written as,

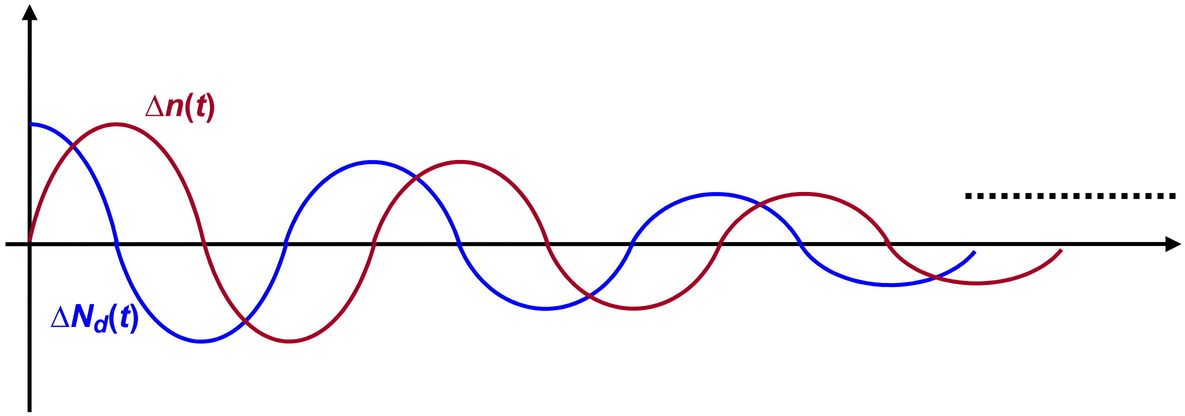
$$\Delta N_d(t) = A e^{-t/\tau_r}\sin(\Omega_r t) + B e^{-t/\tau_r}\cos(\Omega_r t)$$
$$\Delta n(t) = C e^{-t/\tau_r}\sin(\Omega_r t) + D e^{-t/\tau_r}\cos(\Omega_r t)$$

$$\text{where } \Omega_r = \sqrt{\omega_r^2 - \frac{1}{4\tau_r^2}}$$

The constants A,B,C and D are set by the initial conditions. Therefore if the population difference or the photon number are disturbed from their steady state values they return to their steady state values executing damped oscillations which are called relaxation oscillations. Since,

$$\frac{d\Delta n(t)}{dt} = 2g_d n\,\Delta N_d(t)$$

during relaxation oscillations photon number oscillates 90° out of phase with the population difference oscillations. The Figure below depicts the relaxation oscillations that occur when the population difference is suddenly increased from its steady state value (by momentarily increasing the pumping rate, for example).



13.3 Field Amplitude Equation for a Laser and Analogy with Second Order Phase Transitions

In this Section, we look at the field amplitude and phase dynamics in a laser. We suppose that,

$$\langle \hat{a}(t) \rangle = \beta(t) e^{-i\omega_o t}$$

and therefore expand the destruction operator as follows,

$$\hat{a}(t) = \beta(t) e^{-i\omega_o t} + \Delta\hat{a}(t) \qquad \left\{ \text{where } \langle \Delta\hat{a}(t) \rangle = 0 \right.$$

We use the above expansion in the equation,

$$\frac{d\hat{a}(t)}{dt} = \left(-i\omega_o - \frac{1}{2\tau_p} \right)\hat{a}(t) + g_d \left[\hat{N}_2(t) - \hat{N}_1(t) \right]\hat{a}(t) + e^{-i\omega_o t}\, \hat{F}_{sp}(t) + \sqrt{\frac{1}{\tau_p}}\, \hat{S}_{in}(t) e^{-i\omega_o t}$$

and take the average of the resulting equation to obtain,

$$\frac{d\beta(t)}{dt} = \left[g_d\, N_d(t) - \frac{1}{2\tau_p} \right]\beta(t) \qquad \left\{ \text{where } N_d(t) = \langle \hat{N}_2(t) - \hat{N}_1(t) \rangle \right.$$

Assuming $\langle \Delta\hat{a}^{+}(t)\Delta\hat{a}(t) \rangle << |\beta(t)|^2$, the average population difference satisfies,

$$\frac{dN_d(t)}{dt} = N\left(\frac{1}{T_p} - \frac{1}{T_1} \right) - N_d(t)\left(\frac{1}{T_p} + \frac{1}{T_1} \right) - 4 g_d N_d(t) |\beta(t)|^2$$

Suppose we are interested in field dynamics over time scales much longer than the population relaxation times. One can then obtain a simple relation between the average population difference $N_d(t)$ and $|\beta(t)|^2$ by setting $dN_d(t)/dt = 0$ in the above equation. The result is,

$$N_d(t) = N \frac{\left(\frac{1}{T_p} - \frac{1}{T_1} \right)}{\left(\frac{1}{T_p} + \frac{1}{T_1} \right) + 4 g_d |\beta(t)|^2}$$

The above equation can be used to eliminate $N_d(t)$ in the equation for $\beta(t)$ to get,

$$\frac{d\beta(t)}{dt} = \left[g_d N \frac{\left(\frac{1}{T_p} - \frac{1}{T_1}\right)}{\left(\frac{1}{T_p} + \frac{1}{T_1}\right) + 4g_d|\beta(t)|^2} - \frac{1}{2\tau_p} \right] \beta(t)$$

$\beta(t)$ is a complex field amplitude, with an amplitude and a phase, or equivalently, with real and imaginary parts. Let $\beta(t) = x_1(t) + i\, x_2(t)$, where $x_1(t)$ and $x_2(t)$ are real. We can also define a vector $\vec{r}(t)$ in the two dimensional $x_1 - x_2$ plane as,

$$\vec{r}(t) = x_1(t)\,\hat{x} + x_2(t)\,\hat{y}$$

and,

$$r^2(t) = \vec{r}(t)\cdot\vec{r}(t) = x_1^2(t) + x_2^2(t) = |\beta(t)|^2$$

Then the equation for $\beta(t)$ can be written as,

$$\frac{d\vec{r}(t)}{dt} = -\vec{\nabla} V(\vec{r}(t))$$

where,

$$V(\vec{r}) = \frac{r^2}{4\tau_p} - \frac{1}{2}\sqrt{\frac{d}{\frac{1}{T_p}+\frac{1}{T_1}}}\, N\left(\frac{1}{T_p} - \frac{1}{T_1}\right)\tan^{-1}\left[2r\sqrt{\frac{d}{\frac{1}{T_p}+\frac{1}{T_1}}}\right]$$

The dynamical equation, $d\vec{r}/dt = -\vec{\nabla} V(\vec{r})$, shows that the vector $\vec{r}(t)$ moves in the direction of the steepest descent with respect to the function $V(\vec{r})$ and in steady state $\vec{r}(t)$ will be at the point where the function $V(\vec{r})$ has a minimum. Below, we examine the minima of $V(\vec{r})$ as a function of the pumping rate $1/T_p$.

13.3.1 Laser below Threshold ($1/T_p < 1/T_{pth}$)

Recall that the threshold pumping rate is,

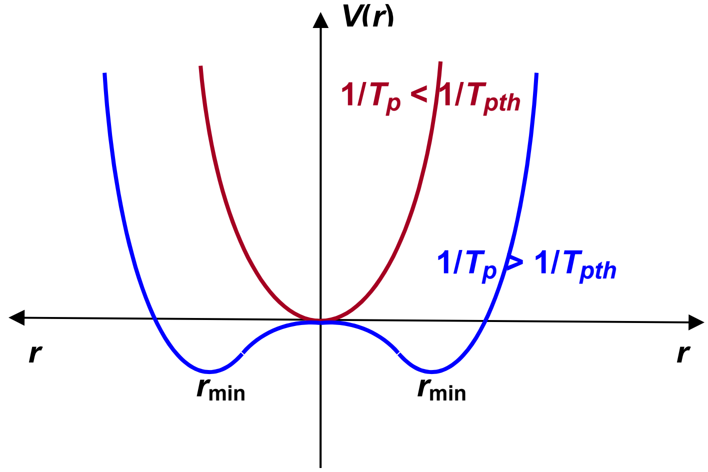
$$\frac{1}{T_{pth}} = \frac{1}{T_1}\frac{(2g_d N\tau_p + 1)}{(2g_d N\tau_p - 1)}$$

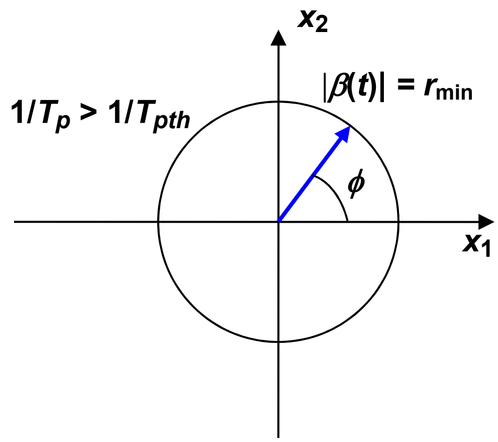
For $1/T_p < 1/T_{pth}$, the function $V(\vec{r})$, plotted below, has only a single minimum at $\vec{r} = 0$. Consequently, below threshold the steady state solution is,

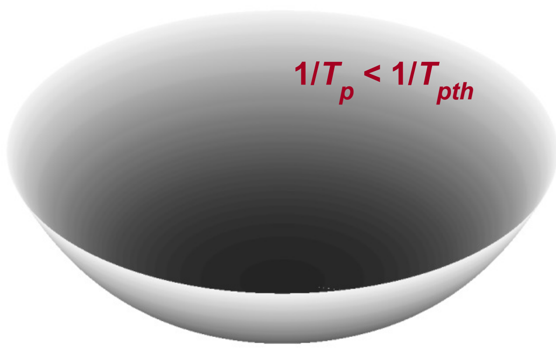
$$\vec{r}(t) = x_1(t)\,\hat{x} + x_2(t)\,\hat{y} = 0$$

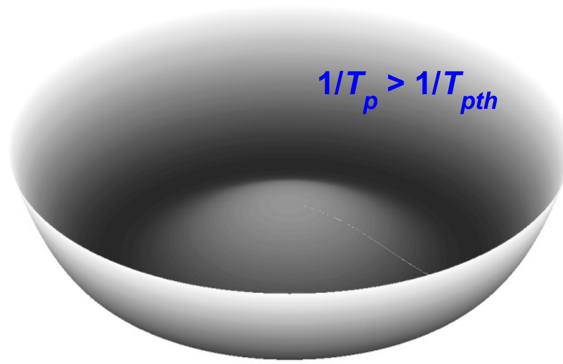
$$\Rightarrow \langle \hat{a}(t)\rangle = \beta(t) e^{-i\omega_o t} = 0$$

Note that although photons due to amplified spontaneous emission are present in the cavity, the average field amplitude is zero since the field has no well-defined phase.









13.3.2 Laser above Threshold ($1/T_p > 1/T_{pth}$)

For pumping rate above the threshold pumping rate, the function $V(\vec{r})$ has a minima for all points in the two dimensional $x_1 - x_2$ plane with values of r equal to $r_{\min}$ where,

$$r_{\min}^2 = \frac{(2g_d N\tau_p - 1)}{4g_d}\left(\frac{1}{T_p} - \frac{1}{T_{pth}}\right)$$

Therefore, in steady state,

$$|\beta(t)|^2 = r_{\min}^2 = \frac{(2g_d N\tau_p - 1)}{4g_d}\left(\frac{1}{T_p} - \frac{1}{T_{pth}}\right) = \text{steady state photon number} = n$$

Note that the function $V(\vec{r})$ is radially symmetric and only depends on the distance from the center (i.e. on $\vec{r}(t)\cdot\vec{r}(t)$ and not on the direction of $\vec{r}(t)$). The contour of minimum values of $V(\vec{r})$ form a circle in the two dimensional $x_1 - x_2$ of radius $r_{\min}$. Recall that,

$$\langle \hat{a}(t)\rangle = \beta(t)e^{-i\omega_o t}$$

Above threshold, the field operator has a non-zero average amplitude and a phase,

$$\beta(t) = x_1(t) + i\,x_2(t) = \sqrt{x_1^2(t) + x_2^2(t)}\;e^{i\tan^{-1}[x_2(t)/x_1(t)]}$$
$$= r(t)e^{i\phi(t)}$$

The laser equations uniquely determine the average amplitude of the field in steady state but not the average phase of the field. In fact, there is no preference for any particular value of the phase $\phi(t)$. Above threshold, the field acquires an average value of the phase that is quite arbitrary. In other words, there is spontaneous breakdown of phase symmetry above threshold. This is quite similar to the breakdown of spin up-down symmetry in ferromagnets below the magnetic phase transition temperature. Although the spin-spin interactions in a ferromagnet can have complete up-down symmetry, the ferromagnetic state below the transition temperature has all spins either pointing in the upward direction or in the downward direction and the actual up or down direction selected is quite arbitrary. Thus, the lasing transition has many features in common with second order phase transitions.

Since there is no restriction on the average phase $\phi(t)$ of the field above threshold, the smallest possible noise or perturbations can send it wandering on the circle shown in the Figure above. But the magnitude of $\beta(t)$ is more or less fixed and equals $\sqrt{n}$. We have already seen that any perturbation in the steady state value of n decays after executing relaxation oscillations.

13.4 Laser Phase Diffusion and the Schawlow-Townes Expression for the Laser Linewidth

13.4.1 Laser below Threshold ($1/T_p < 1/T_{pth}$)

Consider first a laser operating below threshold. The Heisenberg equation for the field operator is,

$$\frac{d\hat{a}(t)}{dt} = \left(-i\omega_o - \frac{1}{2\tau_p}\right)\hat{a}(t) + d\left[\hat{N}_2(t) - \hat{N}_1(t)\right]\hat{a}(t) + e^{-i\omega_o t}\,\hat{F}_{sp}(t) + \sqrt{\frac{1}{\tau_p}}\,\hat{S}_{in}(t)e^{-i\omega_o t}$$

Here, the Langevin operator $\hat{F}_{sp}(t)$ models the spontaneous emission noise and the Langevin operator $\hat{S}_{in}(t)$ models the vacuum fluctuations entering the cavity. If one is interested in phase fluctuations over long time scales (long compared to the population relaxation times) then to a very good approximation one can replace the operator $\hat{N}_2(t) - \hat{N}_1(t)$ by its average time independent value,

$$\left\langle \hat{N}_2(t) - \hat{N}_1(t)\right\rangle = N_d$$

to get,

$$\frac{d\hat{a}(t)}{dt} = \left(-i\omega_o + \left(g_d N_d - \frac{1}{2\tau_p}\right)\right)\hat{a}(t) + e^{-i\omega_o t}\,\hat{F}_{sp}(t) + \sqrt{\frac{1}{\tau_p}}\,\hat{S}_{in}(t)e^{-i\omega_o t}$$

Recall from an earlier Chapter that the spectrum of a quantum state of light is related to the Fourier transform of the first order coherence function,

$$S(\omega) = \int_{-\infty}^{\infty} d\tau\; g_1(t,t+\tau)\,e^{i\omega\tau}$$
$$= \int_{-\infty}^{\infty} d\tau\; \frac{\left\langle \hat{a}^+(t)\hat{a}(t+\tau)\right\rangle}{\sqrt{\left\langle \hat{a}^+(t)\hat{a}(t)\right\rangle \left\langle \hat{a}^+(t+\tau)\hat{a}(t+\tau)\right\rangle}}\,e^{i\omega\tau}$$

The Heisenberg-Langevin equation above can be solved to get,

$$S(\omega) = \frac{\Delta\omega}{(\omega-\omega_o)^2 + \left(\frac{\Delta\omega}{2}\right)^2}$$

where the FWHM linewidth of the spectrum is,

$$\Delta\omega = \frac{1}{\tau_p} - 2g_d\, N_d$$

In steady state, the difference between the photon loss rate and the stimulated emission rate is the spontaneous emission rate,

$$\left(\frac{1}{\tau_p} - 2g_d\, N_d\right) n = 2g_d N_2$$

Therefore,

$$\Delta\omega = \frac{1}{\tau_p} - 2g_d\, N_d = \frac{2g_d N_2}{n} = \text{spontaneous emission rate}$$

As the laser approaches threshold, and the gain begins to approach the loss, the linewidth narrows. Right before threshold, when,

$$2g_d\, N_d \approx \frac{1}{\tau_p}$$

using the definition of the spontaneous emission factor given earlier,

$$n_{sp} = \frac{N_2}{N_2 - N_1}$$

we can write,

$$\Delta\omega = \frac{1}{\tau_p} - 2g_d\, N_d = \frac{2g_d N_2}{n} = \frac{1}{\tau_p}\frac{n_{sp}}{n}$$

13.4.2 Laser above Threshold ($1/T_p > 1/T_{pth}$)

Below threshold, both the in-phase and the out-of-phase quadratures of the noise operators affect the laser linewidth. Above threshold, the amplitude fluctuations are stabilized as a result of gain saturation and the amplitude and phase dynamics are decoupled. Consequently, only the out-of-phase quadratures of the noise operators affect the linewidth. Out-of-phase noise perturbations cause the laser phase to diffuse in time and this phase diffusion determines the laser linewidth.

Consider a laser operating above threshold. Suppose at time t the field operator has an average value given by,

$$\langle \hat{a}(t) \rangle = \beta\, e^{-i\omega_o t} = |\beta|\, e^{i\phi - i\omega_o t}$$

where $|\beta|$ is the average field amplitude and ϕ is the average phase of the field. We would like to study the phase dynamics. The field operators obey the equations,

$$\frac{d\hat{a}(t)}{dt} = \left(-i\omega_o - \frac{1}{2\tau_p}\right)\hat{a}(t) + g_d\left[\hat{N}_2(t) - \hat{N}_1(t)\right]\hat{a}(t) + e^{-i\omega_o t}\,\hat{F}_{sp}(t) + \sqrt{\frac{1}{\tau_p}}\,\hat{S}_{in}(t)e^{-i\omega_o t} \quad (9)$$

$$\frac{d\hat{a}^{+}(t)}{dt} = \left(i\omega_o - \frac{1}{2\tau_p}\right)\hat{a}^{+}(t) + g_d\,\hat{a}^{+}(t)\left[\hat{N}_2(t) - \hat{N}_1(t)\right] + e^{i\omega_o t}\,\hat{F}^{+}_{sp}(t) + \sqrt{\frac{1}{\tau_p}}\,\hat{S}^{+}_{in}(t)e^{i\omega_o t} \quad (10)$$

Recall that one can expand the field operator in terms of quadrature operators that are along and perpendicular to the direction specified by the average phase value,

$$\hat{a}(t) = \left[\hat{x}_{\phi}(t) + i\hat{x}_{\phi+\pi/2}(t)\right] e^{i\phi - i\omega_o t}$$
$$= |\beta|\, e^{i\phi - i\omega_o t} + \left[\Delta\hat{x}_{\phi}(t) + i\Delta\hat{x}_{\phi+\pi/2}(t)\right] e^{i\phi - i\omega_o t}$$

Also recall that the phase fluctuation operator is,

$$\Delta\hat{\phi}(t) = \frac{\Delta\hat{x}_{\phi+\pi/2}(t)}{|\beta|}$$

In order to obtain an equation for the phase fluctuation operator we substitute the expansion of the field operator in terms of the quadrature fluctuation operators in Equations (9) and (10) and subtract the resulting equations. The result is,

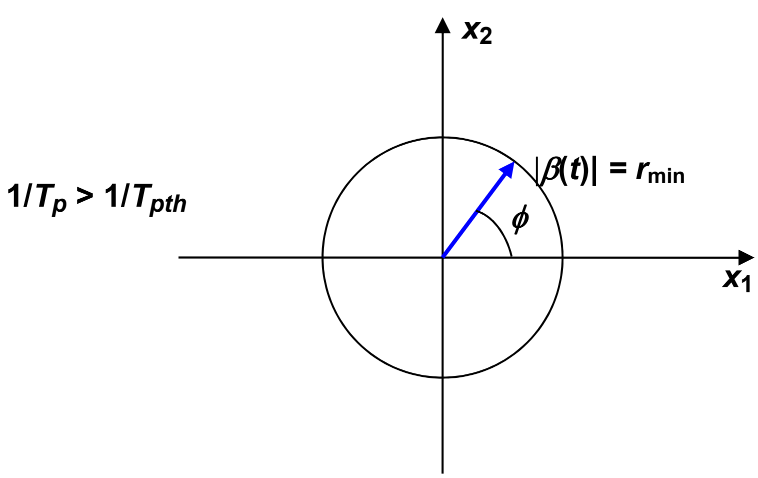
$$\frac{d\,\Delta\hat{\phi}(t)}{dt} = \left[g_d\left[\hat{N}_2(t) - \hat{N}_1(t)\right] - \frac{1}{2\tau_p}\right]\Delta\hat{\phi}(t) + \frac{1}{|\beta|}\left[\frac{\hat{F}_{sp}(t)e^{-i\phi} - \hat{F}_{sp}^{+}(t)e^{i\phi}}{2i}\right]$$
$$+ \frac{1}{|\beta|}\sqrt{\frac{1}{\tau_p}}\left[\frac{\hat{S}_{in}(t)e^{-i\phi} - \hat{S}_{in}^{+}(t)e^{i\phi}}{2i}\right]$$

Above threshold the average value of the population difference is fixed and equal to N_{dth}. Any deviations from this average value are quickly restored. Therefore, if one is interested in phase fluctuations over long time scales (long compared to the population relaxation times) then to a very good approximation one can replace the operator $\hat{N}_2(t) - \hat{N}_1(t)$ by its average value,

$$\left\langle \hat{N}_2(t) - \hat{N}_1(t)\right\rangle = N_d(t) \approx N_{dth} = \frac{1}{2g_d\tau_p}$$

to get a much simpler equation for the phase fluctuation operator,

$$\frac{d\,\Delta\hat{\phi}(t)}{dt} = \frac{1}{|\beta|}\left[\frac{\hat{F}_{sp}(t)e^{-i\phi} - \hat{F}_{sp}^{+}(t)e^{i\phi}}{2i}\right] + \frac{1}{|\beta|}\sqrt{\frac{1}{\tau_p}}\left[\frac{\hat{S}_{in}(t)e^{-i\phi} - \hat{S}_{in}^{+}(t)e^{i\phi}}{2i}\right]$$



The above equation shows that the phase is kicked around by noise just like a particle undergoing diffusive motion. There is no mechanism that restores the phase to a specific value. The noise driving the phase is due to spontaneous emission and also due to vacuum fluctuations entering the cavity from the waveguide. The role played by spontaneous emission can be understood in a semi-classical spirit as follows. Every spontaneously emitted photon has a random phase. Therefore, after every spontaneous emission event the average field phasor gets a random kick whose component tangent to the field phasor contributes to the phase noise. Under the action of the noise, the field phasor wanders randomly on the circle of radius r_{min} where,

$$|\beta|^2 = r_{min}^2 = n$$

The diffusion equation for the phase can be directly integrated to give,

$$\left\langle \left[\Delta\hat{\phi}(t) - \Delta\hat{\phi}(t')\right]^2 \right\rangle = \left[\frac{2g_d N_d (2n_{sp} - 1)}{4\,|\beta|^2} + \frac{1/\tau_p}{4\,|\beta|^2} \right] |t - t'|$$

$$\approx \left[\frac{2g_d N_{dth} (2n_{sp} - 1)}{4\,|\beta|^2} + \frac{1/\tau_p}{4\,|\beta|^2} \right] |t - t'| \qquad \left\{ N_d \approx N_{dth} \text{ above threshold} \right.$$

$$\approx \left[\frac{1/\tau_p\,(2n_{sp} - 1)}{4\,|\beta|^2} + \frac{1/\tau_p}{4\,|\beta|^2} \right] |t - t'| \qquad \left\{ 2g_d N_{dth} = \frac{1}{\tau_p} \text{ above threshold} \right.$$

$$= \frac{1}{2\tau_p} \left(\frac{n_{sp}}{n} \right) |t - t'|$$

Note that the laser phase diffusion has almost equal contributions from spontaneous emission and vacuum fluctuations.

We are now in a apposition to look at the laser linewidth,

$$S(\omega) = \int_{-\infty}^{\infty} d\tau\; g_1(t, t+\tau)\, e^{i\omega\tau}$$

$$= \int_{-\infty}^{\infty} d\tau\; \frac{\left\langle \hat{a}^+(t)\hat{a}(t+\tau) \right\rangle}{\sqrt{\left\langle \hat{a}^+(t)\hat{a}(t) \right\rangle \left\langle \hat{a}^+(t+\tau)\hat{a}(t+\tau) \right\rangle}}\, e^{i\omega\tau}$$

For a laser above threshold we have,

$$\sqrt{\left\langle \hat{a}^+(t)\hat{a}(t) \right\rangle \left\langle \hat{a}^+(t+\tau)\hat{a}(t+\tau) \right\rangle} \approx |\beta|^2$$

and,

$$\left\langle \hat{a}^+(t)\, \hat{a}(t+\tau) \right\rangle \approx |\beta|^2\, e^{-i\omega_o\tau} \left\langle e^{-i\,\Delta\hat{\phi}(t)}\, e^{i\,\Delta\hat{\phi}(t+\tau)} \right\rangle$$

$$= |\beta|^2 e^{-i\omega_o\tau}\, e^{-\frac{\left\langle \left[\Delta\hat{\phi}(t) - \Delta\hat{\phi}(t+\tau)\right]^2 \right\rangle}{2}}$$

$$= |\beta|^2 e^{-i\omega_o\tau}\, e^{-\frac{|\tau|}{4\tau_p}\left(\frac{n_{sp}}{n}\right)}$$

Therefore,

$$S(\omega)=\int_{-\infty}^{\infty} d\tau\ g_1(t,t+\tau)\,e^{i\omega\tau}$$

$$=\int_{-\infty}^{\infty} d\tau\ e^{-\frac{|\tau|}{4\tau_p}\left(\frac{n_{sp}}{n}\right)}\ e^{-i\omega_o\tau}e^{i\omega\tau}$$

$$=\frac{\Delta\omega}{(\omega-\omega_o)^2+\left(\frac{\Delta\omega}{2}\right)^2}$$

The frequency spectrum of the lasing mode is Lorentzian and the full-width at half-maximum (FWHM) laser linewidth $\Delta\omega$ equals,

$$\Delta\omega=\frac{1}{2\tau_p}\left(\frac{n_{sp}}{n}\right)$$

The above expression for the laser linewidth is called the Schawlow-Townes expression (named after the inventors of the laser). Note that the laser linewidth, just after threshold is one-half the laser linewidth just before threshold. All lasers have linewidths that are equal to or greater than the value given by the Schawlow-Townes expression. A large number of average photons in the cavity implies a smaller linewidth. When the number of photons in the cavity is large the phase kick resulting from the addition of each spontaneously emitted photon is small and, therefore, the laser linewidth is narrow. The Schawlow-Townes expression is more often written in terms of the output power P of the laser,

$$\Delta\omega=\frac{\hbar\omega_o}{2\tau_p^2}\left(\frac{n_{sp}}{P}\right)$$

The above relation shows that high power lasers with long cavity photons lifetimes have narrower linewidths.

13.5 Photon Number and Photon Flux Noise of a Laser

13.5.1 Photon Number Noise inside the Laser Cavity

We use the following three equations to find the photon number and photon flux noise of a laser operating above threshold,

$$\frac{d\hat{N}_2(t)}{dt}=\frac{\hat{N}_1(t)}{T_p}-\frac{\hat{N}_2(t)}{T_1}-2g_d\left[\hat{N}_2(t)-\hat{N}_1(t)\right]\hat{a}^+(t)\hat{a}(t)-\hat{F}_N(t)+\hat{F}_P(t)$$
$$-\left\{e^{-i\omega_o t}\,\hat{a}^+(t)\hat{F}_{sp}(t)+e^{i\omega_o t}\,\hat{F}_{sp}^+(t)\hat{a}(t)\right\}$$

$$\frac{d\hat{N}_1(t)}{dt}=-\frac{\hat{N}_1(t)}{T_p}+\frac{\hat{N}_2(t)}{T_1}+2g_d\left[\hat{N}_2(t)-\hat{N}_1(t)\right]\hat{a}^+(t)\hat{a}(t)+\hat{F}_N(t)-\hat{F}_P(t)$$
$$+\left\{e^{-i\omega_o t}\,\hat{a}^+(t)\hat{F}_{sp}(t)+e^{i\omega_o t}\,\hat{F}_{sp}^+(t)\hat{a}(t)\right\}$$

$$\frac{d\hat{n}(t)}{dt} = -\frac{\hat{n}(t)}{\tau_p} + 2g_d\left[\hat{N}_2(t) - \hat{N}_1(t)\right]\hat{n}(t) + \left\{e^{-i\omega_o t}\,\hat{a}^+(t)\hat{F}_{sp}(t) + e^{i\omega_o t}\,\hat{F}_{sp}^+(t)\hat{a}(t)\right\}$$
$$+\sqrt{\frac{1}{\tau_p}}\left\{\hat{a}^+(t)\hat{S}_{in}(t)e^{-i\omega_o t} + e^{i\omega_o t}\hat{S}_{in}^+(t)\hat{a}(t)\right\}$$

Recall that one can write,

$$e^{-i\omega_o t}\,\hat{a}^+(t)\hat{F}_{sp}(t) + e^{i\omega_o t}\,\hat{F}_{sp}^+(t)\hat{a}(t) = 2g_d\left\langle\hat{N}_2(t)\right\rangle + \hat{F}_n(t)$$

where $\hat{F}_n(t)$ is a zero-mean noise source that describes the shot noise associated with the optical transitions,

$$\left\langle\hat{F}_n(t)\hat{F}_n(t')\right\rangle = \left[2g_d N_2(n+1) + 2g_d N_1 n\right]\;\delta(t-t')$$

We also write,

$$\sqrt{\frac{1}{\tau_p}}\left\{\hat{a}^+(t)\hat{S}_{in}(t)e^{-i\omega_o t} + e^{i\omega_o t}\hat{S}_{in}^+(t)\hat{a}(t)\right\} = \hat{F}_v(t)$$

where,

$$\left\langle\hat{F}_v(t)\hat{F}_v(t')\right\rangle = \frac{n}{\tau_p}\delta(t-t')$$

The vacuum fluctuations thus describe the shot noise associated with photon loss from the cavity.

Consider a laser operating above threshold. We assume that,

$$\left.\begin{aligned}\hat{N}_2(t) &= N_2 + \Delta\hat{N}_2(t)\\ \hat{N}_1(t) &= N_1 + \Delta\hat{N}_1(t)\end{aligned}\right\} \quad\Rightarrow\quad \hat{N}_d(t) = N_d + \Delta\hat{N}_d(t)$$
$$\hat{n}(t) = n + \Delta\hat{n}(t)$$

We linearize the non-linear equations for the photon number and the population difference operators. The linearized equations for the fluctuations become,

$$\frac{d}{dt}\begin{bmatrix}\Delta\hat{N}_d(t)\\ \Delta\hat{n}(t)\end{bmatrix} = \begin{bmatrix}-\dfrac{1}{\tau_r} & -\dfrac{2}{\tau_p}\\ 2g_d n & 0\end{bmatrix}\begin{bmatrix}\Delta\hat{N}_d(t)\\ \Delta\hat{n}(t)\end{bmatrix} + \begin{bmatrix}-2\hat{F}_N(t) + 2\hat{F}_P(t) - 2\hat{F}_n(t)\\ \hat{F}_n(t) + \hat{F}_{v(t)}\end{bmatrix}$$

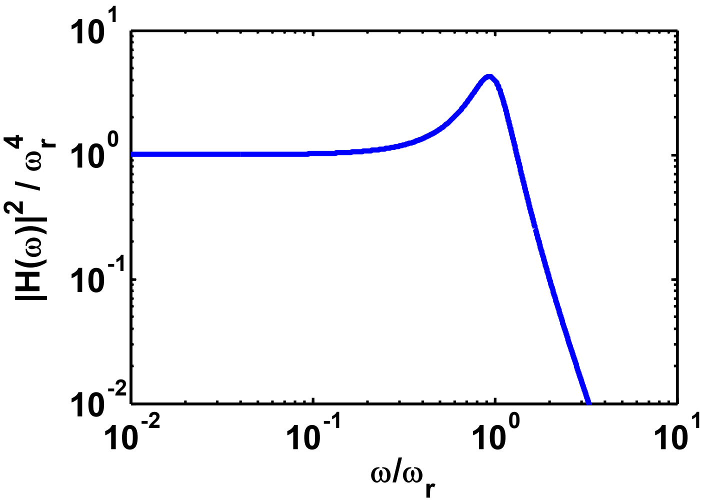
These equations are best solved in the Frequency domain. Upon Fourier transforming we get,

$$\begin{bmatrix}\Delta\hat{N}_d(\omega)\\ \Delta\hat{n}(\omega)\end{bmatrix} = \frac{H(\omega)}{\omega_r^2}\begin{bmatrix}-i\omega & -\dfrac{2}{\tau_p}\\ 2g_d n & -i\omega + \dfrac{1}{\tau_r}\end{bmatrix}\begin{bmatrix}-2\hat{F}_N(\omega) + 2\hat{F}_P(\omega) - 2\hat{F}_n(\omega)\\ \hat{F}_n(\omega) + \hat{F}_v(\omega)\end{bmatrix}$$

Here,

$$H(\omega) = \frac{\omega_r^2}{(\omega_r^2 - \omega^2) - i\,\omega/\tau_r} \qquad\qquad \omega_r^2 = \frac{4g_d n}{\tau_p} \qquad\qquad \frac{1}{\tau_r} = \frac{1}{T_p} + \frac{1}{T_1} + 4g_d n$$

The function $H(\omega)$ is the modulation response function of the laser. It describes the response of a laser to perturbations of different frequency. The shape of the function $|H(\omega)|$ is plotted in the Figure below. The Figure shows that a laser does not respond to perturbations of frequencies much larger than the laser relaxation oscillation frequency ω_r.



The photon number noise spectral density can be found using,

$$S_{\Delta n\Delta n}(\omega)=\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\left\langle \Delta\hat{n}^{*}(\omega')\Delta\hat{n}(\omega)\right\rangle$$

$$S_{\Delta n\Delta n}(\omega)=\frac{|H(\omega)|^{2}}{\omega_r^{4}}\left\{(2g_d n)^2\left[4\frac{N_2}{T_1}+4\frac{N_1}{T_p}\right]\right.$$

$$+\left(\omega^2+\left(\frac{1}{\tau_r}-4g_d n\right)^2\right)\left[2g_d\, N_2(n+1)+2g_d N_1\, n\right]$$

$$\left.+\left(\omega^2+\left(\frac{1}{\tau_r}\right)^2\right)\left[\frac{n}{\tau_p}\right]\right\}$$

The mean square photon number fluctuations can be found by integrating the noise spectral density,

$$\left\langle \Delta\hat{n}^2(t)\right\rangle=R_{\Delta n\Delta n}(\tau=0)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}S_{\Delta n\Delta n}(\omega)$$

Using,

$$n_{sp}=\frac{N_2}{N_2-N_1}\qquad\qquad n_{sp}-1=\frac{N_1}{N_2-N_1}$$

the normalized variance in the photon number is,

$$\frac{\left\langle \Delta\hat{n}^2(t)\right\rangle}{n}=\tau_r\left(\frac{n_{sp}}{T_1}+\frac{n_{sp}-1}{T_p}\right)\qquad\left\{\text{Non - radiative transitions and pump noise}\right.$$

$$+\tau_r\left(\frac{1}{2}+\left(\frac{1}{T_p}+\frac{1}{T_1}\right)^2\frac{\tau_p}{8g_d n}\right)\frac{2n_{sp}-1}{\tau_p}\qquad\left\{\text{Stimulated emission and absorption noise}\right.$$

$$+\tau_r\left(\frac{1}{2}+\left(\frac{1}{\tau_r}\right)^2\frac{\tau_p}{8g_d n}\right)\frac{1}{\tau_p}\qquad\left\{\text{Photon loss noise}\right.$$

Above threshold, the value of n_{sp} is a constant (does not change with pumping) and its value is fixed by the lasing condition,

$$2g_d N_d = \frac{1}{\tau_p}$$

At transparency, n_{sp} is infinite. If the medium is completely inverted then n_{sp} equals unity. In actual lasers, the value of n_{sp} is somewhere between infinity and unity. Here, we will assume that $n_{sp} >> 1$ above threshold.

Photon Number Noise Much Above Threshold: We now look at the photon number noise much above threshold, in the limit of strong pumping when $1/T_p >> 1/T_{pth}, 1/T_1$. The steady state photon number is,

$$n = \frac{(2g_d N\tau_p - 1)}{4g_d}\left(\frac{1}{T_p} - \frac{1}{T_{pth}}\right) \approx \frac{(2g_d N\tau_p - 1)}{4g_d T_p}$$

Since, by assumption, $n_{sp} >> 1$, we have,

$$2g_d N\tau_p = 2g_d N_d \tau_p \left(\frac{N}{N_d}\right) = \left(\frac{N}{N_d}\right) = 2n_{sp} - 1 >> 1$$

and therefore,

$$n \approx \frac{(2g_d N\tau_p - 1)}{4g_d T_p} \approx \frac{N}{2}\frac{\tau_p}{T_p}$$

This also means that,

$$4g_d n >> \frac{1}{T_p}$$

and,

$$\frac{1}{\tau_r} = \frac{1}{T_p} + \frac{1}{T_1} + 4g_d n \approx 4g_d n$$

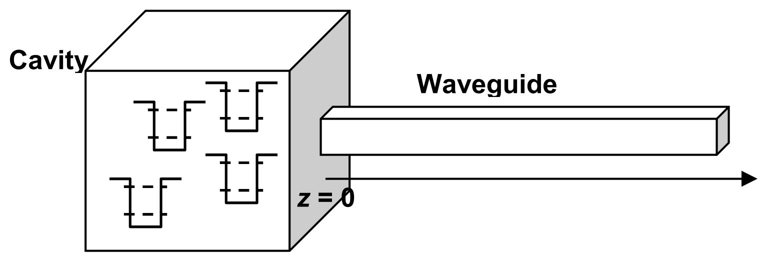
Thus, in the limit $n_{sp} >> 1$ and $1/T_p >> 1/T_{pth}, 1/T_1$, one can approximate the expression for the variance in the photon number as,

$$\begin{aligned}
\frac{\langle \Delta\hat{n}^2(t)\rangle}{n} &\approx \frac{1}{2} && \{\text{Pump noise}\\
&+ 0 && \{\text{Stimulated emission and absorption noise}\\
&+ \frac{1}{2} && \{\text{Photon loss noise}\\
&= 1
\end{aligned}$$

In the assumed limits, the dominant contributions to the laser noise come from the pumping process and from the shot noise associated with photon loss from the cavity. Interestingly, the contribution from the shot noise associated with the stimulated emission and absorption processes has disappeared. The reason for this is as follows. Suppose that in a certain time interval there are more stimulated emission events than given by the average stimulated emission rate. The extra stimulated emission events will end up depleting the gain in the two-level system medium. Consequently, the emission events in the successive time interval will be reduced in number because of the reduced gain in the medium. The result is that the total number of emission events in these two successive time intervals will be more or less close to the value dictated by the average stimulated emission rate. This instantaneous negative feedback from the gain medium helps reduce the noise from stimulated emission and absorption events. Noise from non-radiative transitions is negligible much above threshold compared to the noise from the pumping process. The above expression shows that much above threshold the variance in the photon number approaches the mean photon number. This, as we know, is a characteristic of a coherent state. However, to infer from this result that the quantum state of radiation inside a laser cavity is a coherent state would be a mistake. As we have seen in earlier chapters that a statistical mixture of number states with a Poisson probability distribution will also exhibit the same photon number characteristics.

13.5.2 Photon Flux Noise outside the Laser Cavity

The laser photon noise outside the cavity is different from the noise inside the cavity. Consider the noise form photon loss events. As seen in the previous Section, these events contribute half of the photon number noise inside the cavity much above threshold. The question arises if the photon loss events would also contribute noise to the photon flux noise outside the laser cavity. Suppose that in a certain time interval there are more photon loss events than dictated by the average photon loss rate from the cavity. The extra loss events will end up reducing the number of photons inside the cavity. Consequently, the photon loss events in the successive time interval will be reduced in number because of the reduced number of photons inside the cavity. The result is that the total number of photon loss events in these two successive time intervals will be more or less close to the value dictated by the average photon loss rate. This instantaneous negative feedback from the cavity helps reduce the photon flux noise outside the laser cavity. We will see how this comes out from the math in the discussion that follows.



The field in the waveguide is described by the operators,

$$\hat{S}_{out}(t)e^{-i\omega_o t} = \sqrt{v_g}\,\hat{b}_R(z=0,t)e^{-i\omega_o t}$$
$$= \sqrt{\frac{1}{\tau_p}}\,\hat{a}(t) - \sqrt{v_g}\,\hat{b}_L(z=0,t)e^{-i\omega_o t} = \sqrt{\frac{1}{\tau_p}}\,\hat{a}(t) - \hat{S}_{in}(t)e^{-i\omega_o t}$$

The average photon flux escaping from the laser cavity is,

$$\hat{F}_R(z=0,t) = \frac{\hat{n}(t)}{\tau_p} - \sqrt{\frac{1}{\tau_p}}\left\{\hat{a}^+(t)\hat{S}_{in}(t)e^{-i\omega_o t} + \hat{S}_{in}^+(t)e^{i\omega_o t}\hat{a}(t)\right\} + \hat{S}_{in}^+(t)\hat{S}_{in}(t)$$

$$\left\langle \hat{F}_R(z=0,t)\right\rangle = \left\langle \hat{S}_{out}^+(t)\hat{S}_{out}(t)\right\rangle = \frac{\langle \hat{n}(t)\rangle}{\tau_p}$$

The flux noise operator is,

$$\Delta\hat{F}_R(z=0,t) = \frac{\Delta\hat{n}(t)}{\tau_p} - \hat{F}_v(t) + \hat{S}_{in}^+(t)\hat{S}_{in}(t)$$

The last term will never contribute upon averaging so we can neglect it and write,

$$\Delta\hat{F}_R(z=0,t) = \frac{\Delta\hat{n}(t)}{\tau_p} - \hat{F}_v(t)$$

The above equation is best solved in the frequency domain,

$$\Delta\hat{F}_R(z=0,\omega) = \frac{\Delta\hat{n}(\omega)}{\tau_p} - \hat{F}_v(\omega)$$

Inside the cavity we know that,

$$\begin{bmatrix} \Delta\hat{N}_d(\omega) \\ \Delta\hat{n}(\omega) \end{bmatrix} = \frac{H(\omega)}{\omega_r^2} \begin{bmatrix} -i\omega & -\frac{2}{\tau_p} \\ 2g_d n & -i\omega + \frac{1}{\tau_r} \end{bmatrix} \begin{bmatrix} -2\hat{F}_N(\omega) + 2\hat{F}_P(\omega) - 2\hat{F}_n(\omega) \\ \hat{F}_n(\omega) + \hat{F}_v(\omega) \end{bmatrix}$$

which implies,

$$\Delta\hat{n}(\omega) = \frac{H(\omega)}{\omega_r^2}\left[2g_d n\left(-2\hat{F}_N(\omega) + 2\hat{F}_P(\omega)\right) - \left(4g_d n + i\omega - \frac{1}{\tau_r}\right)\hat{F}_n(\omega) - \left(i\omega - \frac{1}{\tau_r}\right)\hat{F}_v(\omega)\right]$$

And therefore,

$$\Delta\hat{F}_R(z=0,\omega) = \frac{\Delta\hat{n}(\omega)}{\tau_p} - \hat{F}_v(\omega)$$

$$= \frac{H(\omega)}{\omega_r^2\tau_p}\left[2g_d n\left(-2\hat{F}_N(\omega) + 2\hat{F}_P(\omega)\right) - \left(4g_d n + i\omega - \frac{1}{\tau_r}\right)\hat{F}_n(\omega)\right] - \hat{F}_v(\omega)\left[1 + \frac{H(\omega)}{\omega_r^2\tau_p}\left(i\omega - \frac{1}{\tau_r}\right)\right]$$

The above expression is very interesting. If we consider the photon flux noise at high frequencies, such that $\omega >> \omega_r$ and $H(\omega) \approx 0$, we get,

$$\Delta\hat{F}_R(z=0,\omega) = -\hat{F}_v(\omega)$$

Recall that,

$$\sqrt{\frac{1}{\tau_p}}\left\{\hat{a}^+(t)\hat{S}_{in}(t)e^{-i\omega_o t} + e^{i\omega_o t}\hat{S}_{in}^+(t)\hat{a}(t)\right\} = \hat{F}_v(t)$$

which implies,

$$S_{\Delta F_R \Delta F_R}(\omega >> \omega_r) = \int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\left\langle \Delta\hat{F}_R^*(z=0,\omega')\Delta\hat{F}_R(z=0,\omega)\right\rangle = \frac{n}{\tau_p} = \left\langle \hat{F}_R(z=0,t)\right\rangle$$

Thus, at frequencies much higher than the laser relaxation oscillation frequency, the noise in the output photon flux (as seen in the flux noise spectral density) is entirely due to the beating between the field emerging from the cavity and the reflected portion of the vacuum field incident on the cavity, and the resulting noise is exactly at the shot noise level. Next, we look at the photon flux noise spectral density at frequencies smaller than the laser relaxation oscillation frequency, when $\omega << \omega_r$ and $H(\omega) \approx 1$. We assume, as before, $n_{sp} >> 1$ and strong pumping such that, $1/T_p >> 1/T_{pth}, 1/T_1$. First note that the coefficient of the noise source $\hat{F}_v(\omega)$ goes to zero in this limit,

$$\left[1 + \frac{H(\omega)}{\omega_r^2\tau_p}\left(i\omega - \frac{1}{\tau_r}\right)\right] \approx 0$$

This shows that photon loss processes indeed do not contribute to photon flux noise outside the laser cavity at low frequencies. The result for the photon flux noise spectral density at low frequencies is,

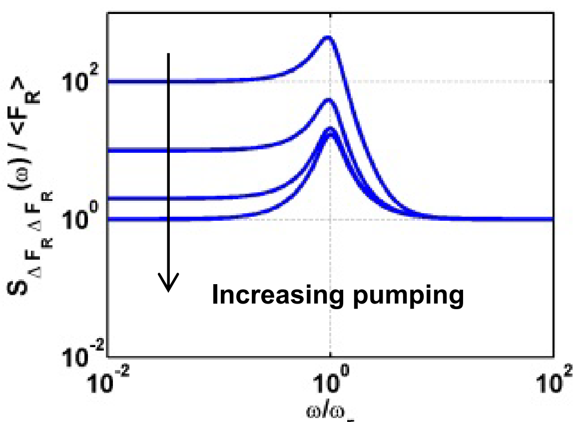
$$\frac{S_{\Delta F_R \Delta F_R}(\omega < \omega_r)}{n/\tau_p} \approx 1 \qquad \{\text{Pump noise}$$

$$+0 \qquad \{\text{Stimulated emission and absorption noise}$$

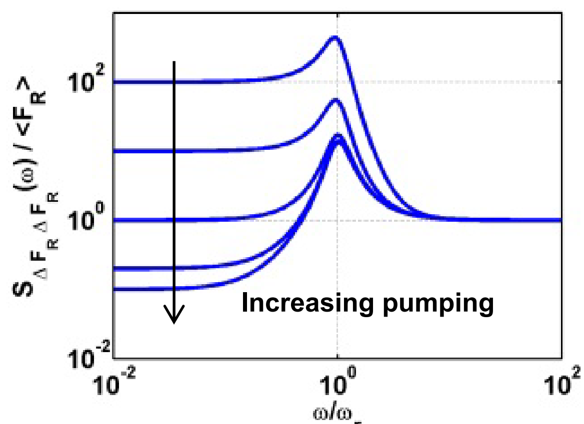
$$+0 \qquad \{\text{Photon loss noise}$$

$$=1$$

Interestingly, at low frequencies the photon flux noise is again at the shot noise level but the noise at low frequencies is entirely due to the pumping process. The noise accompanying pumping is not fundamental. In semiconductor lasers, for example, pumping is performed by an electrical current which takes electrons from the valence band into the conduction band. Electrical current pumping is essentially noiseless (if one ignores the small amount of current noise from the circuit resistances). Therefore, one would expect that the low-frequency photon flux noise of electrically pumped semiconductor lasers to be much smaller than the shot noise level. This turns out to be the case. Photon flux noise from semiconductor lasers has been demonstrated to have values as much as ~13 dB below the shot noise level. Photon flux noise below the shot noise level is a characteristic of amplitude squeezed states. However, to infer from this result that the quantum state of radiation emerging from a semiconductor laser cavity is an amplitude squeezed state would be a mistake. A statistical mixture of number states with a sub-Poisson probability distribution will also exhibit the same photon number characteristics.



Photon flux noise in lasers when the pump has shot noise (solid state and gas lasers)



Photon flux noise in lasers when the pump is noiseless (semiconductor lasers)