Chapter 12: Linear Optical Amplifiers

12.1 Types of Optical Amplifiers

The phrase "linear optical amplifiers" means optical amplifiers in which the amplitude of the output field is linearly related to the input field. For example, parametric amplifiers are linear optical amplifiers even though their operation is based on second or higher order optical nonlinearities.

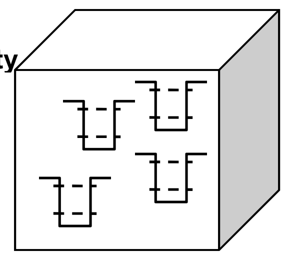
Linear optical amplifiers are of two kinds:

i) **Phase Sensitive Amplifiers:** These amplify only one quadrature of the input field and attenuate the orthogonal quadrature. Example of a phase sensitive amplifier is the degenerate parametric amplifier.

ii) **Phase Insensitive Amplifiers:** These do not distinguish between quadratures and amplify all quadratures in the same way. Examples of phase insensitive amplifiers are ordinary optical amplifiers that are based on population inversion in a gain media.

12.2 Phase Insensitive Optical Amplifiers

Consider a system of N two level systems inside a cavity.



A strong pump light source is used to create and maintain population inversion such that in steady state,

$$\langle \hat{N}_2(t) \rangle > \langle \hat{N}_1(t) \rangle$$

We will look at the amplification of the field corresponding to the cavity radiation mode. We assume that the populations in the upper and lower levels are fixed and are not significantly perturbed in the process of amplification. This assumption is not entirely accurate since field amplification occurs via stimulated emission and the populations must therefore change in the process. This change in population leads to nonlinearities in the amplification process. We assume that the stimulated emission rate is not fast enough to change the populations significantly in the presence of the strong pump. In steady state, we have,

$$g(t) = g_d \langle \hat{N}_2(t) - \hat{N}_1(t) \rangle \qquad n_{sp}(t) = \frac{\langle \hat{N}_2(t) \rangle}{\langle \hat{N}_2(t) - \hat{N}_1(t) \rangle} \qquad (n_{sp}(t) - 1) = \frac{\langle \hat{N}_1(t) \rangle}{\langle \hat{N}_2(t) - \hat{N}_1(t) \rangle}$$

Here, $n_{sp}(t)$ is the population inversion factor. We will assume that both g and n_{sp} are time-independent. The Heisenberg equations for the field operators are,

$$\frac{d\hat{a}(t)}{dt} = -i\omega_o \hat{a}(t) + g\, \hat{a}(t) + e^{-i\omega_o t} \hat{F}_{sp}(t)$$

$$\frac{d\hat{a}^+(t)}{dt} = i\omega_o\, \hat{a}^+(t) + g\, \hat{a}^+(t) + e^{i\omega_o t} \hat{F}^+_{sp}(t)$$

The averages and commutation relations of the noise operators are,

$$\left\langle \hat{F}^{+}_{sp}(t)\hat{F}_{sp}(t')\right\rangle = 2g\, n_{sp}\, \delta(t-t')$$

$$\left\langle \hat{F}_{sp}(t)\hat{F}^{+}_{sp}(t')\right\rangle = 2g\left(n_{sp}-1\right)\delta(t-t')$$

$$\left[\hat{F}_{sp}(t),\hat{F}^{+}_{sp}(t')\right] = -2g\,\delta(t-t')$$

The above are the simplest equations for linear phase insensitive amplification. We have assumed the cavity is lossless. The noise, represented by $\hat{F}_{sp}(t)$, is needed to maintain operator commutation relations. To see this, we write the solution of the above equation as,

$$\hat{a}(t) = \hat{a}(t=0)\, e^{(-i\omega_o+g)t} + e^{-i\omega_o t}\int_0^t e^{g(t-t')}\,\hat{F}_{sp}(t')\,dt'$$

which gives,

$$\left[\hat{a}(t),\hat{a}^{+}(t)\right] = e^{2gt} + \int_0^t dt' \int_0^t dt''\, e^{g(t-t')}\, e^{g(t-t'')}\left[\hat{F}_{sp}(t'),\hat{F}^{+}_{sp}(t'')\right] = 1$$

The above equation shows that field amplification is always accompanied by the addition of noise as described by the operator $\hat{F}_{sp}(t)$.

Suppose the average initial photon number in the cavity at time $t=0$ be $\left\langle \hat{n}(t=0)\right\rangle$.

$$\hat{a}(t) = \hat{a}(t=0)\, e^{(-i\omega_o+g)t} + e^{-i\omega_o t}\int_0^t e^{g(t-t')}\,\hat{F}_{sp}(t')\,dt'$$

The photon number operator at time t is,

$$\begin{aligned}\hat{n}(t) &= \hat{a}^{+}(t)\hat{a}(t)\\ &= \hat{n}(t=0)\, e^{2gt} + \hat{a}^{+}(t=0)\, e^{gt}\int_0^t e^{g(t-t')}\,\hat{F}_{sp}(t')\,dt'\\ &\quad + \int_0^t e^{g(t-t')}\,\hat{F}^{+}_{sp}(t')\,dt' .\, e^{gt}\,\hat{a}(t=0)\\ &\quad + \int_0^t dt_1 \int_0^t dt_2\, e^{g(t-t_1)}\, e^{g(t-t_2)}\,\hat{F}^{+}_{sp}(t_1)\hat{F}_{sp}(t_2)\end{aligned}$$

On taking the average of the above equation we obtain,

$$\begin{aligned}\left\langle \hat{n}(t)\right\rangle &= \left\langle \hat{n}(t=0)\right\rangle e^{2gt} + \int_0^t dt_1 \int_0^t dt_2 \left\langle \hat{F}^{+}_{sp}(t_1)\hat{F}_{sp}(t_2)\right\rangle e^{g(t-t_1)} e^{g(t-t_2)}\\ &= \left\langle \hat{n}(t=0)\right\rangle e^{2gt} + n_{sp}\left(e^{2gt}-1\right)\end{aligned}$$

After time t the photon number gain is e^{2gt}. Let,

$$G = e^{2gt}$$

Then,

$$\left\langle \hat{n}(t)\right\rangle = \left\langle \hat{n}(t=0)\right\rangle G + n_{sp}\left(G-1\right)$$

Amplified Spontaneous Emission (ASE): The first term on the right hand side stands for photon number amplification via stimulated emission. The second term, which is independent of the initial photon number $\left\langle \hat{n}(t=0)\right\rangle$, represents photons that were generated via spontaneous emission and then multiplied due to stimulated emission. This latter contribution is called amplified spontaneous emission (or ASE) and is not desirable. The equation,

$$\left\langle \hat{n}(t)\right\rangle = \left\langle \hat{n}(t=0)\right\rangle G + n_{sp}\left(G-1\right)$$

represents a fundamental relation for linear phase insensitive amplification. If the amplifier has a photon number gain of G then the output must contain $n_{sp}(G-1)$ photons (per mode) due to amplified spontaneous emission irrespective of the details of the amplifier.

Photon Number Noise: The mean square photon number after amplification is,

$$\left\langle \hat{n}^2(t)\right\rangle = \left\langle \hat{n}^2(t=0)\right\rangle G^2 + \left\langle \hat{n}(t=0)\right\rangle \left(4n_{sp}-1\right) G(G-1) + n_{sp}\, G(G-1) + n_{sp}\left(2n_{sp}-1\right)(G-1)^2$$

The variance in the photon number after amplification is,

$$\left\langle \Delta\hat{n}^2(t)\right\rangle = \left\langle \Delta\hat{n}^2(t=0)\right\rangle G^2 + \left\langle \hat{n}(t=0)\right\rangle \left(2n_{sp}-1\right) G(G-1) + n_{sp}(G-1)\left[n_{sp}(G-1)+1\right]$$

The above expression for the variance in the final photon number after amplification consists of the following terms:

i) $\left\langle \Delta\hat{n}^2(t=0)\right\rangle G^2$ = Photon number variance in the input that is amplified by the amplifier.

ii) $n_{sp}(G-1)\left[n_{sp}(G-1)+1\right]$ = Noise due to the variance in the ASE generated photons.

iii) $\left\langle \hat{n}(t=0)\right\rangle \left(2n_{sp}-1\right) G(G-1)$ = Noise due to the beating between the input photons and the ASE photons.

Thermal Photon Number Distribution of ASE: Consider the case when the cavity is completely empty at time $t=0$, i.e. $\left\langle \hat{n}(t=0)\right\rangle = 0$. The average photon number after time t is,

$$\left\langle \hat{n}(t)\right\rangle = n_{sp}(G-1)$$

These photons are generated entirely due to amplified spontaneous emission. The variance in the photon number is,

$$\left\langle \Delta\hat{n}^2(t)\right\rangle = n_{sp}(G-1)\left[n_{sp}(G-1)+1\right] = \left\langle \hat{n}(t)\right\rangle \left[\left\langle \hat{n}(t)\right\rangle + 1\right]$$

The variance in the photon number is related to the mean as in the case of a thermal distribution. In fact, it is not difficult to show that the probability distribution of photon numbers for ASE is exactly a thermal (or Bose-Einstein) distribution,

$$P(m,t) = \frac{1}{1+\left\langle \hat{n}(t)\right\rangle}\left(\frac{\left\langle \hat{n}(t)\right\rangle}{1+\left\langle \hat{n}(t)\right\rangle}\right)^m$$

12.3 Comparison of Optical Amplification with Optical Loss

For comparison, we consider a field undergoing optical loss inside a cavity and described by the equations,

$$\frac{d\hat{a}(t)}{dt} = \left(-i\omega_o - \gamma\right)\hat{a}(t) + \sqrt{2\gamma}\,\hat{S}_{in}(t)e^{-i\omega_o t}$$

$$\frac{d\hat{a}^+(t)}{dt} = \left(i\omega_o - \gamma\right)\hat{a}^+(t) + \sqrt{2\gamma}\,\hat{S}_{in}^+(t)e^{i\omega_o t}$$

The solution is,

$$\hat{a}(t)=\hat{a}(t=0)\,e^{(-i\omega_o-\gamma)t}+\sqrt{2\gamma}\;e^{-i\omega_o t}\int_0^t e^{-\gamma(t-t')}\,\hat{S}_{in}(t')\,dt'$$

The above equation shows that loss is also always accompanied by the addition of noise as described by the second term on the right hand side. The averages and commutation relations of the noise operators are,

$$\left\langle \hat{S}_{in}^{+}(t)\hat{S}_{in}(t')\right\rangle=0$$

$$\left\langle \hat{S}_{in}(t)\hat{S}_{in}^{+}(t')\right\rangle=\delta(t-t')$$

$$\left[\hat{S}_{in}(t),\hat{S}_{in}^{+}(t')\right]=\delta(t-t')$$

The photon number operator is,

$$\begin{aligned}\hat{n}(t)&=\hat{a}^{+}(t)\hat{a}(t)\\&=\hat{n}(t=0)e^{-2\gamma t}+\hat{a}^{+}(t=0)e^{-\gamma t}\sqrt{2\gamma}\int_0^t e^{-\gamma(t-t')}\,\hat{S}_{in}(t')dt'\\&\quad+\sqrt{2\gamma}\int_0^t e^{-\gamma(t-t')}\,\hat{S}_{in}^{+}(t')dt'.\,e^{-\gamma t}\,\hat{a}(t=0)\\&\quad+2\gamma\int_0^t dt_1\int_0^t dt_2\,e^{-\gamma(t-t_1)}\,e^{-\gamma(t-t_2)}\,\hat{S}_{in}^{+}(t_1)\hat{S}_{in}(t_2)\end{aligned}$$

On taking the average of the above equation we obtain,

$$\left\langle \hat{n}(t)\right\rangle=\left\langle \hat{n}(t=0)\right\rangle e^{-2\gamma t}$$

After time t the photon number loss is $e^{-2\gamma t}$. Let $L=e^{-2\gamma t}$. Then,

$$\left\langle \hat{n}(t)\right\rangle=\left\langle \hat{n}(t=0)\right\rangle L$$

Photon Number Noise: The mean square photon number after loss is,

$$\left\langle \hat{n}^2(t)\right\rangle=\left\langle \hat{n}^2(t=0)\right\rangle L^2+\left\langle \hat{n}(t=0)\right\rangle L(1-L)$$

The variance in the photon number after loss is,

$$\left\langle \Delta\hat{n}^2(t)\right\rangle=\left\langle \Delta\hat{n}^2(t=0)\right\rangle L^2+\left\langle \hat{n}(t=0)\right\rangle L(1-L)$$

The above expression for the variance in the final photon number after loss consists of the following terms:

i) $\left\langle \Delta\hat{n}^2(t=0)\right\rangle L^2=$ Photon number variance in the input that is reduced by the loss.

ii) $\left\langle \hat{n}(t=0)\right\rangle L(1-L)=$ Noise introduced as a result of the loss.

12.4 Quadrature Amplification and Noise with a Phase Insensitive Amplifier

The quadrature operator $\hat{x}_\theta(t)$ is defined as,

$$\hat{x}_\theta(t)=\frac{\hat{a}(t)e^{-i\theta}\,e^{i\omega_o t}+\hat{a}(t)e^{i\theta}\,e^{-i\omega_o t}}{2}$$

The equation for $\hat{x}_\theta(t)$ in the presence of gain is,

$$\frac{d\hat{x}_\theta(t)}{dt}=g\,\hat{x}_\theta(t)+\hat{F}_\theta(t)$$

where,

$$\hat{F}_\theta(t)=\frac{\hat{F}_{sp}(t)e^{-i\theta}+\hat{F}_{sp}^{+}(t)e^{i\theta}}{2}=\hat{F}_\theta^{+}(t)$$

The solution is,

$$\hat{x}_\theta(t)=\hat{x}_\theta(t=0)e^{gt}+\int_0^t \hat{F}_\theta(t')e^{g(t-t')}\,dt'$$

$$\Rightarrow\ \left\langle\hat{x}_\theta(t)\right\rangle=\left\langle\hat{x}_\theta(t=0)\right\rangle e^{gt}=\left\langle\hat{x}_\theta(t=0)\right\rangle\sqrt{G}$$

$$\left\langle\hat{x}_\theta^2(t)\right\rangle=\left\langle\hat{x}_\theta^2(t=0)\right\rangle e^{2gt}+\int_0^t dt_1\int_0^t dt_2\ \left\langle\hat{F}_\theta(t_1)\hat{F}_\theta(t_2)\right\rangle e^{g(t-t_1)}e^{g(t-t_2)}$$

$$\left\langle\hat{x}_\theta^2(t)\right\rangle=\left\langle\hat{x}_\theta^2(t=0)\right\rangle G+\frac{(2n_{sp}-1)}{4}(G-1)$$

$$\Rightarrow\left\langle\Delta\hat{x}_\theta^2(t)\right\rangle=\left\langle\Delta\hat{x}_\theta^2(t=0)\right\rangle G+\frac{(2n_{sp}-1)}{4}(G-1)$$

The above equations are true for all values of θ. The variance in the quadrature consists of the following terms:

i) $\left\langle\Delta\hat{x}_\theta^2(t=0)\right\rangle G$ = Quadrature fluctuations in the input that are amplified.

ii) $\frac{(2n_{sp}-1)}{4}(G-1)$ = Quadrature noise that is added by the amplifier.

12.5 Coherent State Amplification

In Chapter 9 we showed that a coherent state remains a coherent state when undergoing loss. Here we discuss how coherent states behave under linear phase insensitive amplification. Suppose the photon number gain of the amplifier is G, and the initial quantum state of radiation is the coherent state $|\alpha\rangle$. It follows that,

$$|\psi(t=0)\rangle=|\alpha\rangle$$

$$\langle\hat{n}(t=0)\rangle=|\alpha|^2$$

$$\left\langle\Delta\hat{n}^2(t=0)\right\rangle=\langle\hat{n}(t=0)\rangle=|\alpha|^2$$

After amplification,

$$\langle\hat{n}(t)\rangle=\langle\hat{n}(t=0)\rangle G+n_{sp}(G-1)$$

$$=|\alpha|^2G+n_{sp}(G-1)$$

$$\left\langle\Delta\hat{n}^2(t)\right\rangle=\left\langle\Delta\hat{n}^2(t=0)\right\rangle G^2+\langle\hat{n}(t=0)\rangle(2n_{sp}-1)G(G-1)+n_{sp}(G-1)\left[n_{sp}(G-1)+1\right]$$

$$=|\alpha|^2G^2+|\alpha|^2(2n_{sp}-1)G(G-1)+n_{sp}(G-1)\left[n_{sp}(G-1)+1\right]$$

Note that,

$$\left\langle\Delta\hat{n}^2(t)\right\rangle>>\langle\hat{n}(t)\rangle$$

Therefore, the final state has a much larger photon number variance than a coherent state. To see this explicitly, consider an ideal amplifier with complete population inversion (i.e. $n_{sp}=1$). Then,

$$\frac{\langle \Delta \hat{n}^2(t) \rangle}{\langle \hat{n}(t) \rangle} = \frac{|\alpha|^2 G^2 + \left(|\alpha|^2 + 1\right) G(G-1)}{|\alpha|^2 G + G - 1}$$

If $|\alpha|^2 >> 1$, then,

$$\frac{\langle \Delta \hat{n}^2(t) \rangle}{\langle \hat{n}(t) \rangle} \approx 2G - 1$$

The right hand side can be much larger than unity.

We look at the quadratures next. For the input state we have,

$$\langle \hat{x}_\theta(t=0) \rangle = \langle \alpha | \hat{x}_\theta | \alpha \rangle = \frac{\alpha\, e^{-i\theta} + \alpha^* e^{i\theta}}{2}$$

$$\langle \Delta \hat{x}_\theta^2(t=0) \rangle = \langle \alpha | \Delta \hat{x}_\theta^2 | \alpha \rangle = \frac{1}{4}$$

After amplification we get,

$$\langle \hat{x}_\theta(t) \rangle = \langle \hat{x}_\theta(t=0) \rangle G = \left(\frac{\alpha\, e^{-i\theta} + \alpha^* e^{i\theta}}{2} \right) G$$

$$\langle \Delta \hat{x}_\theta^2(t) \rangle = \langle \Delta \hat{x}_\theta^2(t=0) \rangle G + \frac{(2n_{sp} - 1)}{4}(G-1) = \frac{G}{4} + \frac{(2n_{sp} - 1)}{4}(G-1)$$

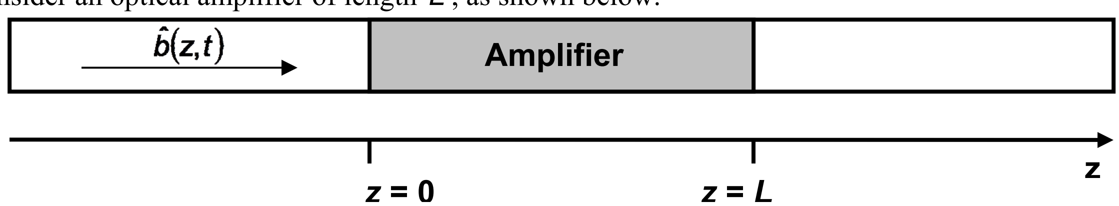
It is obvious that,

$$\langle \Delta \hat{x}_\theta^2(t) \rangle >> \frac{1}{4}$$

And therefore the quadrature variance of an amplified coherent state is much greater than $1/4$.

12.6 A Traveling Wave Optical Amplifier

Consider an optical amplifier of length L, as shown below.



We assume that the amplitude gain per unit length is g and the photon number gain G of the amplifier is therefore $G = e^{2gL}$. In the presence of gain the equation for the operator $\hat{b}(z,t)$ is,

$$\left(\frac{\partial}{\partial z} + \frac{1}{v_g} \frac{\partial}{\partial t} \right) \hat{b}(z,t) = g\, \hat{b}(z,t) + \hat{F}_{sp}(z,t)$$

Conservation of the commutation relation,

$$\left[\hat{b}(z,t), \hat{b}^+(z',t) \right] = \delta(z - z')$$

requires,

$$\left[\hat{F}_{sp}(z,t), \hat{F}_{sp}^+(z',t') \right] = -\frac{2g}{v_g} \delta(z - z')\, \delta(t - t')$$

The averages of the noise operators are,

$$\left\langle \hat{F}_{sp}^{+}(z,t)\hat{F}_{sp}(z',t')\right\rangle = \frac{2g}{v_g} n_{sp}\,\delta(z-z')\delta(t-t')$$

$$\left\langle \hat{F}_{sp}(z,t)\hat{F}_{sp}^{+}(z',t')\right\rangle = \frac{2g}{v_g} (n_{sp}-1)\delta(z-z')\delta(t-t')$$

Solution of the travelling wave equation in the presence of gain is,

$$\hat{b}(z,t) = \hat{b}(z-v_g t,0)e^{v_g g t} + v_g \int_0^t dt'\ e^{v_g g (t-t')}\,\hat{F}_{sp}\left(z-v_g\left(t-t'\right),t'\right)$$

The solution for $z \ge L$ and t such that $z - v_g t \le 0$, is,

$$\hat{b}(z,t) = \hat{b}(z-v_g t,0)e^{g L} + \int_0^L dz'\ e^{g(L-z')}\hat{F}_{sp}\left(z',t-\frac{(z-z')}{v_g}\right)$$

The average flux at the output is,

$$\begin{aligned}
\left\langle \hat{F}(z,t)\right\rangle &= v_g \left\langle \hat{b}^{+}(z,t)\hat{b}(z,t)\right\rangle \\
&= \left\langle \hat{F}(z-v_g t,0)\right\rangle e^{2gL} \\
&\quad + \int_0^L dz' \int_0^L dz''\, e^{g(L-z')}\, e^{g(L-z'')}\ x \\
&\qquad \left\langle \hat{F}_{sp}^{+}\left(z',t-\frac{(z-z')}{v_g}\right)\hat{F}_{sp}\left(z'',t-\frac{(z-z'')}{v_g}\right)\right\rangle \\
&= \left\langle \hat{F}(z-v_g t,0)\right\rangle G + v_g\, n_{sp}\left(G-1\right)\delta\left(z-z\right)
\end{aligned}$$

Recall that,

$$\delta(z-z') = \int_{\beta_o+\frac{\Delta\beta}{2}}^{\beta_o+\frac{\Delta\beta}{2}} \frac{d\beta}{2\pi} e^{i(\beta-\beta_o)(z-z')}$$

And for $z = z'$,

$$\delta\left(z-z\right) = \delta\left(0\right) = \frac{\Delta\beta}{2\pi} = \frac{1}{2\pi}\frac{\Delta\beta}{\Delta\omega}\Delta\omega = \frac{\Delta\omega}{2\pi}\frac{1}{v_g}.$$

Therefore we have,

$$\left\langle \hat{F}\left(z,t\right)\right\rangle = \left\langle \hat{F}\left(z-v_g t,0\right)\right\rangle G + \frac{\Delta\omega}{2\pi} n_{sp}\left(G-1\right) \qquad \left\{\text{for } z > L\,,\ z - v_g t < 0\right.$$

The right hand side consists of the following terms:

i) $\left\langle \hat{F}\left(z-v_g t,0\right)\right\rangle G$ = Amplified input photon flux.

ii) $\frac{\Delta\omega}{2\pi} n_{sp}\left(G-1\right)$ = Photon flux at the amplifier output due to amplified spontaneous emission (ASE).

Note that the ASE flux depends on the optical bandwidth $\Delta\omega$ of interest. This is true as long as the bandwidth $\Delta\omega$ is much smaller than the gain bandwidth. Photons are emitted spontaneously in the entire gain bandwidth and the rate of emission in a small bandwidth $\Delta\omega$ is proportional to $\Delta\omega$. Consequently, ASE photon flux coming out of the amplifier in a bandwidth $\Delta\omega$ is proportional to $\Delta\omega$. Another way to arrive at this result is as follows. The number of modes per unit length of the amplifier is $\Delta\beta/2\pi$. The number of ASE photons per mode at the output is $n_{sp}(G-1)$. Therefore, the ASE photon density (number of ASE photons per unit length) at the output of the amplifier is,

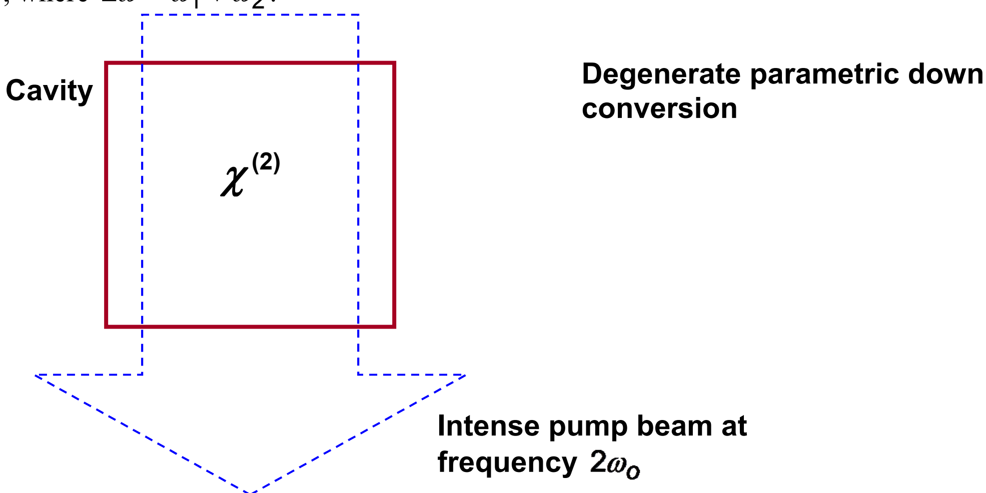
$$\frac{\Delta\beta}{2\pi} \times n_{sp}(G-1)$$

The flux due to ASE photons is therefore,

$$\text{ASE photon flux} = v_g \times \frac{\Delta\beta}{2\pi} \times n_{sp}(G-1) = \frac{\Delta\omega}{2\pi} n_{sp}(G-1)$$

12.7 Phase Sensitive Optical Amplifiers

Phase sensitive amplifiers, as the name suggests, amplify only one quadrature of the input. Degenerate parametric amplification is an example of phase sensitive amplification. One of the simplest interactions in nonlinear optics is where a photon of frequency 2ω splits into two photons, each of frequency ω. This process, known as degenerate parametric down conversion, can happen in a medium with a non-zero second order nonlinear susceptibility $\chi^{(2)}$. Degenerate parametric down conversion is called "degenerate" because the down converted photons have the same frequencies. One can also have parametric down conversions in which a photon of frequency 2ω splits into two photons of frequencies ω_1 and ω_2, where $2\omega = \omega_1 + \omega_2$.



The cavity is assumed to support only a single field mode, and the Hamiltonian is,

$$\hat{H} = \hbar\omega_o \, \hat{a}^+ \hat{a}$$

The cavity contains a medium that has a non-zero second order nonlinear susceptibility $\chi^{(2)}$. The cavity is irradiated with a strong pump field having frequency $2\omega_o$. The pump is assumed to be a continuous wave coherent state. The average amplitude of the pump electric field everywhere inside the cavity is assumed to be $E(t) = \text{Re}\{ E_o \exp(-i2\omega_o t + i2\phi) \}$. The pump frequency is not supported by the cavity and therefore the pump beam photons pass through the cavity. The down converted photons have frequency ω_o, which is the frequency of an eigenmode of the cavity, and therefore the down converted photons stay in the cavity. The photons in the pump at frequency $2\omega_o$ split into two photons of frequency ω_o each, and as time goes on, the population of photons of frequency ω_o will increase in the cavity.

Parametric down conversion consists of both stimulated as well as spontaneous down conversion processes. Here, we are interested in parametric down conversion as an amplification process. We assume that the initial state of the cavity field is $|\psi(t=0)\rangle$. As a result of down conversion the number of photons in the cavity field will increase with time. The Hamiltonian including the parametric interaction is,

$$\hat{H} = \hbar\omega_o\, \hat{a}^+(t)\hat{a}(t) + \frac{i\hbar}{2}\left\{\beta[\hat{a}(t)]^2 \hat{b}_p^+(t) - \beta^* \hat{b}_p(t)\left[\hat{a}^+(t)\right]^2\right\}$$

Here, $\hat{b}_p(t)$ is the field destruction operator for the pump field and the constant μ is proportional to the nonlinear susceptibility $\chi^{(2)}$. It is convenient to replace the operator $\hat{b}_p(t)$ by its average value and write,

$$\beta\left\langle \hat{b}_p(t)\right\rangle = -g\, e^{-i2\omega_o t + i2\phi}$$

to get,

$$\hat{H} = \hbar\omega_o\, \hat{a}^+(t)\hat{a}(t) - i\hbar\frac{g}{2}\left\{e^{i2\omega_o t - i2\phi}\,[\hat{a}(t)]^2 - e^{-i2\omega_o t + i2\phi}\left[\hat{a}^+(t)\right]^2\right\}$$

where g is assumed to be a positive constant.

In the Schrodinger picture, the Hamiltonian is,

$$\hat{H} = \hbar\omega_o\, \hat{a}^+\hat{a} - i\hbar\frac{g}{2}\left\{e^{i2\omega_o t - i2\phi}\,[\hat{a}]^2 - e^{-i2\omega_o t + i2\phi}\left[\hat{a}^+\right]^2\right\}$$

The Heisenberg equations for the cavity field operators are,

$$\frac{d\hat{a}(t)}{dt} = -i\omega_o\hat{a}(t) + g\, e^{-i2(\omega_o t - \phi)}\,\hat{a}^+(t)$$

$$\frac{d\hat{a}^+(t)}{dt} = i\omega_o\hat{a}^+(t) + g\, e^{i2(\omega_o t - \phi)}\,\hat{a}(t)$$

The equations for the two quadrature operators, $\hat{x}_\phi(t)$ and $\hat{x}_{\phi+\pi/2}(t)$, are,

$$\frac{d\,\hat{x}_\phi(t)}{dt} = g\,\hat{x}_\phi(t)$$

$$\frac{d\,\hat{x}_{\phi+\pi/2}(t)}{dt} = -\,g\,\hat{x}_{\phi+\pi/2}(t)$$

Note that one quadrature is amplified and the orthogonal quadrature is attenuated. The quadrature that gets amplified is selected by the phase of the pump. Also note that there are no fundamental noise sources required for the quantum mechanical consistency of the equations. Phase sensitive quadrature amplification can therefore be noiseless. For the quadrature that is amplified we have,

$$\hat{x}_\phi(t) = \hat{x}_\phi(t=0)e^{gt}$$

$$\Rightarrow \quad \left\langle \hat{x}_\phi(t)\right\rangle = \left\langle \hat{x}_\phi(t=0)\right\rangle e^{gt} = \left\langle \hat{x}_\phi(t=0)\right\rangle \sqrt{G} \qquad \left\{G = e^{2gt}\right\}$$

$$\Rightarrow \quad \left\langle \hat{x}_\phi^2(t)\right\rangle = \left\langle \hat{x}_\phi^2(t=0)\right\rangle G$$

$$\Rightarrow \quad \left\langle \Delta\hat{x}_\phi^2(t)\right\rangle = \left\langle \Delta\hat{x}_\phi^2(t=0)\right\rangle G$$

Note that the quadrature noise after amplification is just the amplified input noise. There is no noise added by the amplifier. For the quadrature that is attenuated we have,

$$\hat{x}_{\phi+\pi/2}(t)=\hat{x}_{\phi+\pi/2}(t=0)e^{-gt}$$

$$\Rightarrow \quad \left\langle \hat{x}_{\phi+\pi/2}(t)\right\rangle = \left\langle \hat{x}_{\phi+\pi/2}(t=0)\right\rangle e^{-gt} = \left\langle \hat{x}_{\phi}(t=0)\right\rangle \frac{1}{\sqrt{G}} \qquad \left\{G=e^{2gt}\right\}$$

$$\Rightarrow \quad \left\langle \hat{x}^2_{\phi+\pi/2}(t)\right\rangle = \left\langle \hat{x}^2_{\phi+\pi/2}(t=0)\right\rangle \frac{1}{G}$$

$$\Rightarrow \quad \left\langle \Delta\hat{x}^2_{\phi+\pi/2}(t)\right\rangle = \left\langle \Delta\hat{x}^2_{\phi+\pi/2}(t=0)\right\rangle \frac{1}{G}$$

Comparing quadrature amplification by phase sensitive and phase insensitive amplifiers we see that,

Phase sensitive amplifier: $\left\langle \Delta\hat{x}^2_{\phi}(t)\right\rangle = \left\langle \Delta\hat{x}^2_{\phi}(t=0)\right\rangle G$

Phase insensitive amplifier: $\left\langle \Delta\hat{x}^2_{\phi}(t)\right\rangle = \left\langle \Delta\hat{x}^2_{\phi}(t=0)\right\rangle G + \frac{(2n_{sp}-1)}{4}(G-1)$

Noise-Free Quadrature Amplification of a Coherent State: Suppose the initial state of the cavity field is a coherent state $|\alpha\rangle$, where $\alpha=|\alpha|e^{i\theta}$. In order to amplify the quadrature $\hat{x}_{\theta}(t)$ we choose the phase of the pump beam such that $\phi=\theta$. Then,

$$\left\langle \hat{x}_{\theta}(t=0)\right\rangle = |\alpha|$$

$$\Rightarrow \left\langle \hat{x}_{\theta}(t)\right\rangle = \left\langle \hat{x}_{\theta}(t=0)\right\rangle \sqrt{G} = |\alpha|\sqrt{G}$$

$$\Rightarrow \left\langle \Delta\hat{x}^2_{\theta}(t)\right\rangle = \left\langle \Delta\hat{x}^2_{\theta}(t=0)\right\rangle G = \frac{G}{4}$$

The amplifier seems too good to be true. The catch is that it can only amplify one quadrature and one has to know in advance which quadrature one wants to amplify and set the phase of the pump appropriately. Although degenerate parametric amplifier is a noise-free quadrature amplifier, it is not a noise free photon number amplifier, as shown below.

Photon Number Amplification via Parametric Down Conversion: We now look at photon number amplification using a degenerate parametric amplifier. Since,

$$\hat{a}(t)=\left\{\hat{x}_{\phi}(t)+i\,\hat{x}_{\phi+\pi/2}(t)\right\}e^{-i\omega_o t+i\phi}$$

We get,

$$\hat{a}(t)=\left\{\hat{x}_{\phi}(t=0)\sqrt{G}+i\,\hat{x}_{\phi+\pi/2}(t=0)\frac{1}{\sqrt{G}}\right\}e^{-i\omega_o t+i\phi}$$

$$=\left\{\hat{a}(t=0)\cosh(gt)+\hat{a}^{+}(t=0)e^{2i\phi}\sinh(gt)\right\}e^{-i\omega_o t}$$

Therefore,

$$\hat{n}(t)=\hat{a}^{+}(t)\hat{a}(t)=\hat{n}(t=0)\left[\cosh^2(gt)+\sinh^2(gt)\right]+\sinh^2(gt)$$

$$+\frac{1}{2}\left\{\hat{a}^{+}(t=0)\hat{a}^{+}(t=0)e^{2i\phi}+\hat{a}(t=0)\hat{a}(t=0)e^{-2i\phi}\right\}\sinh(2gt)$$

If we assume that the initial state is a coherent state $|\alpha\rangle$ where $\alpha=|\alpha|e^{i\phi}$, then,

$$\left\langle \hat{n}(t)\right\rangle = |\alpha|^2 G + \frac{1}{4}\left[G+\frac{1}{G}-2\right]$$

The right hand side consists of the following terms:

i) $|\alpha|^2 G$ = Amplified input photon number.

ii) $\frac{1}{4}\left[G+\frac{1}{G}-2\right]$ = Photons due to amplified spontaneous parametric down conversion (ASPD).

Comparing photon number amplification via phase insensitive and phase insensitive amplifiers, we see that for the same photon number gain G, the number of added ASPD photons is smaller than the number of ASE photons.