Chapter 11: Matter-Photon Interactions and Cavity Quantum Electrodynamics

11.1 A Semi-Classical Approach to Particle-Field Interactions

In this Chapter, we will present a quantum theory for the interactions between charged particles and electromagnetic fieldd. Proper handling of the gauge invariance in electromagnetism will be necessary to develop a sensible theory. You have seen in Chapter 5 that the easiest way to quantize radiation is to first choose a gauge and then postulate appropriate commutation relations. But once the field has been quantized, and its Hilbert space constructed, changing the gauge is not easy and something that we will avoid. Therefore, in the discussion that follows the particle interacting with the field will be treated quantum mechanically but the field will be treated classically. This is the so called semi-classical approach. Once we have obtained a suitable Hamiltonian describing the interaction between the particle and the field, we will then chose a gauge and quantize the field in Section 11.2.

In quantum mechanics, in the absence of an electromagnetic field, the kinetic momentum operator $m\hat{\vec{v}}(t)$ of a particle satisfies the following equal-time commutation relations with the particle position operator $\hat{\vec{r}}(t)$,

$$\left[\hat{r}_k(t), m\,\hat{v}_j(t)\right] = i\hbar\delta_{kj}$$

As a result of the above commutation relations, and following the arguments presented in Chapter 1, the momentum of a particle is represented by the gradient operator in the position representation (in the Schrodinger picture),

$$m\hat{\vec{v}} \iff \frac{\hbar}{i}\nabla$$

In the presence of an electromagnetic field, the canonical momentum operator $\hat{\vec{p}}(t)$ of a particle is defined as,

$$\hat{\vec{p}}(t) = m\hat{\vec{v}}(t) + q\vec{A}\left(\hat{\vec{r}}(t),t\right)$$

It is the canonical momentum $\hat{\vec{p}}(t)$, and not the kinetic momentum $m\hat{\vec{v}}(t)$, which obeys the equal-time commutation relations,

$$\left[\hat{r}_k(t), \hat{p}_j(t)\right] = i\hbar\delta_{kj}$$

The above commutation relations give the correct Heisenberg equations for the rate of change of the particle kinetic momentum operator $m\hat{\vec{v}}(t)$ in accordance with the Lorentz force law. The canonical momentum thus plays an important role in describing particle-field interactions in quantum mechanics. In the next Section, we discuss the canonical momentum in more detail.

11.1.1 Canonical and Kinetic Momentum of a Charged Particle

To motivate the ideas behind the canonical momentum, we start from the classical expression for the momentum of the electromagnetic field,

$$\vec{P}(t) = \varepsilon_o\mu_o\int d^3\vec{r}\;\vec{E}(\vec{r},t)\times\vec{H}(\vec{r},t)$$

The electric field can be decomposed into a divergence-free transverse component and a longitudinal component,

$$\vec{E}(\vec{r},t) = \vec{E}_L(\vec{r},t) + \vec{E}_T(\vec{r},t)$$

where,

$$\nabla \cdot \varepsilon_o \vec{E}_L(\vec{r},t) = \rho(\vec{r},t) \qquad \nabla \times \vec{E}_L(\vec{r},t) = 0$$

$$\nabla \cdot \varepsilon_o \vec{E}_T(\vec{r},t) = 0$$

In Fourier-space,

$$\vec{E}(\vec{k},t) = \vec{E}_L(\vec{k},t) + \vec{E}_T(\vec{k},t)$$

The terms "transverse" and "longitudinal" are used because the corresponding fields are orthogonal to and parallel to the direction of the wavevector in Fourier-space,

$$\vec{E}_L(\vec{k},t) = \hat{k}\left[\hat{k}.\vec{E}(\vec{k},t)\right]$$

$$\vec{E}_T(\vec{k},t) = \left[1 - \hat{k}\otimes\hat{k}\right]\vec{E}(\vec{k},t) = \vec{E}(\vec{k},t) - \hat{k}\left[\hat{k}.\vec{E}(\vec{k},t)\right]$$

It follows that,

$$\vec{k}\cdot\vec{E}_L(\vec{k},t) = -i\frac{\rho(\vec{k},t)}{\varepsilon_o} \quad \Rightarrow \quad \vec{E}_L(\vec{k},t) = -i\frac{\hat{k}}{k}\frac{\rho(\vec{k},t)}{\varepsilon_o}$$

$$\hat{k}\cdot\vec{E}_T(\vec{k},t) = 0$$

We also have,

$$\mu_o\vec{H}(r,t) = \nabla\times\vec{A}(r,t)$$

$$\Rightarrow \mu_o\vec{H}(\vec{k},t) = i\hat{k}\times\vec{A}(\vec{k},t)$$

The vector potential can also be divided into transverse and longitudinal parts,

$$\vec{A}(\vec{r},t) = \vec{A}_L(\vec{r},t) + \vec{A}_T(\vec{r},t)$$

With the above definitions, one can write the expression for the field momentum as follows,

$$\vec{P}(t) = \varepsilon_o\mu_o\int d^3\vec{r}\ \vec{E}_L(\vec{r},t)\times\vec{H}(\vec{r},t) \quad + \quad \varepsilon_o\mu_o\int d^3\vec{r}\ \vec{E}_T(\vec{r},t)\times\vec{H}(\vec{r},t)$$

The second term on the right hand side in the above Equation was identified with the momentum of the photons in Chapter 5. Here, we explore the first term, which would be zero if there were no charges and, consequently, if $\vec{E}_L(\vec{r},t)$ were zero. The first term can be written as,

$$\varepsilon_o\mu_o\int d^3\vec{r}\ \vec{E}_L(\vec{r},t)\times\vec{H}(\vec{r},t)$$

$$= \varepsilon_o\mu_o\int\frac{d^3\vec{k}}{(2\pi)^3}\vec{E}_L\left(-\vec{k},t\right)\times\vec{H}\left(\vec{k},t\right) = \varepsilon_o\int\frac{d^3\vec{k}}{(2\pi)^3}\vec{E}_L\left(-\vec{k},t\right)\times\left[i\vec{k}\times\vec{A}\left(\vec{k},t\right)\right]$$

$$= \varepsilon_o\int\frac{d^3\vec{k}}{(2\pi)^3}\, i\,\frac{\rho(-\vec{k},t)}{\varepsilon_o}\hat{k}\times\left[i\vec{k}\times\vec{A}\left(\vec{k},t\right)\right] = \int\frac{d^3\vec{k}}{(2\pi)^3}\rho(-\vec{k},t)\left[1-\hat{k}\times\hat{k}\right]\vec{A}\left(\vec{k},t\right)$$

$$= \int\frac{d^3\vec{k}}{(2\pi)^3}\rho(-\vec{k},t)\,\vec{A}_T\left(\vec{k},t\right)$$

$$= \int d^3\vec{r}\ \rho\left(\vec{r},t\right)\times\vec{A}_T\left(\vec{r},t\right)$$

For a single particle of charge q and position $\vec{r}(t)$, the charge density is,

$$\rho(\vec{r},t) = q\delta\left(\vec{r}-\vec{r}(t)\right)$$

Therefore,

$$\varepsilon_o\mu_o\int d^3\vec{r}\ \vec{E}_L\left(\vec{r},t\right)\times\vec{H}\left(\vec{r},t\right) = q\vec{A}_T\left(\vec{r}(t),t\right)$$

Note that in the expression above the transverse component of the vector potential is evaluated at the location of the particle position. The momentum of the electromagnetic field is then,

$$\vec{P}(t) = q\vec{A}_T(\vec{r}(t),t) \quad + \quad \varepsilon_o \mu_o \int d^3\vec{r}\ \vec{E}_T(\vec{r},t) \times \vec{H}(\vec{r},t)$$

The total momentum of the particle and the field is,

$$\vec{P}_{total}(t) = m\vec{v}(t) \quad + \quad q\vec{A}_T(\vec{r}(t),t) \quad + \quad \varepsilon_o \mu_o \int d^3\vec{r}\ \vec{E}_T(\vec{r},t) \times \vec{H}(\vec{r},t)$$

Interestingly, the first two terms on the right hand side are associated with the particle position and velocity.

11.1.2 Quantum Mechanics of a Charged Particle in the Coulomb Gauge

So far we have not made any choice for the gauge. We now chose a gauge in which the longitudinal component of the vector potential is zero,

$$\vec{A}_L(\vec{r},t) = 0$$

The expression for the total momentum $\vec{P}_{total}(t)$ tells us that if one defines a canonical momentum $\vec{p}(t)$ associated with the particle as,

$$\vec{p}(t) = m\vec{v}(t) + q\vec{A}_T(\vec{r}(t),t) \quad \Longleftrightarrow \quad m\vec{v}(t) = \vec{p}(t) - q\vec{A}_T(\vec{r}(t),t)$$

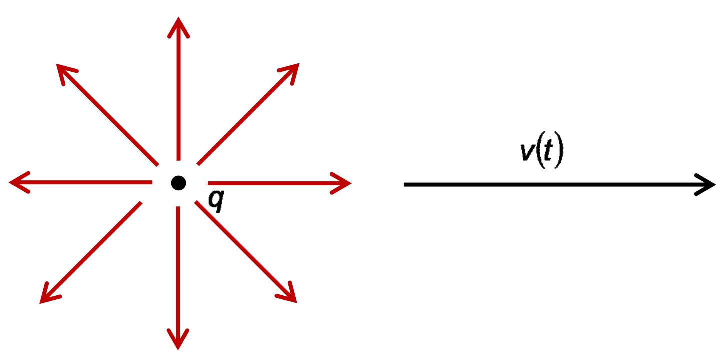
then the total momentum of the particle and the field can be written in a particularly simple form as the sum of the particle canonical momentum $\vec{p}(t)$ and the momentum of the transverse field,

$$\vec{P}_{total}(t) = \vec{p}(t) \quad + \quad \varepsilon_o \mu_o \int d^3\vec{r}\ \vec{E}_T(\vec{r},t) \times \vec{H}(\vec{r},t)$$

Recall that in Chapter 5, in the Coulomb gauge, the momentum of the transverse field was identified with the momentum of the photons. Keeping this in mind, in the Coulomb gauge the canonical momentum $\vec{p}(t)$ of a charged particle is seen to consist of two parts:

i) The kinetic momentum of the charged particle

ii) The momentum of the longitudinal field associated with the charged particle, the source of which is the charged particle itself



Quantization of the Particle Dynamics in the Coulomb Gauge: Taking the charged particle and the longitudinal field "attached" to it as one composite object with a momentum equal to $\vec{p}(t)$, one postulates the following equal-time commutation relations,

$$\left[\hat{r}_k(t), \hat{p}_j(t)\right] = i\hbar\delta_{kj}$$

These commutation relations imply that the canonical momentum of the particle (and not the kinetic momentum) is represented by the gradient operator in the position representation (in the Schrodinger picture),

$$\hat{\vec{p}} \quad \Longleftrightarrow \quad \frac{\hbar}{i}\nabla$$

Particle Hamiltonian in the Coulomb Gauge: Assuming a classical electromagnetic field, the Schrodinger equation for the particle is,

$$i\hbar \frac{\partial |\psi(t)\rangle}{\partial t} = \hat{H}(t)|\psi(t)\rangle$$

Where the Hamiltonian (in the Schrodinger picture) can be written as,

$$\hat{H}(t) = \frac{1}{2} m\hat{\vec{v}}.\hat{\vec{v}} + q\,\phi(\hat{\vec{r}},t) = \frac{\left[\hat{\vec{p}} - q\vec{A}_T(\hat{\vec{r}},t)\right]^2}{2m} + q\,\phi(\hat{\vec{r}},t)$$

In the position representation, the Hamiltonian is,

$$\langle \vec{r}\,|\hat{H}(t)|\psi(t)\rangle = \langle \vec{r}\,|\frac{\left[\hat{\vec{p}} - q\vec{A}_T(\hat{\vec{r}},t)\right]^2}{2m} + q\,\phi(\hat{\vec{r}},t)|\psi(t)\rangle = \left[\frac{\left[\frac{\hbar}{i}\nabla - q\vec{A}_T(\vec{r},t)\right]^2}{2m} + q\,\phi(\vec{r},t)\right]\psi(\vec{r},t)$$

The above expression is valid in the Coulomb gauge. The question now arises what if one makes a different choice for the gauge. This we consider next.

11.1.3 Quantum Mechanics of a Charged Particle in an Arbitrary Gauge

Consider the following gauge transformation,

$$\vec{A}_{new}(\vec{r},t) = \vec{A}(\vec{r},t) + \vec{\nabla}F(\vec{r},t) \qquad \Rightarrow \qquad \vec{A}_{new}(\vec{k},t) = \vec{A}(\vec{k},t) + i\vec{k}.F(\vec{k},t)$$

$$\phi_{new}(\vec{r},t) = \phi(\vec{r},t) - \frac{\partial}{\partial t}F(\vec{r},t)$$

Here, $F(\vec{r},t)$ is an arbitrary single-valued scalar function. The gauge transformation leaves the electric and magnetic fields unchanged. Note also that the gauge transformation affects only the longitudinal component of the vector potential and leaves the transverse component unchanged.

We assume that under the above gauge transformation the quantum state of the particle transforms as,

$$|\psi_{new}(t)\rangle = \hat{T}|\psi(t)\rangle$$

where $\hat{T}$ is a unitary operator. A unitary operator is needed in order to conserve probabilities since we do not expect that a gauge transformation to affect the physical results in any way.

To find the properties of the operator $\hat{T}$ we proceed as follows. Consider the average position of the particle,

$$\langle\psi(t)|\hat{\vec{r}}|\psi(t)\rangle$$

If we make a gauge transformation, we do not expect the average position value to change. Therefore,

$$\langle\psi(t)|\hat{\vec{r}}|\psi(t)\rangle = \langle\psi_{new}(t)|\hat{\vec{r}}|\psi_{new}(t)\rangle = \langle\psi(t)|\hat{T}^{+}\,\hat{\vec{r}}\,\hat{T}|\psi(t)\rangle$$

This implies,

$$\hat{T}^{+}\,\hat{\vec{r}}\,\hat{T} = \hat{\vec{r}} \quad \Rightarrow \quad \left[\hat{\vec{r}},\hat{T}\right] = 0$$

Since the operator $\hat{T}$ commutes with the position operator, and $\hat{T}^{+}\hat{T} = 1$, one may write $\hat{T}$ as,

$$\hat{T} = e^{\frac{i}{\hbar}\alpha(\hat{\vec{r}},t)}$$

It follows that in the position representation the effect of the unitary operator $\hat{T}$ is just a local change of phase of the particle wavefunction,

$$\psi_{new}(\vec{r},t) = \langle \vec{r} \,|\psi_{new}(t)\rangle = \langle \vec{r}\,|\hat{T}|\psi(t)\rangle = e^{\frac{i}{\hbar}\alpha(\vec{r},t)}\psi(\vec{r},t)$$

In order to see how the Hamiltonian transforms under the gauge transformation, we start from the Schrodinger equation,

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle$$

$$\Rightarrow i\hbar\frac{\partial}{\partial t}\hat{T}^{+}\hat{T}|\psi(t)\rangle = \hat{H}(t)\hat{T}^{+}\hat{T}|\psi(t)\rangle$$

$$\Rightarrow i\hbar\frac{\partial\hat{T}^{+}}{\partial t}\hat{T}|\psi(t)\rangle + i\hbar\hat{T}^{+}\frac{\partial}{\partial t}\hat{T}|\psi(t)\rangle = \hat{H}(t)\hat{T}^{+}\hat{T}|\psi(t)\rangle$$

$$\Rightarrow \hbar\hat{T}^{+}\frac{\partial}{\partial t}\hat{T}|\psi(t)\rangle = \left[\hat{H}(t)\hat{T}^{+} - i\hbar\frac{\partial\hat{T}^{+}}{\partial t}\right]\hat{T}|\psi(t)\rangle$$

$$\Rightarrow i\hbar\frac{\partial}{\partial t}\hat{T}|\psi(t)\rangle = \left[\hat{T}\hat{H}(t)\hat{T}^{+} - i\hbar\hat{T}\frac{\partial\hat{T}^{+}}{\partial t}\right]\hat{T}|\psi(t)\rangle$$

$$\Rightarrow i\hbar\frac{\partial}{\partial t}\hat{T}|\psi(t)\rangle = \left[\hat{H}_{new}(t)\right]\hat{T}|\psi(t)\rangle$$

$$\Rightarrow i\hbar\frac{\partial}{\partial t}|\psi_{new}(t)\rangle = \hat{H}_{new}(t)|\psi_{new}(t)\rangle$$

Note that the Schrodinger equation maintains its form under the unitary gauge transformation. Here,

$$\hat{H}_{new}(t) = \hat{T}\hat{H}(t)\hat{T}^{+} - i\hbar\hat{T}\frac{\partial\hat{T}^{+}}{\partial t}$$

We still need to find $\alpha(\hat{\vec{r}},t)$. To find $\alpha(\hat{\vec{r}},t)$, we start from the Coulomb gauge in which,

$$\vec{A}_L(\vec{r},t) = 0$$

$$\Rightarrow \vec{A}(\vec{r},t) = \vec{A}_T(\vec{r},t)$$

and,

$$\left[\hat{r}_k(t),\hat{p}_j(t)\right] = i\hbar\delta_{kj}$$

$$\hat{\vec{p}} \quad \Longleftrightarrow \quad \frac{\hbar}{i}\nabla$$

and we make a gauge transformation,

$$\vec{A}_{new}(\vec{r},t) = \vec{A}_T(\vec{r},t) + \vec{\nabla}F(\vec{r},t)$$

$$\phi_{new}(\vec{r},t) = \phi(\vec{r},t) - \frac{\partial}{\partial t}F(\vec{r},t)$$

The unitary operator transforming the state is,

$$\hat{T} = e^{\frac{i}{\hbar}\alpha(\hat{\vec{r}},t)}$$

Note that since,

$$\left[\hat{r}_k(t),\hat{p}_j(t)\right] = i\hbar\delta_{kj}$$

we have,

$$\hat{T}\,\hat{\vec{p}}\,\hat{T}^{+} = \hat{\vec{p}} - \nabla\alpha(\hat{\vec{r}},t)$$

The transformed Hamiltonian becomes,

$$\hat{H}_{new}(t) = \frac{\left[\hat{\vec{p}} - q\vec{A}_T(\hat{\vec{r}},t) - \nabla\alpha(\hat{\vec{r}},t)\right]^2}{2m} + q\,\phi(\hat{\vec{r}},t) - \frac{\partial\alpha(\hat{\vec{r}},t)}{\partial t}$$

We require that the correct value of $\alpha(\hat{\vec{r}},t)$ should result in the transformed Hamiltonian having the same form as the original Hamiltonian. If,

$$\alpha(\hat{\vec{r}},t) = qF(\hat{\vec{r}},t)$$

then the transformed Hamiltonian will have the same form as the original Hamiltonian,

$$\hat{H}_{new}(t) = \frac{\left[\hat{\vec{p}} - q\vec{A}_{new}(\hat{\vec{r}},t)\right]^2}{2m} + q\,\phi_{new}(\hat{\vec{r}},t)$$

In the position representation the transformed Hamiltonian is,

$$\langle\vec{r}\,|\hat{H}_{new}(t)|\psi_{new}(t)\rangle = \left[\frac{\left[\frac{\hbar}{i}\nabla - q\vec{A}_T(\vec{r},t) - q\nabla F(\vec{r},t)\right]^2}{2m} + q\,\phi(\vec{r},t) - q\frac{\partial F(\vec{r},t)}{\partial t}\right]\psi_{new}(\vec{r},t)$$

$$= \left[\frac{\left[\frac{\hbar}{i}\nabla - q\vec{A}_{new}(\vec{r},t)\right]^2}{2m} + q\,\phi_{new}(\vec{r},t)\right]\psi_{new}(\vec{r},t)$$

Since the first term in the transformed Hamiltonian above represents the kinetic energy of the particle, we must have,

$$m\hat{\vec{v}}(t) = \hat{\vec{p}}(t) - q\vec{A}_{new}(\hat{\vec{r}}(t),t)$$

This suggests that in the new gauge the kinetic momentum of the particle must be related to the canonical momentum by the above equation. In addition, the form of the Hamiltonian in the position representation suggests that the canonical momentum $\hat{\vec{p}}(t)$ is still represented by the gradient operator in the transformed Hamiltonian. Therefore, in the Schrodinger picture,

$$\hat{\vec{p}} \iff \frac{\hbar}{i}\nabla$$

and, therefore, $\hat{\vec{p}}$ satisfies the familiar commutation relation with the particle position operator,

$$\left[\hat{r}_k, \hat{p}_j\right] = i\hbar\delta_{kj}$$

The above arguments show that in any arbitrary gauge the Hamiltonian for a charged particle in a classical electromagnetic field can be written as,

$$\hat{H}(t) = \frac{\left[\hat{\vec{p}} - q\vec{A}(\hat{\vec{r}},t)\right]^2}{2m} + q\,\phi(\hat{\vec{r}},t)$$

where the kinetic and canonical momenta are related by,

$$m\vec{v}(t) = \vec{p}(t) - q\vec{A}(\vec{r}(t),t)$$

and the canonical momentum in the position representation is,

$$\hat{\vec{p}} \iff \frac{\hbar}{i}\nabla$$

Note that in any gauge,

$$\hat{\vec{p}}(t) = m\hat{\vec{v}}(t) + q\vec{A}(\hat{\vec{r}}(t),t)$$

However, only in the Coulomb gauge can the canonical momentum be identified as the sum of the kinetic momentum of the particle and the longitudinal momentum of the field. In a different gauge, the canonical momentum has no such simple interpretation.

Gauge Invariance of the Kinetic Momentum: The term "gauge invariance" could mean either "form invariance" or "value invariance" or both. An operator representing a physical observable must not only have form invariance under a gauge transformation but its value should also not change under a gauge transformation. The kinetic momentum of a particle, for example, possesses both these attributes. To see this consider its expectation value,

$$\langle\psi(t)|m\hat{\vec{v}}|\psi(t)\rangle = \langle\psi(t)|\left[\hat{\vec{p}} - q\vec{A}(\vec{r},t)\right]|\psi(t)\rangle$$

Suppose one now performs a gauge transformation,

$$\vec{A}_{new}(\vec{r},t) = \vec{A}(\vec{r},t) + \vec{\nabla}F(\vec{r},t)$$

$$\phi_{new}(\vec{r},t) = \phi(\vec{r},t) - \frac{\partial}{\partial t}F(\vec{r},t)$$

It follows that,

$$\begin{aligned}
&\langle\psi(t)|\left[\hat{\vec{p}} - q\vec{A}(\vec{r},t)\right]|\psi(t)\rangle \\
&= \langle\psi(t)|\hat{T}^{+}\hat{T}\left[\hat{\vec{p}} - q\vec{A}(\vec{r},t)\right]\hat{T}^{+}\hat{T}|\psi(t)\rangle \\
&= \langle\psi_{new}(t)|\hat{T}\left[\hat{\vec{p}} - q\vec{A}(\vec{r},t)\right]\hat{T}^{+}|\psi_{new}(t)\rangle \\
&= \langle\psi_{new}(t)|\left[\hat{\vec{p}} - q\vec{A}(\vec{r},t) - q\nabla F(\vec{r},t)\right]|\psi_{new}(t)\rangle \\
&= \langle\psi_{new}(t)|\left[\hat{\vec{p}} - q\vec{A}_{new}(\vec{r},t)\right]|\psi_{new}(t)\rangle
\end{aligned}$$

Kinetic Momentum Commutation Relations: In the absence of any electromagnetic field, all components of the kinetic momentum commute,

$$\left[m\hat{v}_k(t), m\hat{v}_j(t)\right] = 0$$

In the presence of an electromagnetic field one obtains,

$$\left[m\hat{v}_k(t), m\hat{v}_j(t)\right] = i\hbar q\mu_o \sum_r \varepsilon_{kjr}\ \hat{H}_r\left(\hat{\vec{r}}(t), t\right)$$

Therefore, in the presence of a magnetic field, different components of the kinetic momentum no longer commute!

The Puzzle of the Hamiltonian: Consider the Hamiltonian,

$$\hat{H}(t) = \frac{\left[\hat{\vec{p}} - q\vec{A}(\hat{\vec{r}},t)\right]^2}{2m} + q\,\phi(\hat{\vec{r}},t)$$

The Hamiltonian governs the time-evolution of the quantum state of the particle via the Schrodinger equation,

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle$$

Under a gauge transformation, the transformed Hamiltonian was found to be,

$$\hat{H}_{new}(t) = \hat{T}\hat{H}(t)\hat{T}^{+} - i\hbar\hat{T}\frac{\partial\hat{T}^{+}}{\partial t} = \frac{\left[\hat{\vec{p}} - q\vec{A}_{new}(\hat{\vec{r}},t)\right]^2}{2m} + q\,\phi_{new}(\hat{\vec{r}},t)$$

and,

$$i\hbar \frac{\partial}{\partial t} |\psi_{new}(t)\rangle = \hat{H}_{new}(t)|\psi_{new}(t)\rangle$$

Therefore, the Hamiltonian is certainly form invariant under a gauge transformation. However, its expectation value changes under a gauge transformation,

$$\begin{aligned}
&\langle \psi(t)|\hat{H}(t)|\psi(t)\rangle \\
&= \langle \psi(t)|\hat{T}^{+}\hat{T}\hat{H}(t)\hat{T}^{+}\hat{T}|\psi(t)\rangle \\
&= \langle \psi_{new}(t)|\hat{T}\hat{H}(t)\hat{T}^{+}|\psi_{new}(t)\rangle \\
&= \langle \psi_{new}(t)|\hat{H}_{new}(t)|\psi_{new}(t)\rangle + \langle \psi_{new}(t)|i\hbar\hat{T}\frac{\partial \hat{T}^{+}}{\partial t}|\psi_{new}(t)\rangle \\
&\neq \langle \psi_{new}(t)|\hat{H}_{new}(t)|\psi_{new}(t)\rangle
\end{aligned}$$

Therefore, the Hamiltonian could not represent the energy of the particle, which is a physical observable. In any gauge, if one subtracts the contribution coming from the scalar potential of the electromagnetic field then one obtains the operator for the energy of the particle. In the present case, the energy of the particle is given by the kinetic energy term in the Hamiltonian,

$$\frac{\left[\hat{\vec{p}} - q\vec{A}(\hat{\vec{r}},t)\right]^2}{2m}$$

which is both form invariant and value invariant under a gauge transformation.

11.1.4 The A.p Interaction

The discussion in the previous Section showed that the operator for the Hamiltonian for a particle in a classical electromagnetic field in the Schrodinger picture is,

$$\hat{H}(t) = \frac{\left[\hat{\vec{p}} - q\vec{A}(\hat{\vec{r}},t)\right]^2}{2m} + q\,\phi\left(\hat{\vec{r}},t\right)$$

Note the distinction between the particle position operator $\hat{\vec{r}}$ and the dummy co-ordinate $\vec{r}$. One can add to the above Hamiltonian an external time-independent potential $qV\left(\hat{\vec{r}}\right)$ that represents the confining potential of a single atom or that of a crystal in which the particle resides,

$$\hat{H}(t) = \frac{\left[\hat{\vec{p}} - q\vec{A}(\hat{\vec{r}},t)\right]^2}{2m} + q\,\phi\left(\hat{\vec{r}},t\right) + qV\left(\hat{\vec{r}}\right)$$

Note that the operator for the energy of the particle is in this case,

$$\frac{\left[\hat{\vec{p}} - q\vec{A}(\hat{\vec{r}},t)\right]^2}{2m} + qV\left(\hat{\vec{r}}\right)$$

One can always choose a gauge in which the scalar potential ϕ of the electromagnetic field is zero and this will be assumed from now onwards. In this gauge, the Hamiltonian operator is the same as the operator for the energy of the particle.

One can divide the Hamiltonian into two parts as shown below,

$$\begin{aligned}
&\hat{H}(t) = \hat{H}_P + \hat{H}_I(t) \\
&\hat{H}_P = \frac{\hat{\vec{p}}^2}{2m} + V(\hat{\vec{r}}) \\
&\hat{H}_I(t) = -\frac{q\hat{\vec{p}}.\vec{A}(\vec{r},t)}{2m} - \frac{q\vec{A}(\vec{r},t).\hat{\vec{p}}}{2m} + \frac{q\vec{A}(\vec{r},t).\vec{A}(\vec{r},t)}{2m}
\end{aligned}$$

The interaction between the field and the particle is described by $\hat{H}_I(t)$. The problem with the above Hamiltonian is that $\hat{H}_P$ does not correspond to the total energy of the particle because $\frac{\hat{\vec{p}}^2}{2m}$ is not the operator for the kinetic energy of the particle in the presence of the electromagnetic field. And, therefore, the eigenstates of $\hat{H}_P$ do not correspond to the energy eigenstates of the particle. Below we derive a form of the Hamiltonian more suitable for quantum optics.

11.1.5 The E.r Interaction: The Electric Dipole Hamiltonian

In most cases of practical interest, the particle wave functions (e.g. wave functions of electrons in atoms) are very localized in space compared to the optical modes. Therefore, one may replace $\vec{A}(\hat{\vec{r}},t)$ by $\vec{A}(\vec{r}_o,t)$ in the kinetic energy term appearing in the Hamiltonian. This is called the long-wavelength approximation. In this approximation, the Hamiltonian is,

$$\frac{\left[\hat{\vec{p}} - q\,\vec{A}(\vec{r}_o,t)\right]^2}{2m}$$

Here, $\vec{r}_o$ is the average location of the particle. This gives,

$$\hat{H}(t) = \frac{\left[\hat{\vec{p}} - q\vec{A}(\vec{r}_o,t)\right]^2}{2m} + qV\left(\hat{\vec{r}}\right)$$

One can now perform a gauge transformation to remove the vector potential from the kinetic energy term of the Hamiltonian. Suppose one chooses,

$$F(\hat{\vec{r}},t) = -\vec{A}(\vec{r}_o,t).\left(\hat{\vec{r}} - \vec{r}_o\right)$$

Then,

$$\vec{A}_{new}(\hat{\vec{r}},t) = \vec{A}(\hat{\vec{r}},t) - \vec{A}(\vec{r}_o,t)$$

$$\phi_{new}(\hat{\vec{r}},t) = 0 + \frac{\partial\vec{A}(\vec{r}_o,t)}{\partial t}.\left(\hat{\vec{r}} - \vec{r}_o\right) = -\vec{E}(\vec{r}_o,t).\left(\hat{\vec{r}} - \vec{r}_o\right)$$

Under this gauge transformation the Hamiltonian becomes,

$$\hat{H}(t) = \frac{\hat{\vec{p}}^2}{2m} + qV(\hat{\vec{r}}) - q\vec{E}(\vec{r}_o,t).\left(\hat{\vec{r}} - \vec{r}_o\right)$$

Now we have,

$$\hat{H}(t) = \hat{H}_P + \hat{H}_I(t)$$

$$\hat{H}_P = \frac{\hat{\vec{p}}^2}{2m} + V(\hat{\vec{r}})$$

$$\hat{H}_I(t) = -q\left(\hat{\vec{r}} - \vec{r}_o\right).\vec{E}(\vec{r}_o,t)$$

Since the vector potential at the location of the particle has been “gauged away”, $\hat{H}_P$ now corresponds to the energy of the particle in the absence of the field. The part $\hat{H}_I(t)$ which describes particle-field interaction is now,

$$-q\left(\hat{\vec{r}} - \vec{r}_o\right).\vec{E}(\vec{r}_o,t)$$

The above interaction Hamiltonian is called the electric dipole Hamiltonian.

11.2 A Quantum Mechanical Approach to Particle-Field Interactions

The final step in building a fully quantum mechanical theory for particle-field interaction is the quantization of the electromagnetic field and the inclusion of the field energy in the Hamiltonian. We do this step by step in the discussion that follows. To be general, we include more than one particle in the discussion. We will start from a classical Hamiltonian and then we will quantize both the particle as well as the field dynamics.

The charges of the particles are q_α and their position and velocity vectors are $\vec{r}_\alpha(t)$ and $\vec{v}_\alpha(t)$, respectively. The charge density can be written as $\rho(\vec{r},t) = \sum_\alpha q_\alpha \delta^3(\vec{r} - \vec{r}_\alpha(t))$. The complete particle-field Hamiltonian must include the kinetic energy of the particles,

$$\sum_\alpha \frac{1}{2} m \vec{v}_\alpha(t).\vec{v}_\alpha(t) = \sum_\alpha \frac{\left[\vec{p}_\alpha(t) - q_\alpha \vec{A}(\vec{r}_\alpha(t),t)\right]^2}{2m}$$

The Hamiltonian must also include the energy of the field,

$$\int d^3\vec{r} \left\{ \frac{1}{2} \varepsilon_o \vec{E}(\vec{r},t).\vec{E}(\vec{r},t) + \frac{1}{2} \mu_o \vec{H}(\vec{r},t).\vec{H}(\vec{r},t) \right\}$$

One might think that the Hamiltonian should also include the potential energy term,

$$\sum_\alpha q_\alpha \phi(\vec{r}_\alpha(t),t)$$

where, $\phi(\vec{r},t)$ is the electromagnetic scalar potential. This turns out not be the case. To understand this, we first note that the electric field can be decomposed into its longitudinal and transverse components,

$$\vec{E}(\vec{r},t) = \vec{E}_L(\vec{r},t) + \vec{E}_T(\vec{r},t)$$

where,

$$\nabla \cdot \varepsilon_o \vec{E}_L(\vec{r},t) = \rho(\vec{r},t) \qquad \nabla \times \vec{E}_L(\vec{r},t) = 0$$

$$\Rightarrow \vec{k} \cdot \vec{E}_L(\vec{k},t) = -i \frac{\rho(\vec{k},t)}{\varepsilon_o} \qquad \Rightarrow \vec{E}_L(\vec{k},t) = -i \frac{\hat{k}}{k} \frac{\rho(\vec{k},t)}{\varepsilon_o}$$

So the longitudinal component of the field energy becomes,

$$\int d^3\vec{r} \, \frac{1}{2} \varepsilon_o \vec{E}_L(\vec{r},t).\vec{E}_L(\vec{r},t)$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2} \varepsilon_o \vec{E}_L(-\vec{k},t).\vec{E}_L(\vec{k},t)$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2} \frac{\rho(-\vec{k},t)\rho(\vec{k},t)}{\varepsilon_o k^2}$$

$$= \frac{1}{2} \int d^3\vec{r} \, \frac{\rho(\vec{r},t)\rho(\vec{r}',t)}{4\pi\varepsilon_o |\vec{r} - \vec{r}'|}$$

$$= \frac{1}{2} \sum_{\alpha\beta} \frac{q_\alpha q_\beta}{4\pi\varepsilon_o |\vec{r}_\alpha(t) - \vec{r}_\beta(t)|}$$

The longitudinal field energy is just the total Coulomb interaction energy of the charged particles, including Coulomb self-interaction. Since we have already included this interaction energy in the term,

$$\int d^3\vec{r} \left\{ \frac{1}{2} \varepsilon_o \vec{E}(\vec{r},t).\vec{E}(\vec{r},t) + \frac{1}{2} \mu_o \vec{H}(\vec{r},t).\vec{H}(\vec{r},t) \right\}$$

the potential energy term mentioned above is not needed. The classical expression for the total particle-field Hamiltonian is therefore,

$$H = \sum_{\alpha} \frac{\left[\vec{p}_\alpha(t) - q_\alpha \vec{A}(\vec{r}_\alpha(t),t)\right]^2}{2m} + \int d^3\vec{r} \left\{ \frac{1}{2} \varepsilon_o \vec{E}(\vec{r},t).\vec{E}(\vec{r},t) + \frac{1}{2}\mu_o \vec{H}(\vec{r},t).\vec{H}(\vec{r},t) \right\}$$

It can be shown that the above Hamiltonian represents the total energy of the particle-field system and is time-independent as the total energy of a system of particles and field ought to be.

In most cases of practical interest, one is interested in the dynamics of a single particle (e.g. electron in an atom), and one would like to exclude Coulomb self-interactions from the description since they tend to generate infinities (but can be renormalized away if one is careful). In such cases, one may write the above Hamiltonian for a single particle and the field as,

$$H = \frac{\left[\vec{p}(t) - q\vec{A}(\vec{r}(t),t)\right]^2}{2m} + qV(\vec{r}(t)) + \int d^3\vec{r} \left\{ \frac{1}{2} \varepsilon_o \vec{E}_T(\vec{r},t).\vec{E}_T(\vec{r},t) + \frac{1}{2}\mu_o \vec{H}(\vec{r},t).\vec{H}(\vec{r},t) \right\}$$

where $V(\vec{r})$ is the static Coulomb potential of all the other particles whose dynamics have not been included in the Hamiltonian, and these other particles are assumed to be stationary. Note also that only the transverse electric field appears in the expression for the field energy. It should be noted that every term in the above Hamiltonian is gauge invariant. Next, we will quantize the above Hamiltonian.

The quantization of the particle dynamics follows the same path as discussed earlier. We postulate the following equal-time commutation rules between the particle position and the canonical momentum operators,

$$\left[\hat{r}_k(t), \hat{p}_j(t)\right] = i\hbar\delta_{kj}$$

Before we quantize the field we need to select a gauge and the most convenient choice is the Coulomb gauge in which the longitudinal vector potential is zero. The electromagnetic field can be quantized following the steps discussed in Chapter 5. The field variables become operators. The resulting fully quantum mechanical Hamiltonian is,

$$\hat{H} = \frac{\left[\hat{\vec{p}}(t) - q\hat{\vec{A}}_T(\vec{r}(t),t)\right]^2}{2m} + qV(\hat{\vec{r}}(t)) + \int d^3\vec{r} \left\{ \frac{1}{2} \varepsilon_o \hat{\vec{E}}_T(\vec{r},t).\hat{\vec{E}}_T(\vec{r},t) + \frac{1}{2}\mu_o \hat{\vec{H}}(\vec{r},t).\hat{\vec{H}}(\vec{r},t) \right\}$$

In the Schrodinger picture, the Hamiltonian is,

$$\hat{H} = \frac{\left[\hat{\vec{p}} - q\hat{\vec{A}}_T(\hat{\vec{r}})\right]^2}{2m} + qV(\hat{\vec{r}}) + \int d^3\vec{r} \left\{ \frac{1}{2} \varepsilon_o \hat{\vec{E}}_T(\vec{r}).\hat{\vec{E}}_T(\vec{r}) + \frac{1}{2}\mu_o \hat{\vec{H}}(\vec{r}).\hat{\vec{H}}(\vec{r}) \right\}$$

In the Schrodinger picture, the Hamiltonian is completely time-independent because the time development of every operator is now included in the description.

11.2.1 The E.r Interaction: The Electric Dipole Hamiltonian

Consider the Hamiltonian,

$$\hat{H} = \frac{\left[\hat{\vec{p}} - q\hat{\vec{A}}_T(\hat{\vec{r}})\right]^2}{2m} + qV(\hat{\vec{r}}) + \int d^3\vec{r} \left\{ \frac{1}{2} \varepsilon_o \hat{\vec{E}}_T(\vec{r}).\hat{\vec{E}}_T(\vec{r}) + \frac{1}{2}\mu_o \hat{\vec{H}}(\vec{r}).\hat{\vec{H}}(\vec{r}) \right\}$$

In most cases of practical interest, the particle wavefunction (e.g. wavefunctions of an electron in an atom) is well localized in space compared to the radiation modes. Therefore, one may replace the operator $\hat{\vec{A}}_T(\hat{\vec{r}},t)$ by $\hat{\vec{A}}_T(\vec{r}_o,t)$ in the kinetic energy term appearing in the Hamiltonian, where $\vec{r}_o$ is the average position of the particle. This is called the long-wavelength approximation. In this approximation, the Hamiltonian in the Schrodinger picture is,

$$\hat{H} = \frac{\left[\hat{\vec{p}} - q\hat{\vec{A}}_T(\vec{r}_o)\right]^2}{2m} + qV(\hat{\vec{r}}) + \int d^3\vec{r}\left\{\frac{1}{2}\,\varepsilon_o\,\hat{\vec{E}}_T(\vec{r}).\hat{\vec{E}}_T(\vec{r}) + \frac{1}{2}\mu_o\hat{\vec{H}}(\vec{r}).\hat{\vec{H}}(\vec{r})\right\}$$

Now we will perform a gauge transformation to get the electric dipole Hamiltonian. Suppose $|\psi\rangle$ is a quantum state of the particle-field system. Under a gauge transformation, represented by the unitary time-independent operator $\hat{T}$, the new state is,

$$|\psi_{new}\rangle = \hat{T}|\psi\rangle$$

Since the Hamiltonian now represents the total energy of the particle-field system, its expectation value must not be gauge-independent,

$$\langle\psi|\hat{H}|\psi\rangle = \langle\psi_{new}|\hat{H}_{new}|\psi_{new}\rangle = \langle\psi|\hat{T}^+\hat{T}\hat{H}\hat{T}^+\hat{T}|\psi\rangle$$

which implies,

$$\hat{H}_{new} = \hat{T}\hat{H}\hat{T}^+$$

Following the semi-classical approach in Section 11.1.5, we assume,

$$\hat{T} = e^{-\frac{i}{\hbar}q\hat{\vec{A}}_T(\vec{r}_o,t).\left(\hat{\vec{r}}-\vec{r}_o\right)}$$

The calculation of the new Hamiltonian is a bit tricky since the operator $\hat{T}$ contains both the particle operators as well as the field creation and destruction operators. The final answer is,

$$\begin{aligned}\hat{H}_{new} &= \hat{T}\hat{H}\hat{T}^+\\ &= \frac{\hat{\vec{p}}^2}{2m} + qV(\hat{\vec{r}}) - q\left(\hat{\vec{r}} - \vec{r}_o\right).\hat{\vec{E}}_T(\vec{r}_o) + \int d^3\vec{r}\left\{\frac{1}{2}\,\varepsilon_o\,\hat{\vec{E}}_T(\vec{r}).\hat{\vec{E}}_T(\vec{r}) + \frac{1}{2}\mu_o\hat{\vec{H}}(\vec{r}).\hat{\vec{H}}(\vec{r})\right\}\\ &\quad + \int\frac{d^3\vec{k}}{(2\pi)^3}\sum_j\ \frac{1}{2\varepsilon_o}\left[\vec{\varepsilon}_j\left(\vec{k}\right).\left(\hat{\vec{r}}-\vec{r}_o\right)\right]^2\end{aligned}$$

The last term represents the dipole self-energy of the particle. It seems to diverge because the long wavelength approximation fails for radiation modes with very large wavevectors. In what follows, we will ignore the dipole self-energy term and restrict ourselves to the Hamiltonian,

$$\hat{H} = \frac{\hat{\vec{p}}^2}{2m} + qV(\hat{\vec{r}}) - q\left(\hat{\vec{r}} - \vec{r}_o\right).\hat{\vec{E}}_T(\vec{r}_o) + \int d^3\vec{r}\left\{\frac{1}{2}\,\varepsilon_o\,\hat{\vec{E}}_T(\vec{r}).\hat{\vec{E}}_T(\vec{r}) + \frac{1}{2}\mu_o\hat{\vec{H}}(\vec{r}).\hat{\vec{H}}(\vec{r})\right\}$$

The above Hamiltonian has three parts,

$$\hat{H} = \hat{H}_P + \hat{H}_I + \hat{H}_F$$

$$\hat{H}_P = \frac{\hat{\vec{p}}^2}{2m} + V(\hat{\vec{r}})$$

$$\hat{H}_I = -q\left(\hat{\vec{r}} - \vec{r}_o\right).\hat{\vec{E}}_T(\vec{r}_o)$$

$$\hat{H}_F = \int d^3\vec{r}\left\{\frac{1}{2}\,\varepsilon_o\hat{\vec{E}}_T(\vec{r}).\hat{\vec{E}}_T(\vec{r}) + \frac{1}{2}\mu_o\hat{\vec{H}}(\vec{r}).\hat{\vec{H}}(\vec{r})\right\}$$

One may also write the above Hamiltonian in the Heisenberg picture,

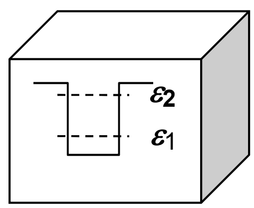
$$\hat{H}(t) = \hat{H}_P(t) + \hat{H}_I(t) + \hat{H}_F(t)$$
$$\hat{H}_P(t) = \frac{\hat{\vec{p}}^2(t)}{2m} + V(\hat{\vec{r}}(t))$$
$$\hat{H}_I(t) = -q\left(\hat{\vec{r}}(t) - \vec{r}_o\right).\hat{\vec{E}}_T(\vec{r}_o,t)$$
$$\hat{H}_F(t) = \int d^3\vec{r}\left\{\frac{1}{2}\,\varepsilon_o\hat{\vec{E}}_T(\vec{r},t).\hat{\vec{E}}_T(\vec{r},t) + \frac{1}{2}\mu_o\hat{\vec{H}}(\vec{r},t).\hat{\vec{H}}(\vec{r},t)\right\}$$

The above Hamiltonian will be used in this Chapter. Note that the electric field appearing in the above Hamiltonian is completely transverse. We will drop the "T" subscript in the remainder of this Chapter.

11.3 A Two-Level System Interacting with Cavity Radiation

Consider the familiar two level system inside a closed cavity which supports a single radiation mode of frequency ω_o.

Cavity



The Hamiltonian is,

$$\hat{H} = \hat{H}_P + \hat{H}_I + \hat{H}_F$$

where,

$$\hat{H}_P = \frac{\hat{\vec{p}}^2}{2m} + qV\left(\hat{\vec{r}}\right) = \varepsilon_1|e_1\rangle\langle e_1| + \varepsilon_2|e_2\rangle\langle e_2|$$
$$= \varepsilon_1\hat{N}_1 + \varepsilon_2\hat{N}_2$$
$$\hat{H}_F = \hbar\omega_o\left(\hat{a}^+\hat{a} + \frac{1}{2}\right)$$
$$\hat{H}_I = -q\left(\hat{\vec{r}} - \vec{r}_o\right)\cdot\hat{\vec{E}}(\vec{r}_o)$$

The interaction part of the Hamiltonian $\hat{H}_I$ can be written in a more suitable form. Recall that the electric field operator (in the Schrodinger picture) is,

$$\hat{\vec{E}}(\vec{r}) = i\sqrt{\frac{\hbar\omega_o}{2\varepsilon_o\varepsilon}}\left(\hat{a} - \hat{a}^+\right)\vec{U}(\vec{r})$$

Therefore,

$$\hat{\vec{E}}(\vec{r}_o) = i\sqrt{\frac{\hbar\omega_o}{2\varepsilon_o\varepsilon}}\left(\hat{a} - \hat{a}^+\right)\vec{U}(\vec{r}_o)$$

The interaction part of the Hamiltonian can be written as,

$$\hat{H}_I = \hat{1}_{\text{particle}}\,\hat{H}_I\,\hat{1}_{\text{particle}}$$
$$= \left\{|e_1\rangle\langle e_1| + |e_2\rangle\langle e_2|\right\}\hat{H}_I\left\{|e_1\rangle\langle e_1| + |e_2\rangle\langle e_2|\right\}$$

We assume that the energy eigenstates of $\hat{H}_P$ have a definite parity and therefore,

$$\langle e_2|\hat{H}_I|e_2\rangle = \langle e_1|\hat{H}_I|e_1\rangle = 0$$

It follows that,

$$\hat{H}_I = \left[k|e_2\rangle\langle e_1| - k^*|e_1\rangle\langle e_2| \right] \left(\hat{a} - \hat{a}^+\right)$$
$$= \left(k\hat{\sigma}_+ - k^*\hat{\sigma}_-\right)\left(\hat{a} - \hat{a}^+\right)$$

where,

$$k = -q\, i\sqrt{\frac{\hbar\omega_o}{2\varepsilon_o\varepsilon}}\,\vec{U}(\vec{r}_o)\,.\,\langle e_2|\left(\hat{\vec{r}} - \vec{r}_o\right)|e_1\rangle$$
$$\hat{\sigma}_+ = |e_2\rangle\langle e_1|$$
$$\hat{\sigma}_- = |e_1\rangle\langle e_2|$$

11.3.1 Hilbert Space

The appropriate Hilbert space now consists of all states of the type,

$$|\psi\rangle = |\phi\rangle_{\text{particle}} \otimes |\chi\rangle_{\text{field}}$$

For example, for a two-level system in state $|e_2\rangle$ and 2 photons in the cavity, the state of the combined system is written as, $|\psi\rangle = |e_2\rangle \otimes |2\rangle$.

11.3.2 Completeness Relation

The completeness relation is,

$$\underbrace{\left\{|e_1\rangle\langle e_1| + |e_2\rangle\langle e_2|\right\}}_{\leftarrow \text{matter part} \rightarrow} \otimes \underbrace{\sum_n |n\rangle\langle n|}_{\leftarrow \text{field part} \rightarrow} = \hat{1}_{\text{particle+field}} \quad .$$

11.3.3 An Isolated Two-Level System inside a Lossless Cavity

Consider a two level system coupled to a single radiation mode inside a lossless optical cavity. The Hamiltonian is,

$$\hat{H} = \varepsilon_1\hat{N}_1 + \varepsilon_2\hat{N}_2 + \hbar\omega_o\,\hat{a}^+\hat{a} + \left(k\hat{\sigma}_+ - k^*\hat{\sigma}_-\right)\left(\hat{a} - \hat{a}^+\right)$$
$$\approx \varepsilon_1\hat{N}_1 + \varepsilon_2\hat{N}_2 + \hbar\omega_o\,\hat{a}^+\hat{a} + \left(k\,\hat{\sigma}_+\hat{a} + k^*\hat{a}^+\hat{\sigma}_-\right)$$

The second line follows from the rotating wave approximation. The above Hamiltonian represents a time-independent completely isolated system. It must have eigenstates and eigenenergies of the form,

$$\hat{H}|E\rangle = E|E\rangle$$

In order to find the eigenstates and the corresponding eigenenergies consider the state $|e_2\rangle \otimes |n\rangle$ (i.e. particle in upper state $|e_2\rangle$ and field in number state n) and check if it is an eigenstate of the Hamiltonian,

$$\hat{H}|e_2\rangle\otimes|n\rangle = (\varepsilon_2 + n\hbar\omega_o)|e_2\rangle\otimes|n\rangle + k^*\sqrt{n+1}|e_1\rangle\otimes|n+1\rangle$$

Thus, $|e_2\rangle\otimes|n\rangle$ is not an eigenstate since $\hat{H}$ acting an $|e_2\rangle\otimes|n\rangle$ generates a new state $|e_1\rangle\otimes|n+1\rangle$. Next, try $|e_1\rangle\otimes|n+1\rangle$,

$$\hat{H}|e_1\rangle\otimes|n+1\rangle = (\varepsilon_1 + (n+1)\hbar\omega_o)|e_1\rangle\otimes|n+1\rangle + k\sqrt{n+1}|e_2\rangle\otimes|n\rangle$$

This also fails. Now, try the linear super position state $|\phi\rangle = a|e_2\rangle\otimes|n\rangle + b|e_1\rangle\otimes|n+1\rangle$

$$\hat{H}|\phi\rangle = a\{(\varepsilon_2 + n\hbar\omega_o)|e_2\rangle \otimes |n\rangle + k^*\sqrt{n+1}|e_1\rangle \otimes |n+1\rangle\}$$
$$+ b\{(\varepsilon_1 + (n+1)\hbar\,\omega_o)|e_1\rangle \otimes |n+1\rangle + k\sqrt{n+1}|e_2\rangle \otimes |n\rangle\}$$
$$= \left[a(\varepsilon_2 + n\hbar\omega_o) + bk\sqrt{n+1}\right]|e_2\rangle \otimes |n\rangle + \left[ak^*\sqrt{n+1} + b(\varepsilon_1 + (n+1)\hbar\,\omega_o)\right]|e_1\rangle \otimes |n+1\rangle$$

If we want $|\phi\rangle$ to be an eigenstate of $\hat{H}$, with eigenvalue E, then we must have $\hat{H}|\phi\rangle = E|\phi\rangle$. This implies the following two equations,

$$a(\varepsilon_2 + n\hbar\omega_o) + b\,k\sqrt{n+1} = Ea$$
$$a\,k^*\sqrt{n+1} + b(\varepsilon_1 + (n+1)\hbar\omega_o) = Eb$$

Proper normalization (i.e. $\langle\phi|\phi\rangle = 1$) requires $|a|^2 + |b|^2 = 1$. The two equations above can be written in a matrix form as,

$$\begin{bmatrix} \varepsilon_2 + n\hbar\omega_o & k\sqrt{n+1} \\ k^*\sqrt{n+1} & \varepsilon_1 + (n+1)\hbar\omega_o \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = E\begin{bmatrix} a \\ b \end{bmatrix}$$

Let the detuning Δ be defined as before,

$$\Delta = \varepsilon_2 - (\varepsilon_1 + \hbar\omega_o) = \Delta\varepsilon - \hbar\omega_o$$

Case of Zero Detuning: Assuming $\Delta = 0$, the matrix above gives two eigenvalues,

$$E_-(n) = \varepsilon_2 + n\hbar\omega_o - |k|\sqrt{n+1}$$
$$E_+(n) = \varepsilon_2 + n\hbar\omega_o + |k|\sqrt{n+1}$$

The corresponding eigenvectors are,

$$|\phi_-(n)\rangle = \frac{1}{\sqrt{2}}\left[|e_2\rangle \otimes |n\rangle - \frac{k^*}{|k|}|e_1\rangle \otimes |n+1\rangle\right]$$
$$|\phi_+(n)\rangle = \frac{1}{\sqrt{2}}\left[|e_2\rangle \otimes |n\rangle + \frac{k^*}{|k|}|e_1\rangle \otimes |n+1\rangle\right]$$

For $n = 0,1,2,3,.....$ the above pairs of states and the corresponding eigenenergies constitute the entire set of eigenstates and eigenenergies of the Hamiltonian $\hat{H}$. The only remaining eigenstate is $|e_1\rangle \otimes |0\rangle$ with energy ε_1. This is the ground state of the system.

Quantum Rabi Oscillations: Suppose one prepares an initial state with the particle in the energy level $|e_2\rangle$ and n photons in the cavity. Therefore, at time $(t = 0)$,

$$|\psi(t=0)\rangle = |e_2\rangle \otimes |n\rangle = \frac{1}{\sqrt{2}}\left[|\phi_-(n)\rangle + |\phi_+(n)\rangle\right]$$

We need to find $|\psi(t)\rangle$. We start from,

$$|\psi(t)\rangle = e^{-i\frac{\hat{H}}{\hbar}t}|\psi(t=0)\rangle$$

$$= \frac{1}{\sqrt{2}} e^{-\frac{i}{\hbar}(\varepsilon_2 + n\hbar\omega_o)t}\left[e^{+i\frac{|k|}{\hbar}\sqrt{n+1}\,t}|\phi_-(n)\rangle + e^{-i\frac{|k|}{\hbar}\sqrt{n+1}\,t}|\phi_+(n)\rangle \right]$$

$$= e^{-\frac{i}{\hbar}(\varepsilon_2 + n\hbar\omega_o)t}\left[\cos\left(\frac{|k|}{\hbar}\sqrt{n+1}\,t\right)|e_2\rangle\otimes|n\rangle - i\frac{k^*}{|k|}\sin\left(\frac{|k|}{\hbar}\sqrt{n+1}\,t\right)|e_1\rangle\otimes|n+1\rangle \right]$$

The state of the system oscillates between $|e_2\rangle\otimes|n\rangle$ and $|e_1\rangle\otimes|n+1\rangle$, and,

$$\left|\langle e_2|\otimes\langle n|\psi(t)\rangle\right|^2 = \cos^2\left(\frac{|k|\sqrt{n+1}}{\hbar}t\right)$$

$$\left|\langle e_1|\otimes\langle n+1|\psi(t)\rangle\right|^2 = \sin^2\left(\frac{|k|\sqrt{n+1}}{\hbar}t\right)$$

The particle in the upper energy level emits a photon and then reabsorbs it after some time. Comparing the above result to semiclassical result obtained earlier where a two-level system subjected to classical radiation exhibited population oscillations that had the time dependence given by,

$$\cos^2\left(\frac{\Omega_R}{2}t\right)$$

we can conclude that,

$$\Omega_R = 2\frac{|k|}{\hbar}\sqrt{n+1}$$

The semiclassical Rabi frequency depended on the strength of the classical field. In the fully quantum result, we have the Rabi frequency proportional to the square root of the number of photons in the field.

Question: What if $|\psi(t=0)\rangle = |e_2\rangle\otimes|\alpha\rangle$ $\{|\alpha\rangle = \text{coherent state}\}$. What is $|\psi(t)\rangle$?

Vacuum Rabi Oscillations: Consider the case when $|\psi(t=0)\rangle = |e_2\rangle\otimes|0\rangle$, i.e. the field is initially in the vacuum state. We get,

$$|\psi(t)\rangle = e^{-i\frac{\varepsilon_2}{\hbar}t}\left[\cos\left(\frac{|k|}{\hbar}t\right)|e_2\rangle\otimes|0\rangle - i\frac{k^*}{|k|}\sin\left(\frac{|k|}{\hbar}t\right)|e_1\rangle\otimes|1\rangle \right]$$

These are what are called vacuum Rabi oscillations. The vacuum Rabi frequency is $2|k|/\hbar$ (assuming $\Delta = 0$). In this case, the particle in the upper energy level emits a photon via spontaneous emission and then reabsorbs it after some time.

Cavity Rabi Splitting: A system consisting of a two-level inside a lossless cavity in the non-interacting case (i.e. no particle-field interaction) has the following energy eigenstates and eigenenergies (assuming $\Delta = 0$),

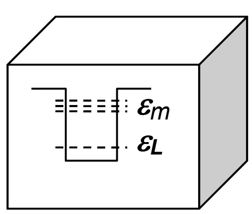
Energies	States
ε_1	$\lvert e_1\rangle\otimes\lvert 0\rangle$
ε_2	$\lvert e_1\rangle\otimes\lvert 1\rangle$
ε_2	$\lvert e_2\rangle\otimes\lvert 0\rangle$
$\varepsilon_2+\hbar\omega_o$	$\lvert e_1\rangle\otimes\lvert 2\rangle$
$\varepsilon_2+\hbar\omega_o$	$\lvert e_2\rangle\otimes\lvert 1\rangle$
$\vdots$	
$\varepsilon_2+n\hbar\omega_o$	$\lvert e_1\rangle\otimes\lvert n+1\rangle$
$\varepsilon_2+n\hbar\omega_o$	$\lvert e_2\rangle\otimes\lvert n\rangle$
$\vdots$	

When the interaction between the particle and the field is present the energy spectrum is as follows,

Energies	States
ε_1	$\lvert e_1\rangle\otimes\lvert 0\rangle$
$\varepsilon_2-\lvert k\rvert$	$\lvert\varphi_-(0)\rangle$
$\varepsilon_2+\lvert k\rvert$	$\lvert\varphi_+(0)\rangle$
$\varepsilon_2+\hbar\omega_o-\lvert k\rvert\sqrt{2}$	$\lvert\varphi_-(1)\rangle$
$\varepsilon_2+\hbar\omega_o+\lvert k\rvert\sqrt{2}$	$\lvert\varphi_+(1)\rangle$
$\vdots$	
$\varepsilon_2+n\hbar\omega_o-\lvert k\rvert\sqrt{n+1}$	$\lvert\varphi_-(n)\rangle$
$\varepsilon_2+n\hbar\omega_o+\lvert k\rvert\sqrt{n+1}$	$\lvert\varphi_+(n)\rangle$
$\vdots$	

The interaction lifts the degeneracy between the states $\lvert\varepsilon_2\rangle\otimes\lvert n\rangle$ and $\lvert\varepsilon_1\rangle\otimes\lvert n+1\rangle$ and splits the two degenerate energies $\{\varepsilon_2+n\hbar\omega_o,\varepsilon_1+(n+1)\hbar\omega_o\}$ into $\{\varepsilon_2+n\hbar\omega_o-\lvert k\rvert\sqrt{n+1},\varepsilon_2+n\hbar\omega_o+\lvert k\rvert\sqrt{n+1}\}$. This is called Rabi splitting. For $n=0$, this is called vacuum Rabi splitting. The magnitude of the vacuum Rabi splitting is $2\lvert k\rvert$.

11.3.4 Optical Transitions and Fermi's Golden Rule



Consider an energy level coupled to a continuum of higher energy levels via interaction with the cavity mode,

$$\hat{H}\approx\varepsilon_L\hat{N}_L+\sum_m\varepsilon_m\hat{N}_m+\hbar\omega_o\,\hat{a}^+\hat{a}+\sum_m\left(k_m\,\hat{\sigma}_{+m}\hat{a}+k_m^*\hat{a}^+\hat{\sigma}_{-m}\right)$$

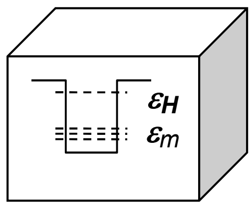
Suppose the particle is initially in the lower energy level and there are n photons inside the cavity. So the initial state of the system is $|e_L\rangle \otimes |n\rangle$. The transition rate to the higher energy levels is given by the Fermi's Golden Rule,

$$R_{\uparrow} = \frac{2\pi}{\hbar}\sum_m \left|\langle n-1| \otimes \langle e_m|\hat{H}_I|e_L\rangle \otimes |n\rangle\right|^2 \delta(\varepsilon_L + \hbar\omega_o - \varepsilon_m)$$

$$= \frac{2\pi}{\hbar}\sum_m \left|\langle n-1| \otimes \langle e_m|k_m\hat{\sigma}_{+m}\hat{a}|e_L\rangle \otimes |n\rangle\right|^2 \delta(\varepsilon_L + \hbar\omega_o - \varepsilon_m)$$

$$= \frac{2\pi}{\hbar}\sum_m |k_m|^2 n\, \delta(\varepsilon_L + \hbar\omega_o - \varepsilon_m)$$

Note that the transition rate is proportional to the number of photons in the cavity.



Now consider an energy level coupled to a continuum of lower energy levels via interaction with the cavity mode,

$$\hat{H} \approx \varepsilon_H \hat{N}_H + \sum_m \varepsilon_m \hat{N}_m + \hbar\omega_o\, \hat{a}^+\hat{a} + \sum_m \left(k_m\, \hat{\sigma}_{+m}\hat{a} + k_m^*\hat{a}^+\hat{\sigma}_{-m}\right)$$

Suppose the particle is initially in the higher energy level and there are n photons inside the cavity. So the initial state of the system is $|e_H\rangle \otimes |n\rangle$. The transition rate to the lower energy levels is given by the Fermi's Golden Rule,

$$R_{\downarrow} = \frac{2\pi}{\hbar}\sum_m \left|\langle n+1| \otimes \langle e_m|\hat{H}_I|e_H\rangle \otimes |n\rangle\right|^2 \delta(\varepsilon_H - \hbar\omega_o - \varepsilon_m)$$

$$= \frac{2\pi}{\hbar}\sum_m \left|\langle n+1| \otimes \langle e_m|k_m^*\hat{a}^+\hat{\sigma}_{-m}|e_H\rangle \otimes |n\rangle\right|^2 \delta(\varepsilon_H - \hbar\omega_o - \varepsilon_m)$$

$$= \frac{2\pi}{\hbar}\sum_m |k_m|^2 (n+1)\, \delta(\varepsilon_H - \hbar\omega_o - \varepsilon_m)$$

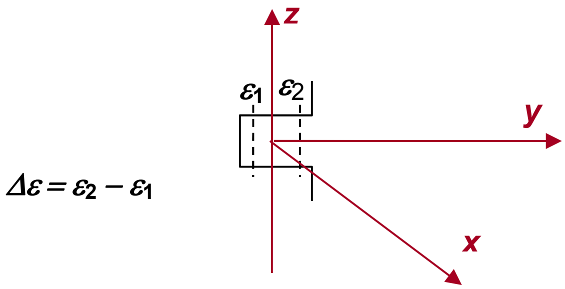
Note that the transition rate is proportional to one plus the number of photons in the cavity. The transition rate includes contributions from both stimulated and spontaneous emission processes. The factor $(n+1)$ that results from the quantum mechanical treatment is an agreement with the result Einstein obtained via thermodynamic arguments well before the quantum theory of radiation was developed.

11.4 Spontaneous Emission Rate for a Two-Level System in Free Space

We have seen in the previous Section that when the final states are a continuum, one can use Fermi's Golden Rule to calculate transition rates. In the examples considered, the photon final state was a single state but the possible final states for the particle belonged to a continuum. Another useful example is when the particle final state is a single state but the photon final states belong to a continuum. This is the case when an atom emits a photon in free space. Consider a two level system interacting with radiation in free space. Recall that in free space the electric field operator can be expanded as,

$$\hat{\vec{E}}(\vec{r},t) = V\int \frac{d^3\vec{k}}{(2\pi)^3}\sum_j i\sqrt{\frac{\hbar\omega_k}{2\varepsilon_o}}\left[\hat{a}_j(\vec{k},t) - \hat{a}_j^+(-\vec{k},t)\right]\frac{e^{i\vec{k}.\vec{r}}}{\sqrt{V}}\hat{\varepsilon}_j(\vec{k})$$

We assume that the two level system is located at the origin and oriented such that only the z-component of the field has a non-zero dipole matrix element.



The Hamiltonian is,

$$\hat{H} \approx \varepsilon_1\hat{N}_1 + \varepsilon_2\hat{N}_2 + V\int\frac{d^3\vec{k}}{(2\pi)^3}\sum_j \hbar\omega_k\ \hat{a}_j^+(\vec{k})\hat{a}_j(\vec{k}) + V\int\frac{d^3\vec{k}}{(2\pi)^3}\sum_j M_j(\vec{k})\hat{\sigma}_+\hat{a}_j(\vec{k}) + M_j^*(\vec{k})\hat{a}_j^+(\vec{k})\hat{\sigma}_-$$

where,

$$M_j(\vec{k}) = -q\, i\sqrt{\frac{\hbar\omega_k}{2\varepsilon_o}}\ \hat{\varepsilon}_j(\vec{k}).\langle e_2|\hat{z}|e_1\rangle\frac{1}{\sqrt{V}}$$

Suppose the particle is initially in the higher energy level and the radiation is sitting in the vacuum state. Therefore,

$$|\psi_{\text{initial}}\rangle = |e_2\rangle\otimes|0\rangle$$

The final state could correspond to any one of the states in which the particle is in the lower energy level and there is one spontaneously emitted photon in any one of the many radiation modes,

$$|\psi_{\text{final}}\rangle = |e_1\rangle\otimes|1\rangle_{\vec{k},j}$$

Since the final states form a continuum one can use Fermi's Golden Rule. The answer is,

$$R_\downarrow = \frac{2\pi}{\hbar}V\int\frac{d^3\vec{k}}{(2\pi)^3}\sum_j \left|{}_{\vec{k},j}\langle 1|\otimes\langle e_2|\hat{H}_I|e_2\rangle\otimes|0\rangle\right|^2\delta(\varepsilon_2-\hbar\omega_k-\varepsilon_1)$$

$$= \frac{2\pi}{\hbar}V\int\frac{d^3\vec{k}}{(2\pi)^3}\sum_j q^2\left(\frac{\hbar\omega_k}{2\varepsilon_o}\right)\left|\hat{\varepsilon}_j(\vec{k}).\langle e_2|\hat{z}|e_1\rangle\right|^2\frac{1}{V}\delta(\varepsilon_2-\hbar\omega_k-\varepsilon_1)$$

$$= \frac{2\pi}{\hbar^2}\frac{q^2\left|\langle e_2|z|e_1\rangle\right|^2}{3}\left(\frac{\Delta\varepsilon}{2\varepsilon_o}\right)D(\omega)\Big|_{\omega=\Delta\varepsilon/\hbar}$$

The quantity $D(\omega)$ is the density of radiation modes (i.e. the number of radiation modes per unit volume per unit frequency interval) in free space at frequency ω and is given by,

$$D(\omega) = \frac{\omega^2}{\pi^2c^3}$$

The spontaneous emission rate in free space is therefore proportional to the photon density of states in free space. The photon density of states can be significantly modified in various microscale structures, such as photonic crystals and microcavities, thereby altering the spontaneous emission rate. In fact, with the help of suitably designed structures the spontaneous emission can be enhanced several fold compared to the spontaneous emission rate in free space or even completely suppressed.

11.5 Dynamics in the Presence of Weak Decoherence for a Two-Level System Coupled to a Cavity Radiation Mode

The analysis in the previous section produced the exact eigenstates of the particle-field Hamiltonian. Decoherence is expected to quickly destroy the linear superposition in the eigenstates,

$$|\phi_-(n)\rangle = \frac{1}{\sqrt{2}}\left[|e_2\rangle \otimes |n\rangle - \frac{k^*}{|k|}|e_1\rangle \otimes |n+1\rangle\right]$$

$$|\phi_+(n)\rangle = \frac{1}{\sqrt{2}}\left[|e_2\rangle \otimes |n\rangle + \frac{k^*}{|k|}|e_1\rangle \otimes |n+1\rangle\right]$$

and the state of the system would collapse into either $|\varepsilon_2\rangle \otimes |n\rangle$ or $|\varepsilon_1\rangle \otimes |n+1\rangle$. In the presence of decoherence, one has two options:

a) Use a suitable basis set to expand the density operator

b) Work in the Heisenberg picture

The former is useful when the decoherence rate is not much faster than the vacuum Rabi splitting $2|k|/\hbar$ (case of weak decoherence) The latter is more generally applicable and useful but is not as transparent as the former.

11.5.1 The Density Matrix Approach

Suppose the density matrix of the system consisting of a two level system interacting with a single radiation mode of a cavity is $\hat{\rho}(t)$. Suppose the initial state is $|e_2\rangle \otimes |n\rangle$. Therefore,

$$\hat{\rho}(t=0) = |e_2\rangle \otimes |n\rangle \langle n| \otimes \langle e_2|$$

The Hamiltonian is,

$$\hat{H} = \varepsilon_1 \hat{N}_1 + \varepsilon_2 \hat{N}_2 + \hbar\omega_o \hat{a}^+\hat{a} + \left(k\,\hat{\sigma}_+\hat{a} + k^*\hat{a}^+\hat{\sigma}_-\right)$$

Given the initial state, it is reasonable to assume that the Hilbert space relevant to the problem consists of only two states, $|e_2\rangle \otimes |n\rangle$ and $|e_1\rangle \otimes |n+1\rangle$. One can therefore express the density operator in the basis consisting of only these two states. Let,

$$|e_2\rangle \otimes |n\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad |e_1\rangle \otimes |n+1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

And in the matrix representation,

$$\hat{\rho}(t=0) = |e_2\rangle \otimes |n\rangle \langle n| \otimes \langle e_2| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The equation for the density operator is,

$$i\hbar\frac{\partial \hat{\rho}(t)}{\partial t} = \left[\hat{H}, \hat{\rho}(t)\right]$$

On taking the matrix elements of the above equation with respect to the basis states one obtains,

$$\frac{\partial \rho_{22}(t)}{\partial t} = \frac{i}{\hbar}\left[k^*\sqrt{n+1}\,\rho_{21}(t) - k\sqrt{n+1}\,\rho_{12}(t)\right] = -\frac{\partial \rho_{11}(t)}{\partial t}$$

$$\frac{\partial \rho_{12}(t)}{\partial t} = \left(\frac{i}{\hbar}\Delta - \frac{1}{T_2}\right)\rho_{12}(t) - \frac{i}{\hbar}k^*\sqrt{n+1}\,\left[\rho_{22}(t) - \rho_{11}(t)\right]$$

$$\frac{\partial \rho_{21}(t)}{\partial t} = \left(-\frac{i}{\hbar}\Delta - \frac{1}{T_2}\right)\rho_{21}(t) + \frac{i}{\hbar}k\sqrt{n+1}\,\left[\rho_{22}(t) - \rho_{11}(t)\right]$$

We have included the effects of decoherence in the above equations in the off-diagonal components of the density matrix. We have assumed that decoherence collapses any superposition between the states $|e_2\rangle \otimes |n\rangle$ and $|e_1\rangle \otimes |n+1\rangle$. The solution of the density matrix equations, subject to the initial condition, and assuming zero detuning, is,

$$\rho_{22}(t) = 1 - \rho_{11}(t) = \frac{1}{2} + \frac{e^{-\frac{t}{2T_2}}}{2}\left[\cos\Omega t + \frac{1}{2\Omega T_2}\sin\Omega t\right]$$

where,

$$\Omega = \sqrt{\Omega_R^2 - \left(\frac{1}{2T_2}\right)^2} \qquad \Omega_R = 2\frac{|k|}{\hbar}\sqrt{n+1}$$

As $t \to \infty$, the off-diagonal components of the density matrix go to zero, and,

$$\hat{\rho}(t \to \infty) = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

11.5.2 Decoherence in the Heisenberg Picture

Adding the effects of decoherence in the Heisenberg picture in a quantum mechanically consistent way requires some care. We start from the non-interacting Hamiltonian for a two level system,

$$\hat{H} = \varepsilon_1 \hat{N}_1 + \varepsilon_2 \hat{N}_2$$

one can write the following Heisenberg equations for $\hat{N}_1(t)$, $\hat{N}_2(t)$, $\hat{\sigma}_+(t)$ and $\hat{\sigma}_-(t)$,

$$\frac{d\hat{N}_1(t)}{dt} = 0 \qquad \frac{d\hat{N}_2(t)}{dt} = 0$$

$$\frac{d\hat{\sigma}_+(t)}{dt} = \frac{i}{\hbar}(\varepsilon_2 - \varepsilon_1)\hat{\sigma}_+(t)$$

$$\frac{d\hat{\sigma}_-(t)}{dt} = -\frac{i}{\hbar}(\varepsilon_2 - \varepsilon_1)\hat{\sigma}_-(t)$$

We know that decoherence makes the off diagonal components of the density matrix go to zero. So we try to model decoherence phenomenologically by modifying the equations for $\hat{\sigma}_+(t)$ and $\hat{\sigma}_-(t)$ and adding decay terms as shown below,

$$\left.\begin{aligned} \frac{d\hat{\sigma}_+(t)}{dt} &= i\frac{\Delta\varepsilon}{\hbar}\hat{\sigma}_+(t) - \frac{\hat{\sigma}_+(t)}{T_2} \\ \frac{d\hat{\sigma}_-(t)}{dt} &= -i\frac{\Delta\varepsilon}{\hbar}\hat{\sigma}_-(t) - \frac{\hat{\sigma}_-(t)}{T_2} \end{aligned}\right\} \qquad \text{where, } \Delta\varepsilon = \varepsilon_2 - \varepsilon_1$$

The decay terms destroy the quantum mechanical consistency of the equations. To see this, note that,

$$\hat{\sigma}_+(t + \Delta t) = \hat{\sigma}_+(t)\left[1 + \left(i\frac{\Delta\varepsilon}{\hbar} - \frac{1}{T_2}\right)\Delta t\right]$$

$$\hat{\sigma}_-(t + \Delta t) = \hat{\sigma}_-(t)\left[1 + \left(-i\frac{\Delta\varepsilon}{\hbar} - \frac{1}{T_2}\right)\Delta t\right]$$

Multiply the above two equations, and obtain their commutator keeping all terms that are of first order in 'Δt',

$$\left[\hat{\sigma}_+(t+\Delta t),\hat{\sigma}_-(t+\Delta t)\right]=\left[\hat{\sigma}_+(t),\hat{\sigma}_-(t)\right]\left[1-\frac{2}{T_2}\Delta t\right]$$

$$\Rightarrow \hat{N}_2(t+\Delta t)-\hat{N}_1(t+\Delta t)=\left[\hat{N}_2(t)-\hat{N}_1(t)\right]\left[1-\frac{2}{T_2}\Delta t\right]$$

$$\Rightarrow \left\langle \hat{N}_2(t+\Delta t)\right\rangle-\left\langle \hat{N}_1(t+\Delta t)\right\rangle \neq \left\langle \hat{N}_2(t)\right\rangle-\left\langle \hat{N}_1(t)\right\rangle$$

But since,

$$\frac{d\hat{N}_1(t)}{dt}=0 \qquad \frac{d\hat{N}_2(t)}{dt}=0$$

The result $\left\langle \hat{N}_2(t+\Delta t)\right\rangle-\left\langle \hat{N}_1(t+\Delta t)\right\rangle \neq \left\langle \hat{N}_2(t)\right\rangle-\left\langle \hat{N}_1(t)\right\rangle$ cannot be correct.

One can restore quantum mechanical consistency of the equations by adding Langevin noise sources to the equations for $\hat{\sigma}_+(t)$ and $\hat{\sigma}_-(t)$,

$$\left.\begin{aligned}\frac{d\hat{\sigma}_+}{dt}(t)&=i\frac{\Delta\varepsilon}{\hbar}\hat{\sigma}_+(t)-\frac{1}{T_2}\hat{\sigma}_+(t)+\hat{F}_+(t)\,e^{i\omega_o t}\\ \frac{d\hat{\sigma}_-}{dt}(t)&=-i\frac{\Delta\varepsilon}{\hbar}\hat{\sigma}_-(t)-\frac{1}{T_2}\hat{\sigma}_-(t)+\hat{F}_-(t)\,e^{-i\omega_o t}\end{aligned}\right\} \qquad \text{where } \left[\hat{F}_+(t)\right]^+=\hat{F}_-(t)$$

The exponential factors $e^{\pm i\omega_o t}$ are convenient, but not necessary, since they will make the algebra simpler in Sections that follow. As always, we make the assumptions,

a) Noise operators act in their own Hilbert Space.

b) System operators at time t commute with the noise operators at time t' where $t'>t$.

c) $\left\langle \hat{F}_+(t)\right\rangle=\left\langle \hat{F}_-(t)\right\rangle=0$

d) $\left\langle \hat{F}_+(t_1)\hat{F}_-(t_2)\right\rangle=A(t_1)\delta(t_1-t_2)$ and $\left\langle \hat{F}_-(t_1)\hat{F}_+(t_2)\right\rangle=B(t_1)\delta(t_1-t_2)$

e) $\left\langle \left[\hat{F}_+(t_1),\hat{F}_-(t_2)\right]\right\rangle=\left[A(t_1)-B(t_1)\right]\delta(t_1-t_2)$

The solution of the equations to order Δt in time is,

$$\hat{\sigma}_+(t+\Delta t)=\hat{\sigma}_+(t)\left[1+\left(i\frac{\Delta\varepsilon}{\hbar}-\frac{1}{T_2}\right)\Delta t\right]+\int_t^{t+\Delta t}\hat{F}_+(t')e^{i\omega_o t'}dt'$$

$$\hat{\sigma}_-(t+\Delta t)=\hat{\sigma}_-(t)\left[1+\left(-i\frac{\Delta\varepsilon}{\hbar}-\frac{1}{T_2}\right)\Delta t\right]+\int_t^{t+\Delta t}\hat{F}_-(t')e^{-i\omega_o t'}dt'$$

We now find the values of $A(t)$ and $B(t)$ as follows.

1) We first multiply the first equation by the second equation from the right side and note that,

$$\hat{N}_2(t)=\hat{\sigma}_+(t)\hat{\sigma}_-(t)$$

to get,

$$\hat{N}_2(t+\Delta t)=\hat{N}_2(t)\left[1-\frac{2}{T_2}\Delta t\right]+\hat{\sigma}_+(t)\left[\int_t^{t+\Delta t}\hat{F}_-(t')e^{-i\omega_o t'}dt'\right]+\left[\int_t^{t+\Delta t}\hat{F}_+(t')e^{i\omega_o t'}dt'\right]\hat{\sigma}_-(t)$$
$$+\int_t^{t+\Delta t}dt'\int_t^{t+\Delta t}dt''\,\hat{F}_+(t')\hat{F}_-(t'')e^{i\omega_o t'}e^{-i\omega_o t''}$$

We take the average on both sides of the above equation to get,

$$\Delta t\left\langle \frac{d\hat{N}_2(t)}{dt}\right\rangle = -\frac{2}{T_2}\left\langle \hat{N}_2(t)\right\rangle \Delta t + \int_t^{t+\Delta t} dt' \int_t^{t+\Delta t} dt" \left\langle \hat{F}_+(t')\hat{F}_-(t")\right\rangle e^{i\omega_o t'} e^{-i\omega_o t"}$$

$$\left\langle \frac{d\hat{N}_2(t)}{dt}\right\rangle = -\frac{2}{T_2}\left\langle \hat{N}_2(t)\right\rangle + A(t)$$

$$\Rightarrow A(t) = \frac{2}{T_2}\left\langle \hat{N}_2(t)\right\rangle + \left\langle \frac{d\hat{N}_2(t)}{dt}\right\rangle = \frac{2}{T_2}\left\langle \hat{N}_2(t)\right\rangle$$

This implies,

$$\left\langle \hat{F}_+(t)\hat{F}_-(t')\right\rangle = A(t)\delta(t-t') = \left[\frac{2}{T_2}\left\langle \hat{N}_2(t)\right\rangle + \left\langle \frac{d\hat{N}_2(t)}{dt}\right\rangle\right]\delta(t-t')$$

2) We now multiply the first equation by the second equation from the left side and note that,

$$\hat{N}_1(t) = \hat{\sigma}_-(t)\hat{\sigma}_+(t)$$

to get,

$$B(t) = \frac{2}{T_2}\left\langle \hat{N}_1(t)\right\rangle + \left\langle \frac{d\hat{N}_1(t)}{dt}\right\rangle = \frac{2}{T_2}\left\langle \hat{N}_1(t)\right\rangle$$

and therefore,

$$\left\langle \hat{F}_-(t)\hat{F}_+(t')\right\rangle = B(t)\delta(t-t') = \left[\frac{2}{T_2}\left\langle \hat{N}_1(t)\right\rangle + \left\langle \frac{d\hat{N}_1(t)}{dt}\right\rangle\right]\delta(t-t')$$

3) Finally, we find commutator at time $t+\Delta t$,

$$\left[\hat{\sigma}_+(t+\Delta t),\hat{\sigma}_-(t+\Delta t)\right] = \left[\hat{\sigma}_+(t),\hat{\sigma}_-(t)\right]\left(1-\frac{2}{T_2}\Delta t\right) + \int_t^{t+\Delta t} dt' \int_t^{t+\Delta t} dt'' \left[\hat{F}_+(t'),\hat{F}_-(t'')\right] e^{i\omega_o(t'-t")}$$

$$\Rightarrow \left[\hat{N}_2(t+\Delta t)-\hat{N}_1(t+\Delta t)\right] = \left[\hat{N}_2(t)-\hat{N}_1(t)\right]\left(1-\frac{2}{T_2}\Delta t\right) + \int_t^{t+\Delta t} dt' \int_t^{t+\Delta t} dt'' \left[\hat{F}_+(t'),\hat{F}_-(t'')\right] e^{i\omega_o(t'-t")}$$

In order to preserve the commutation relation we must have,

$$\left[\hat{F}_+(t'),\hat{F}_-(t")\right] = \left[\hat{F}_+(t'),\hat{F}_-(t")\right] = \left[\frac{d\hat{N}_2}{dt} - \frac{d\hat{N}_1}{dt} + \frac{2}{T_2}\hat{N}_2(t') - \frac{2}{T_2}\hat{N}_1(t')\right]\delta(t'-t")$$

Similarly, one can show that,

$$\left\langle \hat{F}_+(t_1)\hat{F}_+(t_2)\right\rangle = \left\langle \hat{F}_-(t_1)\hat{F}_-(t_2)\right\rangle = 0$$

In most cases of practical interest, when decoherence is faster than the rate of change of the populations, one can use the following approximations,

$$\left\langle \hat{F}_+(t)\hat{F}_-(t')\right\rangle = A(t)\delta(t-t') \approx \frac{2}{T_2}\left\langle \hat{N}_2(t)\right\rangle \delta(t-t')$$

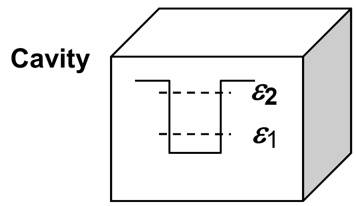
$$\left\langle \hat{F}_-(t)\hat{F}_+(t')\right\rangle = B(t)\delta(t-t') \approx \frac{2}{T_2}\left\langle \hat{N}_1(t)\right\rangle \delta(t-t')$$

$$\left[\hat{F}_+(t'),\hat{F}_-(t")\right] = \left[\hat{F}_+(t'),\hat{F}_-(t")\right] \approx \frac{2}{T_2}\left[\hat{N}_2(t') - \hat{N}_1(t')\right]\delta(t'-t")$$

11.5.3 Dynamics in the Presence of Strong Decoherence for a Two-Level System Coupled to a Cavity Radiation Mode

We now consider a two level system interacting with a single mode of radiation inside a cavity. The Hamiltonian is,

$$\hat{H} = \varepsilon_1 \hat{N}_1 + \varepsilon_2 \hat{N}_2 + \hbar\omega_o \hat{a}^+ \hat{a} + \left(k\,\hat{\sigma}_+ \hat{a} + k^* \hat{a}^+ \hat{\sigma}_- \right)$$



Earlier we saw that in the absence of decoherence the exact eigenstates and eigenenergies of the Hamiltonian could be obtained. Here we consider the case when decoherence is large. In the presence of decoherence, the Heisenberg equations are,

$$\frac{d\hat{N}_2(t)}{dt} = \frac{-i}{\hbar}\left[k\hat{\sigma}_+(t)\hat{a}(t) - k^*\hat{a}^+(t)\hat{\sigma}_-(t) \right]$$

$$\frac{d\hat{N}_1(t)}{dt} = \frac{i}{\hbar}\left[k\hat{\sigma}_+(t)\hat{a}(t) - k^*\hat{a}^+(t)\hat{\sigma}_-(t) \right]$$

$$\frac{d\hat{\sigma}_+(t)}{dt} = \left(i\frac{\Delta\varepsilon}{\hbar} - \frac{1}{T_2} \right)\hat{\sigma}_+(t) - \frac{i}{\hbar}k^*\,\hat{a}^+(t)\left[\hat{N}_2(t) - \hat{N}_1(t) \right] + \hat{F}_+(t)e^{i\omega_o t}$$

$$\frac{d\hat{\sigma}_-(t)}{dt} = \left(-i\frac{\Delta\varepsilon}{\hbar} - \frac{1}{T_2} \right)\hat{\sigma}_-(t) + \frac{i}{\hbar}k\left[\hat{N}_2(t) - \hat{N}_1(t) \right]\hat{a}(t) + \hat{F}_-(t)e^{-i\omega_o t}$$

$$\frac{d\hat{a}(t)}{dt} = -i\omega_o\hat{a}(t) - \frac{i}{\hbar}k^*\hat{\sigma}_-(t)$$

$$\frac{d\hat{a}^+(t)}{dt} = -i\omega_o\hat{a}^+(t) + \frac{i}{\hbar}k\,\hat{\sigma}_+(t)$$

Start from the equation for $\hat{\sigma}_-(t)$,

$$\frac{d\hat{\sigma}_-(t)}{dt} = \left(-i\frac{\Delta\varepsilon}{\hbar} - \frac{1}{T_2} \right)\hat{\sigma}_-(t) + \frac{i}{\hbar}k\left[\hat{N}_2(t) - \hat{N}_1(t) \right]\hat{a}(t) + \hat{F}_-(t)e^{-i\omega_o t}$$

Because of fast decoherence, $\hat{\sigma}_-(t)$ is expected to reach its steady state value pretty fast. But we need to factor out the fast time dependence before we find the steady state value. We write,

$$\hat{\sigma}_-(t) = \hat{\chi}_-(t)e^{-i\omega_o t}$$

and then find the equation for $\hat{\chi}_-(t)$ and set it to zero,

$$0 = \frac{d\hat{\chi}_-(t)}{dt} = \left(i\left(\omega_o - \frac{\Delta\varepsilon}{\hbar} \right) - \frac{1}{T_2} \right)\hat{\chi}_-(t) + \frac{i}{\hbar}k\left[\hat{N}_2(t) - \hat{N}_1(t) \right]\hat{a}(t)e^{i\omega_o t} + \hat{F}_-(t)$$

This gives,

$$\hat{\chi}_-(t) = \frac{-k\left[\hat{N}_2(t) - \hat{N}_1(t) \right]\hat{a}(t)e^{i\omega_o t} + i\hbar\hat{F}_-(t)}{(\hbar\omega_o - \Delta\varepsilon) + i\hbar/T_2}$$

$$\Rightarrow \hat{\sigma}_-(t) = -k\frac{\left[\hat{N}_2(t)-\hat{N}_1(t)\right]\hat{a}(t)}{(\hbar\omega_o-\Delta\varepsilon)+i\hbar/T_2}+\frac{i\hbar\hat{F}_-(t)e^{-i\omega_o t}}{(\hbar\omega_o-\Delta\varepsilon)+i\hbar/T_2}$$

We use the above value of $\hat{\sigma}_-(t)$ in the equation for the field operator to get,

$$\frac{d\hat{a}(t)}{dt} = -i\omega_o\hat{a}(t) + i\frac{|k|^2}{\hbar}\frac{\left[\hat{N}_2(t)-\hat{N}_1(t)\right]}{(\hbar\omega_o-\Delta\varepsilon)+i\hbar/T_2}\hat{a}(t) + \frac{k^*\hat{F}_-(t)e^{-i\omega_o t}}{(\hbar\omega_o-\Delta\varepsilon)+i\hbar/T_2}$$

Consider the average value of the second term on the right hand side,

$$i\frac{|k|^2}{\hbar}\frac{\left\langle\left[\hat{N}_2(t)-\hat{N}_1(t)\right]\right\rangle}{(\hbar\omega_o-\Delta\varepsilon)+i\hbar/T_2}$$

$$=\frac{|k|^2}{\hbar}\left(\frac{\hbar/T_2}{(\hbar\omega_o-\Delta\varepsilon)^2+(\hbar/T_2)^2}+i\frac{(\hbar\omega_o-\Delta\varepsilon)}{(\hbar\omega_o-\Delta\varepsilon)^2+(\hbar/T_2)^2}\right)\left\langle\left[\hat{N}_2(t)-\hat{N}_1(t)\right]\right\rangle$$

$$= g(t) - i\Delta\omega_o$$

Here, $g(t)$ can be identified with the gain provided by the two level system and $\Delta\omega_o$ can be identified with the change in the cavity mode frequency due to interaction with the two-level system (also called frequency pulling). We will ignore this frequency shift. We write the gain term as,

$$\frac{|k|^2}{\hbar}\frac{\hbar/T_2}{(\hbar\omega_o-\Delta\varepsilon)^2+(\hbar/T_2)^2}\left[\hat{N}_2(t)-\hat{N}_1(t)\right] = g_d\left[\hat{N}_2(t)-\hat{N}_1(t)\right]$$

g_d is called the differential gain. The equation for the operator $\hat{a}^+(t)$ can be obtained similarly. We finally have,

$$\frac{d\hat{a}(t)}{dt} = -i\omega_o\hat{a}(t) + g_d\left[\hat{N}_2(t)-\hat{N}_1(t)\right]\hat{a}(t) + \hat{F}_{sp}(t)e^{-i\omega_o t}$$

$$\frac{d\hat{a}^+(t)}{dt} = +i\omega_o\hat{a}^+(t) + g_d\,\hat{a}^+(t)\left[\hat{N}_2(t)-\hat{N}_1(t)\right] + \hat{F}_{sp}^+(t)e^{i\omega_o t}$$

where we have defined two new noise sources,

$$\hat{F}_{sp}(t) = \frac{k^*\hat{F}_-(t)}{(\hbar\omega_o-\Delta\varepsilon)+i\hbar/T_2}$$

$$\hat{F}_{sp}^+(t) = \frac{k\hat{F}_+(t)}{(\hbar\omega_o-\Delta\varepsilon)-i\hbar/T_2}$$

The correlation functions of the noise sources are,

$$\left\langle\hat{F}_{sp}^+(t)\hat{F}_{sp}(t')\right\rangle = |k|^2\frac{\left\langle\hat{F}_+(t)\hat{F}_-(t')\right\rangle}{(\hbar\omega_o-\Delta\varepsilon)^2+(\hbar/T_2)^2} = 2\frac{|k|^2}{\hbar}\frac{\hbar/T_2}{(\hbar\omega_o-\Delta\varepsilon)^2+(\hbar/T_2)^2}\left\langle\hat{N}_2(t)\right\rangle\delta(t-t')$$

$$\left\langle\hat{F}_{sp}(t)\hat{F}_{sp}^+(t')\right\rangle = |k|^2\frac{\left\langle\hat{F}_-(t)\hat{F}_+(t')\right\rangle}{(\hbar\omega_o-\Delta\varepsilon)^2+(\hbar/T_2)^2} = 2\frac{|k|^2}{\hbar}\frac{\hbar/T_2}{(\hbar\omega_o-\Delta\varepsilon)^2+(\hbar/T_2)^2}\left\langle\hat{N}_1(t)\right\rangle\delta(t-t')$$

Their commutator is,

$$\left[\hat{F}_{sp}(t),\hat{F}_{sp}^+(t')\right] = |k|^2\frac{\left\langle\left[\hat{F}_-(t),\hat{F}_+(t')\right]\right\rangle}{(\hbar\omega_o-\Delta\varepsilon)^2+(\hbar/T_2)^2} = -2\frac{|k|^2}{\hbar}\frac{(\hbar/T_2)}{(\hbar\omega_o-\Delta\varepsilon)^2+(\hbar/T_2)^2}\left[\hat{N}_2(t)-\hat{N}_1(t)\right]\delta(t-t')$$

$$= 2g_d\left[\hat{N}_2(t)-\hat{N}_1(t)\right]\delta(t-t')$$

Photon Number Equation: Using,

$$\frac{d\hat{a}(t)}{dt} = -i\omega_o \hat{a}(t) + g_d\left[\hat{N}_2(t) - \hat{N}_1(t)\right]\hat{a}(t) + e^{-i\omega_o t}\hat{F}_{sp}(t)$$

$$\frac{d\hat{a}^+(t)}{dt} = i\omega_o \hat{a}^+(t) + g_d\, \hat{a}^+(t)\left[\hat{N}_2(t) - \hat{N}_1(t)\right] + e^{i\omega_o t}\hat{F}^+_{sp}(t)$$

we can derive an equation for the photon number operator $\hat{n}(t) = \hat{a}^+(t)\hat{a}(t)$,

$$\frac{d\hat{n}(t)}{dt} = 2g_d\left[\hat{N}_2(t) - \hat{N}_1(t)\right]\hat{n}(t) + \left\{\hat{a}^+(t)\hat{F}_{sp}(t)e^{-i\omega_o t} + \hat{F}^+_{sp}(t)e^{i\omega_o t}\hat{a}(t)\right\}$$

The term $2g_d\hat{N}_2(t)\hat{n}(t)$ on the right hand side represents stimulated emission. The term $2g_d\hat{N}_1(t)\hat{n}(t)$ represents stimulated absorption. The spontaneous emission part is hiding in the noise term,

$$\left\{\hat{a}^+(t)\hat{F}_{sp}(t)e^{-i\omega_o t} + \hat{F}^+_{sp}(t)\hat{a}(t)e^{i\omega_o t}\right\}$$

and comes out when we take its average. So the noise is not zero-mean. Note that terms like $\left\langle\hat{a}^+(t)\hat{F}_{sp}(t)\right\rangle$ and $\left\langle\hat{F}^+_{sp}(t)\hat{a}(t)\right\rangle$ are not trivial to evaluate, since $\hat{a}(t)$ depends on $\hat{F}_{sp}(t')$ for all times $t' \le t$. To see this explicitly, integrate the equations for the field operators from $t-\Delta t$ to t and then take the limit $\Delta t \to 0$. This procedure gives,

$$\hat{a}(t) = \lim_{\Delta t \to 0}\; \hat{a}(t-\Delta t) + \int_{t-\Delta t}^{t} dt'\,\hat{F}_{sp}(t')e^{-i\omega_o t'}$$

$$\hat{a}^+(t) = \lim_{\Delta t \to 0}\; \hat{a}^+(t-\Delta t) + \int_{t-\Delta t}^{t} dt'\,\hat{F}^+_{sp}(t')e^{i\omega_o t'}$$

Now we evaluate,

$$\left\langle\hat{F}^+_{sp}(t)e^{i\omega_o t}\hat{a}(t)\right\rangle = e^{i\omega_o t}\left\langle\hat{F}^+_{sp}(t)\hat{a}(t-\Delta t)\right\rangle + e^{i\omega_o t}\int_{t-\Delta t}^{t} dt'\left\langle\hat{F}^+_{sp}(t)\hat{F}_{sp}(t')\right\rangle e^{-i\omega_o t'}$$

$$= \frac{|k|^2}{\hbar}\frac{\hbar/T_2}{(\hbar\omega_o - \Delta\varepsilon)^2 + (\hbar/T_2)^2}\left\langle\hat{N}_2(t)\right\rangle = g_d\left\langle\hat{N}_2(t)\right\rangle$$

Similarly,

$$\left\langle\hat{a}^+(t)\hat{F}_{sp}(t)e^{-i\omega_o t}\right\rangle = \frac{|k|^2}{\hbar}\frac{\hbar/T_2}{(\hbar\omega_o - \Delta\varepsilon)^2 + (\hbar/T_2)^2}\left\langle\hat{N}_2(t)\right\rangle$$

And therefore we find,

$$\left\langle\hat{a}^+(t)\hat{F}_{sp}(t)e^{-i\omega_o t} + \hat{F}^+_{sp}(t)e^{i\omega_o t}\hat{a}(t)\right\rangle = 2\frac{|k|^2}{\hbar}\frac{\hbar/T_2}{(\hbar\omega_o - \Delta\varepsilon)^2 + (\hbar/T_2)^2}\left\langle\hat{N}_2(t)\right\rangle$$

$$= 2g_d\left\langle\hat{N}_2(t)\right\rangle$$

The answer is indeed the average spontaneous emission rate. Since the last term in the photon number equation is not zero mean, one can define a zero-mean photon number noise source as,

$$\hat{F}_n(t) = \left\{\hat{a}^+(t)\hat{F}_{sp}(t)e^{-i\omega_o t} + \hat{F}^+_{sp}(t)e^{i\omega_o t}\hat{a}(t)\right\} - \left\langle\hat{a}^+(t)\hat{F}_{sp}(t)e^{-i\omega_o t} + \hat{F}^+_{sp}(t)e^{i\omega_o t}\hat{a}(t)\right\rangle$$

And write the photon number equation as,

$$\frac{d\hat{n}(t)}{dt} = 2g_d\left[\hat{N}_2(t) - \hat{N}_1(t)\right]\hat{n}(t) + 2g_d\left\langle\hat{N}_2(t)\right\rangle + \hat{F}_n(t)$$

The correlation of the photon number noise source is,

$$\left\langle \hat{F}_n(t)\hat{F}_n(t')\right\rangle = \left[2g_d\left\langle \hat{N}_2(t)\right\rangle\left(\left\langle \hat{n}(t)\right\rangle + 1\right) + 2g_d\left\langle \hat{N}_1(t)\right\rangle\left\langle \hat{n}(t)\right\rangle\right]\delta(t-t')$$

The above correlation function shows that the every process (stimulated emission, stimulated absorption, spontaneous emission) contributes shot noise to the photon number. This fact will be used later in understanding the fluctuations in a laser.

11.6 Non-Radiative Transitions

We now include non-radiative transitions in the rate equations for a two-level system. We will work in the Heisenberg picture. Starting from the non-interacting Hamiltonian for a two level system,

$$\hat{H} = \varepsilon_1\hat{N}_1 + \varepsilon_2\hat{N}_2$$

One can write the following Heisenberg equations for $\hat{N}_1(t)$ and $\hat{N}_2(t)$,

$$\frac{d\hat{N}_2(t)}{dt} = -\frac{\hat{N}_2(t)}{T_1}$$

$$\frac{d\hat{N}_1(t)}{dt} = \frac{\hat{N}_2(t)}{T_1}$$

The question arises if the above equations are quantum mechanically consistent? Let us check,

$$\frac{d\hat{N}_2(t)}{dt} = -\frac{\hat{N}_2(t)}{T_1}$$

$$\Rightarrow \hat{N}_2(t+\Delta t) = \hat{N}_2(t)\left(1-\frac{\Delta t}{T_1}\right)$$

Multiply the above equation by itself to get,

$$\left(\hat{N}_2(t+\Delta t)\right)^2 = \left(\hat{N}_2(t)\right)^2\left(1-\frac{2\Delta t}{T_1}\right) \qquad \left\{\left(\hat{N}_2(t)\right)^2 = \hat{N}_2(t)\right.$$

$$\Rightarrow \hat{N}_2(t+\Delta t) = \hat{N}_2(t)\left(1-\frac{2\Delta t}{T_1}\right)$$

The above two equations for $\hat{N}_2(t+\Delta t)$ cannot both be right, and our analysis is not correct. We modify the original equation and add a zero-mean Langevin noise source as follows,

$$\frac{d\hat{N}_2(t)}{dt} = -\frac{\hat{N}_2(t)}{T_1} - \hat{F}_N(t)$$

Note that $\hat{F}_N(t)$ is Hermitian and therefore commutation relations cannot be used to determine its properties. The above equation implies,

$$\hat{N}_2(t+\Delta t) = \hat{N}_2(t)\left(1-\frac{\Delta t}{T_1}\right) - \int_t^{t+\Delta t} dt_1\,\hat{F}_N(t_1)$$

Multiply the above equation by itself to get,

$$\hat{N}_2(t+\Delta t) = \hat{N}_2(t)\left[1-\frac{2\Delta t}{T_1}\right] - 2\hat{N}_2(t)\left[1-\frac{\Delta t}{T_1}\right]\int_t^{t+\Delta t}\hat{F}_N(t_1)dt_1$$
$$+\int_t^{t+\Delta t} dt_1 \int_t^{t+\Delta t} dt_2\,\hat{F}_N(t_1)\hat{F}_N(t_2)$$

Take average and assume,

$$\left\langle \hat{F}_N(t_1)\hat{F}_N(t_2)\right\rangle = c(t_1)\,\delta(t_1-t_2)$$

to get,

$$\left\langle \hat{N}_2(t+\Delta t)\right\rangle = \left\langle \hat{N}_2(t)\right\rangle\left(1-\frac{2\Delta t}{T_1}\right) + c(t)\,\Delta t$$

If we want,

$$\left\langle \hat{N}_2(t+\Delta t)\right\rangle = \left\langle \hat{N}_2(t)\right\rangle\left(1-\frac{\Delta t}{T_1}\right)$$

we must have,

$$c(t) = \frac{\left\langle \hat{N}_2(t)\right\rangle}{T_1} = \text{average relaxation rate}$$

Thus, non-radiative transitions seem to have a shot noise behavior. This should not come as a surprise since transitions occurs randomly in time.

We also want,

$$\hat{N}_1(t) + \hat{N}_2(t) = \hat{1}$$

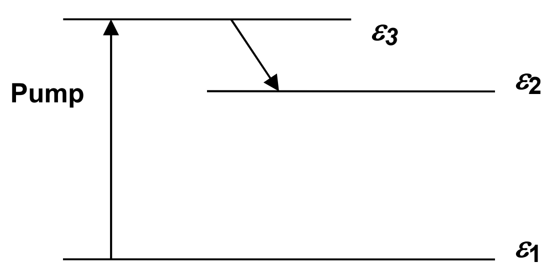
$$\Rightarrow \frac{d}{dt}\left[\hat{N}_1(t) + \hat{N}_2(t)\right] = 0$$

This is possible only if the equation for $\hat{N}_1(t)$ is,

$$\frac{d\hat{N}_1(t)}{dt} = +\frac{\hat{N}_2(t)}{T_1} + \hat{F}_N(t)$$

11.7 External Pumping

An external pump (usually a high power laser) can be used to transfer particles from the lower energy level to the higher energy level. This is achieved in a three level scheme as shown below. The relaxation rate from level 3 to level 2 is assumed to be very fast so that all particles transferred by the pump from the lower level into the uppermost level relax into level 2 immediately.



In the presence of pumping, the equations for the two-level system are,

$$\frac{d\hat{N}_2(t)}{dt} = \frac{\hat{N}_1(t)}{T_p} - \frac{\hat{N}_2(t)}{T_1} - \hat{F}_N(t) + \hat{F}_P(t)$$

$$\frac{d\hat{N}_1(t)}{dt} = -\frac{\hat{N}_1(t)}{T_p} + \frac{\hat{N}_2(t)}{T_1} + \hat{F}_N(t) - \hat{F}_P(t)$$

Here, $\hat{F}_P(t)$ is the noise source that models the noise in the pumping process and its correlation function can be found using the same methods as were used in the case of $\hat{F}_N(t)$ and the result is,

$$\left\langle \hat{F}_p(t_1)\hat{F}_p(t_2)\right\rangle = \frac{\left\langle \hat{N}_1(t)\right\rangle}{T_p}\delta(t_1 - t_2)$$

The equations for the populations,

$$\frac{d\hat{N}_2(t)}{dt} = \frac{\hat{N}_1(t)}{T_p} - \frac{\hat{N}_2(t)}{T_1} - \hat{F}_N(t) + \hat{F}_P(t)$$

$$\frac{d\hat{N}_1(t)}{dt} = -\frac{\hat{N}_1(t)}{T_p} + \frac{\hat{N}_2(t)}{T_1} + \hat{F}_N(t) - \hat{F}_P(t)$$

plus the equations for coherences,

$$\frac{d\hat{\sigma}_+(t)}{dt} = \left(i\frac{\Delta\varepsilon}{\hbar} - \frac{1}{T_2}\right)\hat{\sigma}_+(t) + \hat{F}_+(t)$$

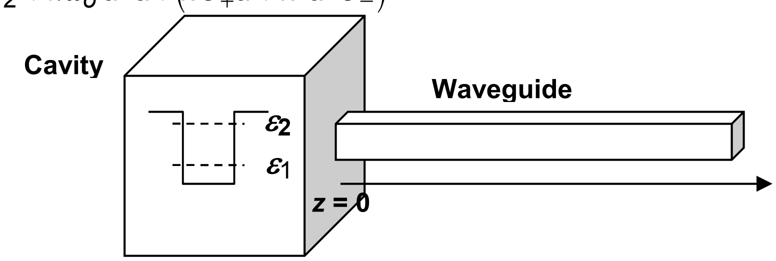
$$\frac{d\hat{\sigma}_-(t)}{dt} = \left(-i\frac{\Delta\varepsilon}{\hbar} - \frac{1}{T_2}\right)\hat{\sigma}_-(t) + \hat{F}_-(t)$$

complete the description of an isolated two-level system.

11.8 A Two Level System Interacting with Cavity Radiation: The Full Set of Equations

Consider the familiar two level system inside a closed cavity which supports a single radiation mode of frequency ω_o. The Hamiltonian is,

$$\hat{H} = \varepsilon_1\hat{N}_1 + \varepsilon_2\hat{N}_2 + \hbar\omega_o\,\hat{a}^+\hat{a} + \left(k\,\hat{\sigma}_+\hat{a} + k^*\hat{a}^+\hat{\sigma}_-\right)$$



We include the effects of cavity loss, relaxation, decoherence, and pumping in the Heisenberg equations. The resulting equations are,

$$\frac{d\,\hat{N}_2(t)}{dt} = \frac{\hat{N}_1(t)}{T_p} - \frac{\hat{N}_2(t)}{T_1} - \frac{i}{\hbar}\left[k\hat{\sigma}_+(t)\hat{a}(t) - k^*\hat{a}^+(t)\hat{\sigma}_-(t)\right] - \hat{F}_N(t) + \hat{F}_P(t)$$

$$\frac{d\hat{N}_1(t)}{dt} = -\frac{\hat{N}_1(t)}{T_p} + \frac{\hat{N}_2(t)}{T_1} + \frac{i}{\hbar}\left[k\hat{\sigma}_+(t)\hat{a}(t) - k^*\hat{a}^+(t)\hat{\sigma}_-(t)\right] + \hat{F}_N(t) - \hat{F}_P(t)$$

$$\frac{d\hat{\sigma}_+(t)}{dt} = \left(i\frac{\Delta\varepsilon}{\hbar} - \frac{1}{T_2} \right)\hat{\sigma}_+(t) - \frac{i}{\hbar}k^*\,\hat{a}^+(t)\left[\hat{N}_2(t) - \hat{N}_1(t)\right] + \hat{F}_+(t)e^{i\omega_o t}$$

$$\frac{d\hat{\sigma}_-(t)}{dt} = \left(-i\frac{\Delta\varepsilon}{\hbar} - \frac{1}{T_2} \right)\hat{\sigma}_-(t) + \frac{i}{\hbar}k\left[\hat{N}_2(t) - \hat{N}_1(t)\right]\hat{a}(t) + \hat{F}_-(t)e^{-i\omega_o t}$$

$$\frac{d\hat{a}(t)}{dt} = \left(-i\omega_o - \frac{1}{2\tau_p} \right)\hat{a}(t) - \frac{i}{\hbar}k^*\hat{\sigma}_-(t) + \sqrt{\frac{1}{\tau_p}}\,\hat{S}_{in}(t)e^{-i\omega_o t}$$

$$\frac{d\hat{a}^+(t)}{dt} = \left(i\omega_o - \frac{1}{2\tau_p} \right)\hat{a}^+(t) + \frac{i}{\hbar}k\,\hat{\sigma}_+(t) + \sqrt{\frac{1}{\tau_p}}\,\hat{S}^+_{in}(t)e^{i\omega_o t}$$

Here,

$$\hat{S}_{in}(t) = \sqrt{v_g}\,\hat{b}_L(z=0,t)e^{-i\omega_o t}$$

$$\hat{S}^+_{in}(t) = \sqrt{v_g}\,\hat{b}^+_L(z=0,t)e^{i\omega_o t}$$

The photons coming out of the cavity are described by the equations,

$$\hat{S}_{out}(t)e^{-i\omega_o t} = \sqrt{v_g}\,\hat{b}_R(z=0,t)e^{-i\omega_o t} = \sqrt{\frac{1}{\tau_p}}\,\hat{a}(t) - \sqrt{v_g}\,\hat{b}_L(z=0,t)e^{-i\omega_o t}$$

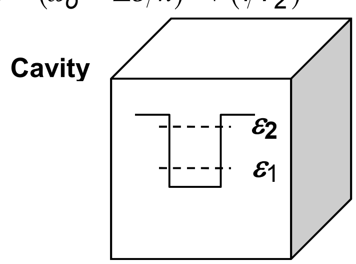
$$\hat{S}^+_{out}(t)e^{i\omega_o t} = \sqrt{v_g}\,\hat{b}^+_R(z=0,t)e^{i\omega_o t} = \sqrt{\frac{1}{\tau_p}}\,\hat{a}^+(t) - \sqrt{v_g}\,\hat{b}^+_L(z=0,t)e^{i\omega_o t}$$

The above equations constitute the complete set of equations needed to describe the quantum behavior of a two-level system in an optical cavity.

11.9 Cavity Enhanced Spontaneous Emission and the Purcell Effect

In the presence of strong decoherence we saw that the spontaneous emission rate in a closed cavity is given by the relation,

$$R_\downarrow = 2g_d = 2\frac{|k|^2}{\hbar^2}\frac{1/T_2}{(\omega_o - \Delta\varepsilon/\hbar)^2 + (1/T_2)^2}$$



Since,

$$|k|^2 = q^2\left(\frac{\hbar\omega_o}{2\varepsilon_o\varepsilon}\right)\left|\vec{U}(\vec{r}_o)\,.\,\langle e_2|\hat{\vec{r}}|e_1\rangle\right|^2$$

the coupling parameter $|k|$ can be made large by decreasing the size of the cavity (and therefore the mode volume). We assume that the two-level system is oriented such that it has a non-zero dipole matrix element with only the z-component of the cavity field. We also assume that the cavity field is polarized in the z-direction at the location of the two level system. Define mode volume V_p as,

$$\frac{1}{V_p} = \vec{U}^*(\vec{r}_o)\,.\,\vec{U}(\vec{r}_o)$$

Then the spontaneous emission rate on resonance (i.e. $\omega_o = \Delta\varepsilon/\hbar$) becomes,

$$R_{\downarrow}(\text{Cavity}) = 2\frac{|k|^2}{\hbar^2}\frac{1/T_2}{(\omega_o - \Delta\varepsilon/\hbar)^2 + (1/T_2)^2}$$
$$= \frac{2\pi}{\hbar^2}\frac{q^2\,|\langle e_2|z|e_1\rangle|^2}{3}\left(\frac{\hbar\omega_o}{2\varepsilon_o}\right)\left(\frac{3T_2}{\pi\,\varepsilon\,V_p}\right)$$

It is interesting to compare the cavity spontaneous emission rate to the spontaneous emission rate in free space.

$$R_{\downarrow}(\text{Free space}) = \frac{2\pi}{\hbar^2}\frac{q^2\,|\langle e_2|z|e_1\rangle|^2}{3}\left(\frac{\Delta\varepsilon}{2\varepsilon_o}\right)D(\omega)\Big|_{\omega=\Delta\varepsilon/\hbar}$$

The spontaneous emission rate in the cavity will be larger provided,

$$\frac{3T_2}{\pi\,\varepsilon\,V_p} > D(\omega)\Big|_{\omega=\Delta\varepsilon/\hbar}$$

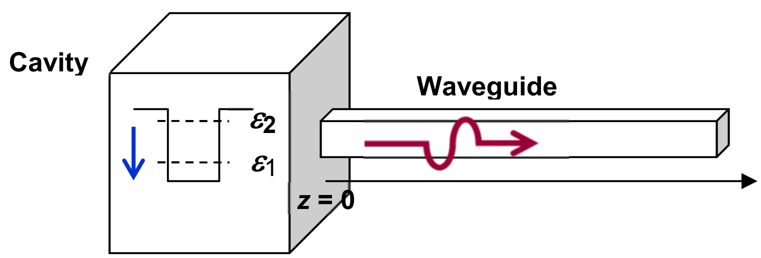
The result above is valid provided the decoherence time T_2 is much shorter than the inverse vacuum Rabi frequency $\hbar/2|k|$ and also much shorter than the cavity photon lifetime τ_p, i.e.,

$$T_2 << \frac{\hbar}{2|k|}\,,\,\tau_p$$

Below we consider what happens when the cavity photon lifetime becomes shorter than the decoherence time and also much shorter than the inverse vacuum Rabi frequency $\hbar/2|k|$, i.e.,

$$\tau_p << \frac{\hbar}{2|k|}\,,\,T_2$$

We consider a two level system inside a cavity coupled to waveguide, as shown below. The strongest coupling is not between the cavity field and the two level system, but between the cavity field and the waveguide. As a result of this coupling, the cavity mode becomes hybridized with the propagating modes of the waveguide.



In the first calculation step we will ignore the coupling between the cavity field and the two-level system and only consider it later as a perturbation in order to calculate the spontaneous emission rate. Consider first the Heisenberg equation for the cavity field operator,

$$\frac{d\hat{a}(t)}{dt} = \left(-i\omega_o - \frac{1}{2\tau_p}\right)\hat{a}(t) + \sqrt{\frac{v_g}{\tau_p}}\, \hat{b}_L(z=0,t)e^{-i\omega_o t}$$

We know that,

$$\hat{b}_L(z=0,t)e^{-i\omega_o t} = L\int_{-\beta_o-\Delta\beta/2}^{-\beta_o+\Delta\beta/2}\frac{d\beta}{2\pi}\,\hat{a}(\beta)\frac{e^{-i\omega(\beta)t}}{\sqrt{L}} = \frac{L}{v_g}\int_{\omega_o-\Delta\omega/2}^{\omega_o+\Delta\omega/2}\frac{d\omega}{2\pi}\,\hat{a}_L(\omega)\frac{e^{-i\omega t}}{\sqrt{L}}$$

Here the subscript "L" has been introduced in the notation for the field operator to explicitly indicate that the operator is for the mode moving in the left direction. We get,

$$\frac{d\hat{a}(t)}{dt} = \left(-i\omega_o - \frac{1}{2\tau_p}\right)\hat{a}(t) + \sqrt{\frac{1}{v_g\tau_p}}\, L\int_{\omega_o-\Delta\beta/2}^{\omega_o+\Delta\beta/2}\frac{d\omega}{2\pi}\,\hat{a}_L(\omega)\frac{e^{-i\omega t}}{\sqrt{L}}$$

The solution for t large is,

$$\hat{a}(t) \approx \sqrt{\frac{1}{v_g\tau_p}}\, L\int_{\omega_o-\Delta\omega/2}^{\omega_o+\Delta\omega/2}\frac{d\omega}{2\pi}\,\frac{i}{(\omega-\omega_o)+i/2\tau_p}\,\hat{a}_L(\omega)\frac{e^{-i\omega t}}{\sqrt{L}}$$

The relation between the right propagating modes and the cavity field is,

$$\hat{b}_R(z=0,t)\,e^{-i\omega_o t} = \sqrt{\frac{1}{v_g\tau_p}}\;\hat{a}(t) - \hat{b}_L(z=0,t)e^{-i\omega_o t}$$

Using the value of the cavity field calculated above, and valid for large times, we obtain,

$$\frac{L}{v_g}\int_{\omega_o-\Delta\omega/2}^{\omega_o+\Delta\omega/2}\frac{d\omega}{2\pi}\,\hat{a}_R(\omega)\frac{e^{-i\omega t}}{\sqrt{L}} = -\frac{L}{v_g}\int_{\omega_o-\Delta\omega/2}^{\omega_o+\Delta\omega/2}\frac{d\omega}{2\pi}\,\frac{(\omega-\omega_o)\tau_p - i/2}{(\omega-\omega_o)\tau_p + i/2}\,\hat{a}_L(\omega)\frac{e^{-i\omega t}}{\sqrt{L}}$$

$$\Rightarrow \hat{a}_R(\omega) = -\frac{(\omega-\omega_o)\tau_p - i/2}{(\omega-\omega_o)\tau_p + i/2}\,\hat{a}_L(\omega)$$

The above equation expresses the relation between the operators for the right and left moving modes. We use it to get,

$$\hat{a}(t) \approx -\sqrt{\frac{1}{v_g\tau_p}}\, L\int_{\omega_o-\Delta\omega/2}^{\omega_o+\Delta\omega/2}\frac{d\omega}{2\pi}\,\frac{i}{(\omega-\omega_o)-i/2\tau_p}\,\hat{a}_R(\omega)\frac{e^{-i\omega t}}{\sqrt{L}}$$

In the Schrodinger picture,

$$\hat{a} \approx -\sqrt{\frac{1}{v_g\tau_p}}\, L\int_{\omega_o-\Delta\omega/2}^{\omega_o+\Delta\omega/2}\frac{d\omega}{2\pi}\,\frac{i}{(\omega-\omega_o)-i/2\tau_p}\,\hat{a}_R(\omega)\frac{1}{\sqrt{L}}$$

The above equation shows that the cavity mode is "made up of" waveguide modes of different frequencies that come into the cavity, resonate for a while, and then leave. The expression above also shows that the waveguide modes with frequencies ω close to the resonance frequency ω_o of the closed cavity mode are the most important. We now calculate the spontaneous emission rate. The interaction part of the Hamiltonian can be written as,

$$\left(k\,\hat{\sigma}_+\hat{a} + k^*\hat{a}^+\hat{\sigma}_-\right) = -k\sqrt{\frac{1}{v_g\tau_p}}\, L\int_{\omega_o-\Delta\omega/2}^{\omega_o+\Delta\omega/2}\frac{d\omega}{2\pi}\,\frac{i}{(\omega-\omega_o)-i/2\tau_p}\,\hat{\sigma}_+\hat{a}_R(\omega)\frac{1}{\sqrt{L}}$$

$$+ k^* i\sqrt{\frac{1}{v_g\tau_p}}\, L\int_{\omega_o-\Delta\omega/2}^{\omega_o+\Delta\omega/2}\frac{d\omega}{2\pi}\,\frac{i}{(\omega-\omega_o)+i/2\tau_p}\,\hat{a}_R^+(\omega)\,\hat{\sigma}_-\,\frac{1}{\sqrt{L}}$$

The initial state of the system is,

$$|\psi_{\text{initial}}\rangle = |e_2\rangle\otimes|0\rangle$$

The final state could correspond to any one of the states in which the particle is in the lower energy level and there is one spontaneously emitted photon in any one of the waveguide modes moving in the right direction,

$$|\psi_{\text{final}}\rangle = |e_1\rangle \otimes |1\rangle_\omega^R$$

Using Fermi's Golden rule, the spontaneous emission rate becomes,

$$R_\downarrow(\text{Cavity}) = \frac{2|k|^2}{\hbar} \int_{\omega_o - \Delta\omega/2}^{\omega_o + \Delta\omega/2} \frac{d\omega}{2\pi} \frac{\pi/\tau_p}{(\omega - \omega_o)^2 + (1/2\tau_p)^2} \delta(\varepsilon_2 - \hbar\omega - \varepsilon_1)$$

$$= 2\frac{|k|^2}{\hbar^2} \frac{(1/2\tau_p)}{(\omega_o - \Delta\varepsilon/\hbar)^2 + (1/2\tau_p)^2}$$

The spontaneous emission rate on resonance (i.e. $\omega_o = \Delta\varepsilon/\hbar$) is,

$$R_\downarrow(\text{Cavity}) = \frac{2\pi}{\hbar^2} \frac{q^2\, |\langle e_2|z|e_1\rangle|^2}{3} \left(\frac{\hbar\omega_o}{2\varepsilon_o}\right)\left(\frac{6\,\tau_p}{\pi\,\varepsilon\, V_p}\right) \qquad \left\{ \tau_p \ll \frac{\hbar}{2|k|}, T_2 \right.$$

We can again compare the cavity spontaneous emission rate to the spontaneous emission rate in free space.

$$R_\downarrow(\text{Free space}) = \frac{2\pi}{\hbar^2} \frac{q^2\, |\langle e_2|z|e_1\rangle|^2}{3} \left(\frac{\Delta\varepsilon}{2\varepsilon_o}\right) D(\omega)\Big|_{\omega = \Delta\varepsilon/\hbar}$$

The spontaneous emission rate in the cavity will be larger provided,

$$\frac{6\,\tau_p}{\pi\,\varepsilon\, V_p} > D(\omega)\Big|_{\omega = \Delta\varepsilon/\hbar}$$

The enhancement expressed by the expression above is called Purcell enhancement. We can also write the condition for Purcell enhancement in terms of the cavity quality factor Q,

$$\frac{6\,Q}{\pi\,\omega_o\,\varepsilon\, V_p} > D(\omega)\Big|_{\omega = \Delta\varepsilon/\hbar}$$

A large Q cavity with a small mode volume V_p can enhance spontaneous emission rates by factors as large as 50-100.

In the more general case where the decoherence time and the cavity photon lifetime are comparable and both are much shorter than the inverse Rabi frequency, the following result can be derived with little additional effort,

$$R_\downarrow(\text{Cavity}) = \frac{2|k|^2}{\hbar^2} \int_{\omega_o - \Delta\omega/2}^{\omega_o + \Delta\omega/2} \frac{d\omega}{\pi} \frac{1/2\tau_p}{(\omega - \omega_o)^2 + (1/2\tau_p)^2} \frac{1/T_2}{(\omega - \Delta\varepsilon/\hbar)^2 + (1/T_2)^2} \qquad \left\{ \tau_p, T_2 \ll \frac{\hbar}{2|k|} \right.$$

The integral over frequency can be performed in the complex plane and the final result is,

$$R_\downarrow(\text{Cavity}) = 2\frac{|k|^2}{\hbar^2} \frac{(1/2\tau_p + 1/T_2)}{(\omega_o - \Delta\varepsilon/\hbar)^2 + (1/2\tau_p + 1/T_2)^2} \qquad \left\{ \tau_p, T_2 \ll \frac{\hbar}{2|k|} \right.$$

$$= \frac{2\pi}{\hbar^2} \frac{q^2\, |\langle e_2|z|e_1\rangle|^2}{3} \left(\frac{\hbar\omega_o}{2\varepsilon_o}\right)\left(\frac{3}{\pi\,\varepsilon\, V_p}\right)\left(\frac{1}{1/2\tau_p + 1/T_2}\right) \qquad \{\omega_o = \Delta\varepsilon/\hbar$$

It can be seen that the above expression reduces to the correct results in the two limits, $T_2 << \hbar/2|k|$, τ_p and $\tau_p << \hbar/2|k|$, T_2, discussed earlier.

11.10 Weak and Strong Coupling Regimes in Quantum Cavity Electrodynamics

The above discussion shows that matter-photon interactions in a cavity can be categorized as follows:

i) Strong Coupling Regime: Here,

$$\frac{2|k|}{\hbar} >> \frac{1}{2\tau_p} + \frac{1}{T_2}$$

and quantum mechanical coherence between matter and radiation degrees of freedom is important.

i) Weak Coupling Regime: Here,

$$\frac{2|k|}{\hbar} << \frac{1}{2\tau_p} + \frac{1}{T_2}$$

and quantum mechanical coherence between matter and radiation degrees of freedom is not important. In this regime, matter and radiation interactions can generally be described in terms of stimulated and spontaneous transitions.

11.11 A Collection of *N* Two-Level Systems Interacting with Cavity Radiation

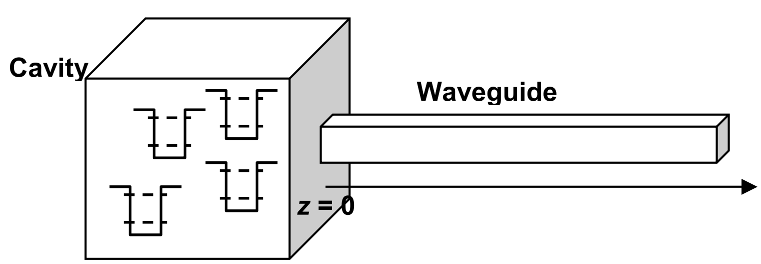
Consider a collection of N two-level systems interacting with a single radiation mode inside a cavity and described by the Hamiltonian,

$$\hat{H} = \sum_{j=1}^{N} \left\{ \varepsilon_1 \hat{N}_{1j} + \varepsilon_2 \hat{N}_{2j} \right\} + \hbar\omega_o \hat{a}^+ \hat{a} + \sum_{j=1}^{N} \left\{ k\,\hat{\sigma}_{+j}\,\hat{a} + k^* \hat{a}^+ \hat{\sigma}_{-j} \right\}$$

where,

$$\hat{N}_{1j} = |e_1\rangle_j \; {}_j\langle e_1| \qquad \hat{N}_{2j} = |e_2\rangle_j \; {}_j\langle e_2|$$

$$\hat{\sigma}_{+j} = |e_2\rangle_j \; {}_j\langle e_1| \qquad \hat{\sigma}_{-j} = |e_1\rangle_j \; {}_j\langle e_2|$$



If we define,

$$\hat{N}_1 = \sum_{j=1}^{N} \hat{N}_{1j} \qquad \hat{N}_2 = \sum_{j=1}^{N} \hat{N}_{2j}$$

$$\Rightarrow \hat{N}_1 + \hat{N}_2 = \sum_{j=1}^{N} \left(\hat{N}_{1j} + \hat{N}_{2j} \right) = \sum_{j=1}^{N} 1 = N$$

$$\hat{\sigma}_+ = \sum_{j=1}^{N} \hat{\sigma}_{+j} \qquad \hat{\sigma}_- = \sum_{j=1}^{N} \hat{\sigma}_{-j}$$

then with these definitions, the Hamiltonian can be written in the following familiar form,

$$\hat{H} = \varepsilon_1 \hat{N}_1 + \varepsilon_2 \hat{N}_2 + \hbar\omega_o \hat{a}^+ \hat{a} + \left(k\,\hat{\sigma}_+ \hat{a} + k^* \hat{a}^+ \hat{\sigma}_- \right)$$

The only fact one has to bear in mind when working with the above Hamiltonian is that the sum $\hat{N}_1 + \hat{N}_2$ equals N in the case of N two-level systems and 1 in the case of a single two-level system. The Heisenberg equations, as given in the previous section, remain unchanged.

The initial state of the system is specified by specifying the initial state of all the N two-level systems as well as the state of the radiation mode. For example, if the density operator of the j-th two-level system at $t=0$ is $\hat{\rho}_j\left(t=0\right)$, then the density operator $\hat{\rho}\left(t=0\right)$ for the full system is given by,

$$\hat{\rho}\left(t=0\right) = \hat{\rho}_1\left(t=0\right) \otimes \hat{\rho}_2\left(t=0\right) \otimes \hat{\rho}_3\left(t=0\right) \otimes \ldots\ldots \otimes \hat{\rho}_N\left(t=0\right) \otimes \hat{\rho}_{rad}\left(t=0\right)$$