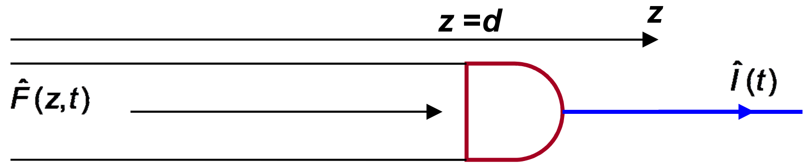
# Chapter 10: Coherences, Correlation Functions, and Detection

## 10.1 Direct Photodetection

The description in the previous Chapters of propagating quantum states is well suited for modeling the process of photodetection. Consider a photodetector attached to the end of a waveguide (or fiber), as shown below. In direct photodetection, the time-dependent photon flux is measured via the photocurrent.




We assume that the photodetector is ideal (100% efficient and infinite bandwidth) and all photons entering the photodetector are converted into electrons. Real photodetectors have efficiencies that can be as high as 95%. A photodetector with less than 100% efficiency can always be modeled as a beam splitter followed by an ideal photodetector. The current at the output of the photodetector is described by an operator $\hat{I}(t)$. The current operator for an ideal photodetector is related to the photon flux operator by the relation,

$$\hat{I}(t) = q\,\hat{F}(z_d,t) = q\,v_g\,b^{+}(z_d,t)\,b(z_d,t)$$

Note that the only source of noise in the ideal photodetector current is the noise in the photon flux.

**Example: A CW Coherent State:** Suppose the incoming radiation is a continuous wave coherent state having frequency $\omega_o$, average power $P_o$ and some arbitrary phase $\phi$,

$$\alpha(z) = \sqrt{\frac{P_o}{v_g \hbar\omega_o}}\, e^{i\phi}$$

The average photon flux at any location is,

$$\left\langle \hat{F}(z,t) \right\rangle = \frac{P_o}{\hbar\omega_o}$$

The flux is independent of the location. The photon flux noise correction function was calculated earlier and the result was,

$$\left\langle \Delta\hat{F}(z,t_1)\,\Delta\hat{F}(z,t_2) \right\rangle = \frac{P_o}{h\omega_o}\,\delta(t_1 - t_2)$$

For a coherent state, the photon flux exhibited shot noise. Since,

$$\hat{I}(t) = q\,\hat{F}(z_d,t)$$

the average current is,

$$\left\langle \hat{I}(t) \right\rangle = q\left\langle \hat{F}(z_d,t) \right\rangle = \frac{q}{\hbar\omega_o} P_o$$

The current noise correlation function is,

$$\left\langle \Delta\hat{I}(t_1)\Delta\hat{I}(t_2) \right\rangle = q^2 \left\langle \Delta\hat{F}(z,t_1)\,\Delta\hat{F}(z,t_2) \right\rangle = q^2 \frac{P_o}{h\omega_o}\,\delta(t_1 - t_2) = q\left\langle \hat{I}(t) \right\rangle \delta(t_1 - t_2)$$

It follows that the current noise spectral density is,

$$S_{\Delta I\,\Delta I}(\omega)=\int_{-\infty}^{\infty} d\tau\, e^{i\omega\tau}\,\left\langle \Delta\hat{I}(t+\tau)\Delta\hat{I}(t)\right\rangle$$
$$= q^2\frac{P_o}{\hbar\omega_o} = q\left\langle \hat{I}(t)\right\rangle$$

The photodetector current is said to exhibit shot noise when the current noise spectral density is flat with a value equal to the charge of an electron times the average photodetector current. The example shows that the photodetector is a useful tool for measuring and characterizing photon flux noise.

**Note:** In cases were operator ordering can make a difference, the symmetric correlation function is used,

$$\frac{\left\langle \hat{I}(t_1)\hat{I}(t_2)+\hat{I}(t_2)\hat{I}(t_1)\right\rangle}{2}$$

**Example: An Amplitude Squeezed State:** Consider a CW squeezed state defined by,

$$|\psi(t=0)\rangle = |\alpha(z),\varepsilon(z)\rangle = \hat{T}(\alpha)\,\hat{S}(\varepsilon)|0\rangle$$

where,

$$\hat{S}(\varepsilon)=e^{\int_{-\infty}^{\infty} dz'\left(\frac{\varepsilon^*(z')}{2}\left(\hat{b}(z',0)\right)^2-\frac{\varepsilon(z')}{2}\left(\hat{b}^+(z',0)\right)^2\right)}$$

$$\hat{T}(\alpha)=e^{\int_{-\infty}^{\infty} dz'\left(\alpha(z')\hat{b}^+(z',0)-\alpha^*(z')\hat{b}(z',0)\right)}$$

and,

$$\varepsilon(z)=r\,e^{i2\phi} = \text{constant}$$
$$\alpha(z)=|\alpha|e^{i\phi} = \text{constant}$$

The average current is,

$$\left\langle \hat{I}(t)\right\rangle = q\left\langle \hat{F}(z_d,t)\right\rangle = qv_g|\alpha|^2 + qv_g\sinh^2(r)\frac{\Delta\beta}{2\pi}$$

The current noise correlation function is,

$$\left\langle \Delta\hat{I}(t_1)\Delta\hat{I}(t_2)\right\rangle = q^2\left\langle \Delta\hat{F}(z,t_1)\,\Delta\hat{F}(z,t_2)\right\rangle = q^2v_g|\alpha|^2e^{-2r}\delta(t_1-t_2)$$
$$+2q^2v_g\sinh^2(r)\cosh^2(r)\frac{\Delta\beta}{2\pi}\delta(t_1-t_2)$$

If $|\alpha|^2 >> \Delta\beta$, then,

$$\left\langle \Delta\hat{I}(t_1)\Delta\hat{I}(t_2)\right\rangle \approx q\left\langle \hat{I}(t)\right\rangle e^{-2r}\delta(t_1-t_2)$$

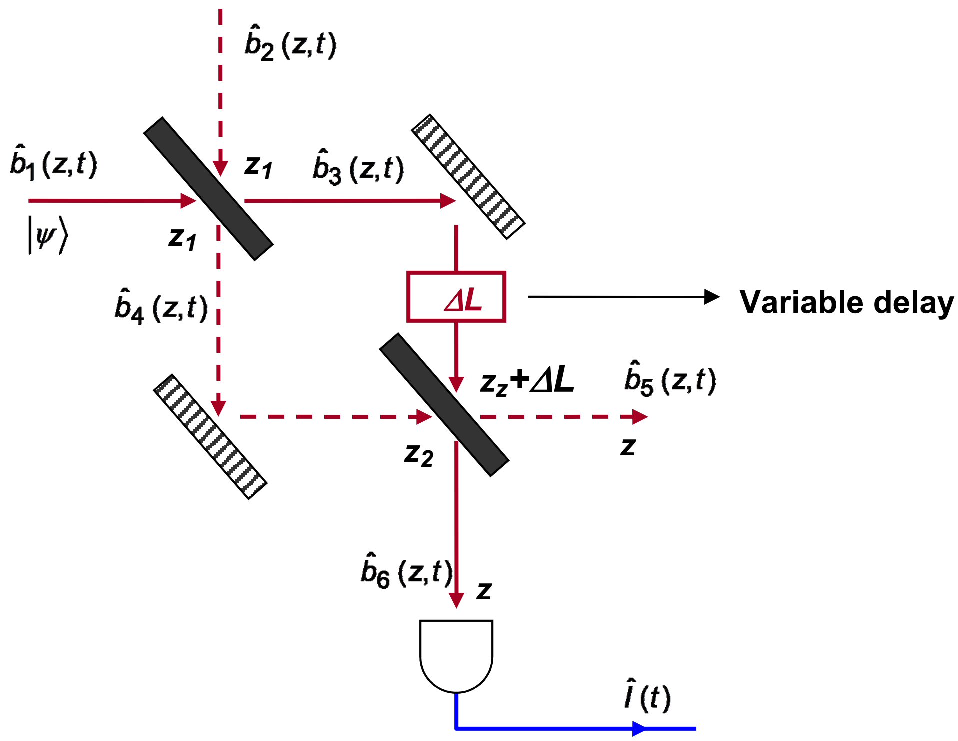
We see that the current noise is suppressed below the shot noise value for CW amplitude squeezed states. For CW phase squeezed states, we would have seen current noise fluctuations increase above the shot noise value.

## 10.2 Coherence and Correlation Functions

The coherence and statistical properties of radiation are described using correlation functions. The essential ideas are best illustrated through an example. Consider the following interferometer. The radiation state to be characterized is input from port 1 of the first beam splitter. The average photocurrent from the detector placed right after the second beam splitter is measured as function of the path difference

$\Delta L$ between the two arms of the interferometer. The path difference $\Delta L$ corresponds to a time delay $\tau$ of $\Delta L / v_g$. We assume that the average photon flux at the input is constant (independent of time) and equals $F_o$,

$$\left\langle \hat{F}_1(z_1,t) \right\rangle = F_o$$




The beam splitter relation for the first beam splitter is,

$$\begin{bmatrix} \hat{b}_3(z_1,t)\, e^{i\beta_o z_1} \\ b_4(z_1,t)\, e^{i\beta_o z_1} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \hat{b}_1(z_1,t)\, e^{i\beta_o z_1} \\ \hat{b}_2(z_1,t)\, e^{i\beta_o z_1} \end{bmatrix}$$

For the second beam splitter it is,

$$\begin{bmatrix} \hat{b}_5(z_2,t)\, e^{i\beta_o z_2} \\ b_6(z_2+\Delta L,t)\, e^{i\beta_o (z_2+\Delta L)} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \hat{b}_4(z_2,t)\, e^{i\beta_o z_2} \\ \hat{b}_3(z_2+\Delta L,t)\, e^{i\beta_o (z_2+\Delta L)} \end{bmatrix}$$

The average photocurrent from the detector for path difference $\Delta L = v_g \tau$ at time $t + (z_2 - z_1)/v_g + \tau$ can be written as,

$$\frac{\left\langle \hat{I}_\tau\left(t + \frac{z_2 - z_1}{v_g} + \tau\right) \right\rangle}{q} = \left\langle \hat{F}_6\left(z_2 + \Delta L, t + \frac{z_2 - z_1}{v_g} + \tau\right) \right\rangle$$

$$= \frac{1}{4}\left[ \left\langle \hat{F}_1(z_1, t+\tau) \right\rangle + \left\langle \hat{F}_1(z_1,t) \right\rangle + v_g \left\langle \hat{b}_1^+(z_1, t+\tau) \hat{b}_1(z_1,t) \right\rangle e^{i\beta_o \Delta L} + v_g \left\langle \hat{b}_1^+(z_1,t) \hat{b}_1(z_1, t+\tau) \right\rangle e^{-i\beta_o \Delta L} \right]$$

$$= \frac{1}{4}\left[ 2F_o + v_g \left\langle \hat{b}_1^+(z_1, t+\tau) \hat{b}_1(z_1,t) \right\rangle e^{i\beta_o \Delta L} + v_g \left\langle \hat{b}_1^+(z_1,t) \hat{b}_1(z_1, t+\tau) \right\rangle e^{-i\beta_o \Delta L} \right]$$

$$= \frac{F_o}{2}\left[1 + \frac{v_g\left\langle \hat{b}_1^+(z_1,t+\tau)\hat{b}_1(z_1,t)\right\rangle}{F_o}\frac{e^{i\beta_o\Delta L}}{2} + \frac{v_g\left\langle \hat{b}_1^+(z_1,t)\hat{b}_1(z_1,t+\tau)\right\rangle}{F_o}\frac{e^{-i\beta_o\Delta L}}{2}\right]$$

$$= \frac{F_o}{2}\left[1 + \frac{v_g\left\langle \hat{b}_1^+(z_1,t+\tau)\hat{b}_1(z_1,t)\right\rangle}{F_o}\frac{e^{i\beta_o v_g\tau}}{2} + \frac{v_g\left\langle \hat{b}_1^+(z_1,t)\hat{b}_1(z_1,t+\tau)\right\rangle}{F_o}\frac{e^{-i\beta_o v_g\tau}}{2}\right]$$

Suppose that the incoming radiation was a continuous wave coherent state having frequency $\omega_o$, average power $P_o$ and some arbitrary phase $\phi$,

$$\alpha(z) = \sqrt{\frac{P_o}{v_g\hbar\omega_o}}\,e^{i\phi} = \sqrt{\frac{F_o}{v_g}}\,e^{i\phi}$$

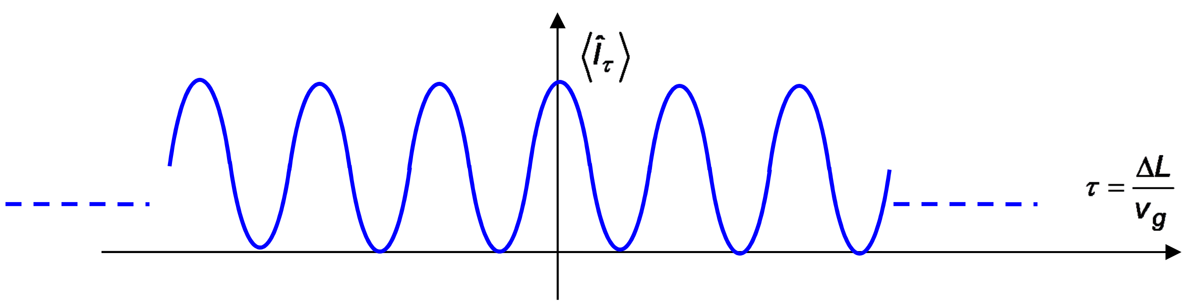
Then,

$$\left\langle \hat{b}_1^+(z_1,t+\tau)\hat{b}_1(z_1,t)\right\rangle = \left\langle \hat{b}_1^+(z_1,t)\hat{b}_1(z_1,t+\tau)\right\rangle = \frac{F_o}{v_g}$$

and,

$$\frac{\left\langle \hat{I}_\tau\left(t + \frac{z_2 - z_1}{v_g} + \tau\right)\right\rangle}{q} = \frac{F_o}{2}\left[1 + \cos(\beta_o v_g \tau)\right]$$

As the path difference $\Delta L$ is varied, the photocurrent will increase and decrease in a periodic way as a result of the interference at the output between the radiation beams that took two different paths through the interferometer. These interference "fringes" in the phtotocurrent are depicted in the figure below.




Now assume that the phase $\phi$ of the coherent state is not constant but varies in space,

$$\alpha(z) = \sqrt{\frac{F_o}{v_g}}\,e^{i\phi(z)}$$

Assume further that the phase varies randomly and the phase-phase correlation function is,

$$\left\langle e^{-i\phi(z)} e^{i\phi(z')}\right\rangle = e^{-\frac{|z-z'|}{L\phi}}$$

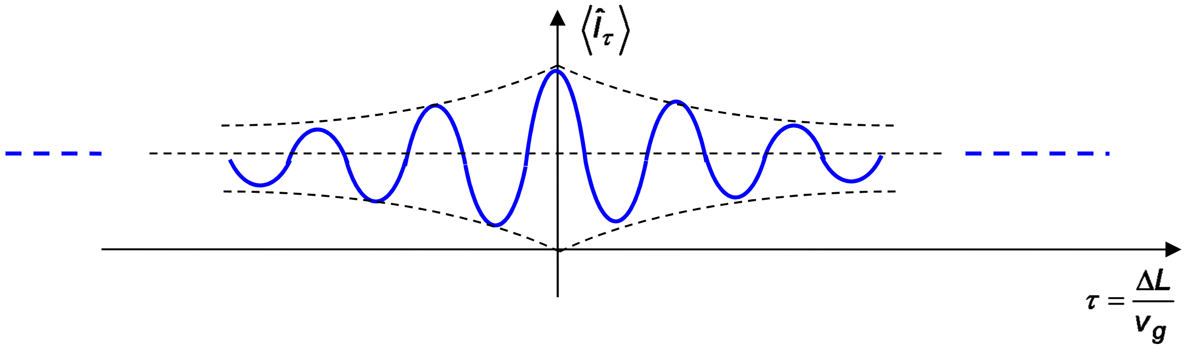
Here, $L_\phi$ is the phase coherence length. It follows that,

$$\left\langle \hat{b}_1^+(z_1,t+\tau)\hat{b}_1(z_1,t)\right\rangle = \left\langle \hat{b}_1^+(z_1-v_g\tau,t)\hat{b}_1(z_1,t)\right\rangle = \frac{F_o}{v_g}\left\langle e^{-i\phi(z_1-v_g\tau)}e^{i\phi(z_1)}\right\rangle = \frac{F_o}{v_g}e^{-\frac{|\Delta L|}{L_\phi}}$$

$$\left\langle \hat{b}_1^+(z_1,t)\hat{b}_1(z_1,t+\tau)\right\rangle = \frac{F_o}{v_g}e^{-\frac{|\Delta L|}{L_\phi}}$$

The average photocurrent is,

$$\frac{\left\langle \hat{I}_\tau\left(t+\frac{z_2-z_1}{v_g}+\tau\right)\right\rangle}{q} = \frac{F_o}{2}\left[1+e^{-\frac{v_g|\tau|}{L_\phi}}\cos(\beta_o v_g \tau)\right]$$




One can see that now the interference fringes decrease as a function of the path difference. The correlation functions, $\left\langle \hat{b}_1^+(z_1,t+\tau)\hat{b}_1(z_1,t)\right\rangle$ and $\left\langle \hat{b}_1^+(z_1,t)\hat{b}_1(z_1,t+\tau)\right\rangle$, are therefore a measure of the temporal coherence of the radiation. In the following Sections, we formalize the notion of coherence functions.

### 10.2.1 First Order Coherence Function

The normalized first order coherence function $g_1(t_1,t_2)$ at any location $z$ is defined as,

$$g_1(z:t_1,t_2) = \frac{\left\langle \hat{b}^+(z,t_1)e^{i\omega_o t_1}\hat{b}(z,t_2)e^{-i\omega_o t_2}\right\rangle}{\sqrt{\left\langle \hat{b}^+(z,t_1)\hat{b}(z,t_1)\right\rangle\left\langle \hat{b}^+(z,t_2)\hat{b}(z,t_2)\right\rangle}}$$

According to the Cauchy-Schwartz inequality for correlation functions,

$$\left|\left\langle \hat{b}^+(z,t_1)e^{i\omega_o t_1}\hat{b}(z,t_2)e^{-i\omega_o t_2}\right\rangle\right|^2 \le \left\langle \hat{b}^+(z,t_1)\hat{b}(z,t_1)\right\rangle \left\langle \hat{b}^+(z,t_2)\hat{b}(z,t_2)\right\rangle$$

Therefore,

$$|g_1(z:t_1,t_2)| \le 1$$

For continuous wave coherent states, $|g_1(z:t_1,t_1)| = 1$, implying complete first order temporal coherence. In general, if for a quantum state the correlation function $\left\langle \hat{b}^+(z_1,t_1)\hat{b}(z_1,t_2)\right\rangle$ factorizes, i.e.,

$$\left\langle \hat{b}^+(z_1,t_1)\hat{b}(z_1,t_2)\right\rangle = \left\langle \hat{b}^+(z_1,t_1)\right\rangle \left\langle \hat{b}(z_1,t_2)\right\rangle$$

then $|g_1(z:t_1,t_1)| = 1$ and the state will have complete first order coherence.

### 10.2.2 Radiation Spectrum, First Order Coherence Function and Fourier Transform Spectroscopy

The normalized spectrum $S(\omega)$ of a radiation source is related to the first order coherence function as,

$$S(\omega)=\int_{-\infty}^{\infty} d\tau \quad g_1(z:t,t+\tau)\ e^{i\omega\tau}$$

For example, for a coherent state we get,

$$\begin{aligned}S(\omega)&=\int_{-\infty}^{\infty} d\tau \quad g_1(z:t,t+\tau)\ e^{i\omega\tau}\\&=\int_{-\infty}^{\infty} d\tau \quad e^{-i\omega_o\tau}\ e^{i\omega\tau}\\&=2\pi\ \delta(\omega-\omega_o)\end{aligned}$$

For a coherent state whose phase is randomly varying in space we get,

$$\begin{aligned}S(\omega)&=\int_{-\infty}^{\infty} d\tau \quad g_1(z:t,t+\tau)\, e^{i\omega\tau}\\&=\int_{-\infty}^{\infty} d\tau \quad e^{-\frac{c|\tau|}{L_\phi}}\, e^{-i\omega_o\tau}\ e^{i\omega\tau}\\&=2\pi\ \frac{c/(L_\phi\pi)}{(\omega-\omega_o)^2+(c/L_\phi)^2}\end{aligned}$$

The finite phase coherence length broadens the spectral linewidth from a delta function to a Lorentzian of full width at half maximum (FWHM) equal to $2c/L_\phi$ .

The interferometer, discussed above, is commonly used to measure the spectrum of radiation. This technique is called Fourier Transform Spectroscopy. Assume radiation propagation in free-space for which the dispersion relation is, $\omega=\beta\, c$, and, $v_g=c$. The Fourier transform of the average photocurrent $\langle \hat{I}_\tau \rangle$ for a constant average photon flux (or constant average power) radiation, as a function of the path difference $\Delta L=c\tau$ , can be written as follows (assuming a non-dispersive medium),

$$\int_{-\infty}^{\infty} d\tau \ \langle \hat{I}_\tau \rangle\, e^{i\omega\tau}=\frac{F_o}{2}\left[2\pi\ \delta(\omega)+S(-\omega)+S(\omega)\right]$$

Therefore, the Fourier transform of the average photocurrent can be used to obtain the spectrum of the radiation.

### 10.2.3 Second Order Coherence Function

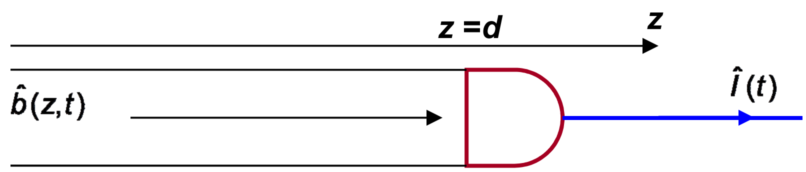
The first order coherence function might not be able to distinguish between different states of radiation. For example, coherent state with a randomly varying phase and radiation from a broadband thermal source passed through a narrow band filter, which has a Lorentzian transmission spectrum, might exhibit identical first order coherence functions. Higher order coherence functions can help in such cases. Here, we only consider the second order coherence function.

Earlier, we have seen that the photon flux correlation function,

$$\langle \hat{F}(z,t_1)\,\hat{F}(z,t_2)\rangle=v_g^2\,\langle \hat{b}^+(z,t_1)\,\hat{b}(z,t_1)\,\hat{b}^+(z,t_2)\,\hat{b}(z,t_2)\rangle$$

can be used to characterize the fluctuations in the photon arrival times. It can also be measured by measuring the correlation function of the photodetector current,

$$\left\langle \hat{I}(t_1)\,\hat{I}(t_2)\right\rangle = q^2\left\langle \hat{F}(z_d,t_1)\,\hat{F}(z_d,t_2)\right\rangle = q^2 v_g^2\left\langle \hat{b}^+(z_d,t_1)\,\hat{b}(z_d,t_1)\,\hat{b}^+(z_d,t_2)\,\hat{b}(z_d,t_2)\right\rangle$$




The normalized second order coherence function is closely related to the photon flux correlation function and is defined as,

$$g_2(z:t_1,t_2) = \frac{\left\langle \hat{b}^+(z,t_2)\,\hat{b}^+(z,t_1)\,\hat{b}(z,t_1)\,\hat{b}(z,t_2)\right\rangle}{\left\langle \hat{b}_1^+(z,t_1)\hat{b}(z,t_1)\right\rangle\left\langle \hat{b}_1^+(z,t_2)\hat{b}(z,t_2)\right\rangle} = g(z:t_2,t_1)$$

The symmetry of $g_2$ under the interchange of the time variables is not obvious. Notice that,

$$\begin{aligned}
&\left\langle \hat{b}^+(z,t_2)\,\hat{b}^+(z,t_1)\,\hat{b}(z,t_1)\,\hat{b}(z,t_2)\right\rangle \\
&= \left\langle \hat{b}^+(z-v_g t_2,0)\,\hat{b}^+(z-v_g t_1,0)\,\hat{b}(z-v_g t_1,0\;\;\hat{b}(z-v_g t_2,0)\right\rangle \\
&= \left\langle \hat{b}^+(z-v_g t_1,0)\,\hat{b}^+(z-v_g t_2,0)\,\hat{b}(z-v_g t_2,0\;\;\hat{b}(z-v_g t_1,0)\right\rangle \qquad \left\{ \text{Using equal - time commutation relations} \right. \\
&= \left\langle \hat{b}^+(z,t_1)\,\hat{b}^+(z,t_2)\,\hat{b}(z,t_2)\,\hat{b}(z,t_1)\right\rangle
\end{aligned}$$

And, therefore, $g_2(z:t_1,t_2) = g(z:t_2,t_1)$.

The relationship between the second order coherence function and the photon flux correlation can be established as follows. Recall the equal-space commutation relation,

$$\left[\hat{b}(z,t),\hat{b}^+(z,t')\right] = \frac{1}{v_g}\,\delta(t-t')$$

Therefore,

$$\begin{aligned}
&\left\langle \hat{b}^+(z,t_2)\,\hat{b}^+(z,t_1)\,\hat{b}(z,t_1)\,\hat{b}(z,t_2)\right\rangle \\
&= \left\langle \hat{b}^+(z,t_1)\,\hat{b}^+(z,t_2)\,\hat{b}(z,t_1)\,\hat{b}(z,t_2)\right\rangle \\
&= \left\langle \hat{b}^+(z,t_1)\,\hat{b}(z,t_1)\,\hat{b}^+(z,t_2)\,\hat{b}(z,t_2)\right\rangle - \frac{\delta(t_1-t_2)}{v_g}\left\langle \hat{b}^+(z,t_1)\,\hat{b}(z,t_2)\right\rangle
\end{aligned}$$

It follows that,

$$g_2(z:t_1,t_2) = \frac{\left\langle \hat{F}(z,t_1)\,\hat{F}(z,t_2)\right\rangle - \left\langle \hat{F}(z,t_1)\right\rangle \delta(t_1-t_2)}{\left\langle \hat{F}(z,t_1)\right\rangle\left\langle \hat{F}(z,t_2)\right\rangle}$$

Assuming a constant average photon flux radiation state with,

$$\left\langle \hat{F}(z_d,t_1)\right\rangle = F_o$$

$$\left\langle \hat{I}(t)\right\rangle = qF_o$$

we can write the current correlation function in terms of the second order coherence function,

$$\left\langle \hat{i}(t_1)\,\hat{i}(t_2)\right\rangle = q^2\left\langle \hat{F}(z_d,t_1)\,\hat{F}(z_d,t_2)\right\rangle$$
$$= q^2 v_g^2 \left\langle \hat{b}^+(z_d,t_1)\,\hat{b}(z_d,t_1)\,\hat{b}^+(z_d,t_2)\,\hat{b}(z_d,t_2)\right\rangle$$
$$= q^2 F_o\left[\delta(t_1-t_2)+g_2(z_d:t_1,t_2)F_o\right]$$

A state has complete second order coherence if the correlation function,

$$\left\langle \hat{b}^+(z,t_2)\hat{b}^+(z,t_1)\,\hat{b}(z,t_1)\,\hat{b}(z,t_2)\right\rangle$$

factorizes,

$$\left\langle \hat{b}^+(z,t_2)\hat{b}^+(z,t_1)\,\hat{b}(z,t_1)\,\hat{b}(z,t_2)\right\rangle = \left\langle \hat{b}^+(z,t_2)\right\rangle\left\langle \hat{b}^+(z,t_1)\right\rangle\left\langle \hat{b}(z,t_1)\right\rangle\left\langle \hat{b}(z,t_2)\right\rangle$$

and $g_2(z:t_1,t_2)=1$. For example, for a continuous wave coherent state, $g_2(z:t_1,t_2)=1$.

### 10.2.4 Photon Bunching and Photon Anti-Bunching

The photon flux correlation function can be written as,

$$\left\langle \hat{F}(z_d,t+\tau)\,\hat{F}(z_d,t)\right\rangle = F_o\,\delta(\tau)+F_o^2\,g_2(z_d:\tau)$$

If $g_2(z_d:\tau)<g_2(z_d:0)$, then it means that photons have a tendency to arrive together. In other words, if the arrival of one photon is detected then there is an increased likelihood of another photon arriving soon afterwards. This phenomenon is called photon bunching. Light from thermal sources exhibits photon bunching. On the other hand, if $g_2(z_d:\tau)>g_2(z_d:0)$, then it means that photons have a tendency not to arrive together. If the arrival of one photon is detected then there is a reduced likelihood of another photon arriving soon afterwards. This situation is called photon anti-bunching. For example, consider a CW amplitude squeezed state (Section 10.1) with an average flux $F_o \approx v_g\,|\alpha|^2$ (assuming $|\alpha|^2 >> \Delta\beta$). The second order coherence function is,

$$g_2(z_d:\tau)\approx 1-\frac{1-e^{-2r}}{F_o}\delta(\tau)$$

Since $g_2(z_d:\tau)>g_2(z_d:0)$, a CW amplitude squeezed state exhibits photon anti-bunching on a very short time scale. In most practical cases, the delta function above is replaced by a function with some temporal width that represents the time scale over which photons are anti-bunched.

**Large Time Limit**: For a continuous wave coherent state we have,

$$g_2(z:\tau)=1$$

A continuous wave coherent state has shot noise and the arrival times of different photons are completely uncorrelated. For any real radiation state, $g_2(z:\tau)$ approaches unity for large $\tau$ since photon arrival times are not correlated if the photons are well separated in time (any physical source emitting these photon is unlikely to have correlations that live infinitely long), We can write,

$$g_2(z:\tau >> \tau_c)=1$$

The timescale $\tau_c$, during which $g_2(z:\tau)$ differs from unity, represents the timescale over which the photon arrival times are correlated in a radiation state.

### 10.2.5 Photon Counting Measurements: Poissonian, Super-Poissonian, and Sub-Poissonian Statistics

The photon flux noise correlation function can be written as,

$$\left\langle \Delta\hat{F}(z_d,t+\tau)\,\Delta\hat{F}(z_d,t)\right\rangle = F_o\,\delta(\tau)+F_o^2\left[g_2(z_d:\tau)-1\right]$$

Suppose one counts the photons arriving at the detector in time interval $T$. The average number of photons counted in time $T$ is,

$$\left\langle \hat{N}(z_d) \right\rangle = \int_{-T/2}^{T/2} \left\langle \hat{F}(z_d,t) \right\rangle dt = F_o T$$

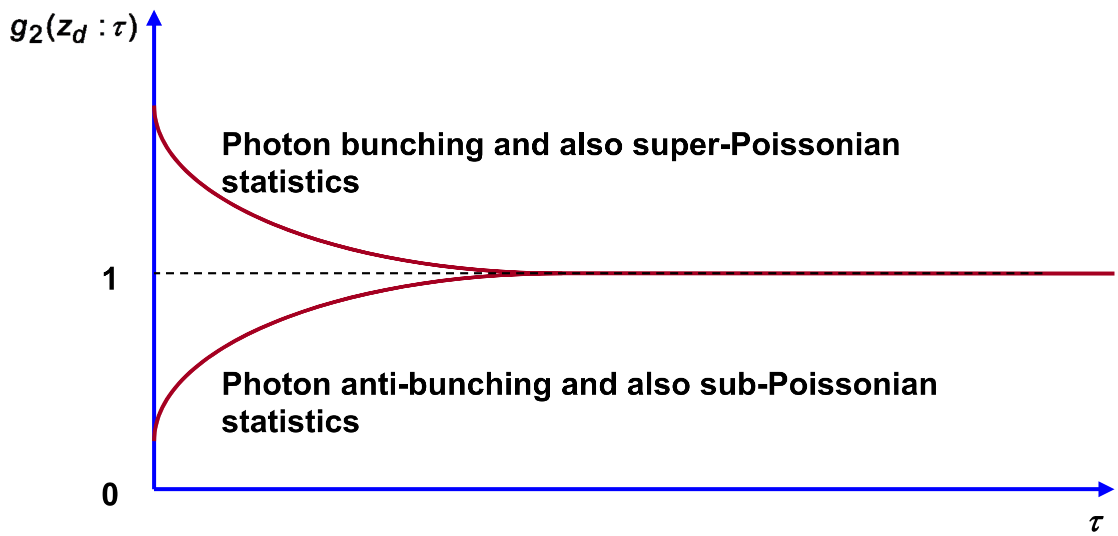
The variance in the photon number counted in time interval $T$ is,

$$\begin{aligned}\left\langle \Delta\hat{N}^2(z_d) \right\rangle &= \left\langle \hat{N}^2(z_d) \right\rangle - \left\langle \hat{N}(z_d) \right\rangle^2 \\ &= \int_{-T/2}^{T/2} dt_1 \int_{-T/2}^{T/2} dt_2 \left\langle \Delta\hat{F}(z_d,t_1)\, \Delta\hat{F}(z_d,t_2) \right\rangle dt \\ &= \int_{-T}^{T} d\tau \; (T-|\tau|) \left\{ F_o\, \delta(\tau) + F_o^2 \left[ g_2(z_d:\tau) - 1 \right] \right\} \\ &= \left\langle \hat{N}(z_d) \right\rangle + F_o^2 \int_{-T}^{T} d\tau \; (T-|\tau|) \left[ g_2(z_d:\tau) - 1 \right]\end{aligned}$$

In the limit $T \to \infty$, one obtains,

$$\frac{\left\langle \Delta\hat{N}^2(z_d) \right\rangle}{\left\langle \hat{N}(z_d) \right\rangle} = 1 + F_o \int_{-\infty}^{\infty} d\tau \quad \left[ g_2(z_d:\tau) - 1 \right]$$

.




**Poissonian Statistics:** If $g_2(z:\tau) = 1$, then,

$$\left\langle \Delta\hat{N}^2(z_d) \right\rangle = \left\langle \hat{N}^2(z_d) \right\rangle - \left\langle \hat{N}(z_d) \right\rangle^2 = \left\langle \hat{N}(z_d) \right\rangle$$

The variance in the photon number is equal to the mean and the radiation has Poissonian statistics. For example, a continuous wave coherent state has Poissonian statistics.

**Super-Poissonian Statistics:** If $g_2(z:\tau) > 1$, then,

$$\left\langle \Delta\hat{N}^2(z_d) \right\rangle = \left\langle \hat{N}^2(z_d) \right\rangle - \left\langle \hat{N}(z_d) \right\rangle^2 > \left\langle \hat{N}(z_d) \right\rangle$$

The variance in the photon number is larger than the mean and the radiation has super-Poissonian statistics. For example, radiation from a thermal source has super-Poissonian statistics.

**Sub-Poissonian Statistics:** If $g_2(z:\tau)<1$, then,

$$\left\langle \Delta\hat{N}^2(z_d)\right\rangle = \left\langle \hat{N}^2(z_d)\right\rangle - \left\langle \hat{N}(z_d)\right\rangle^2 < \left\langle \hat{N}(z_d)\right\rangle$$

The variance in the photon number is smaller than the mean and the radiation has sub-Poissonian statistics. For example, consider a CW amplitude squeezed state with an average flux $F_o \approx v_g |\alpha|^2$ (assuming $|\alpha|^2 >> 1$). The second order coherence function is,

$$g_2(z_d:\tau) \approx 1 - \frac{1-e^{-2r}}{F_o}\delta(\tau)$$

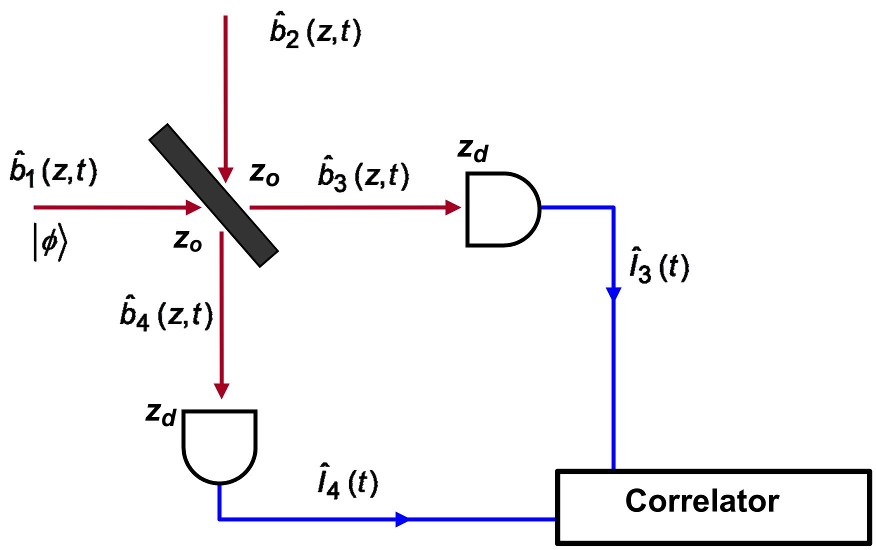
and,

$$\left\langle \Delta\hat{N}^2(z_d)\right\rangle \approx \left\langle \hat{N}(z_d)\right\rangle e^{-2r} < \left\langle \hat{N}(z_d)\right\rangle$$

Therefore, a CW amplitude squeezed state exhibits sub-Poission statistics.

### 10.2.6 Measurement of Second Order Coherence: The Hanburry Brown and Twiss Technique

Consider the arrangement shown in the Figure below.




The radiation comes in from port 1 of a 50-50 beam splitter,

$$|\psi(t=0)\rangle = |\phi\rangle_1 \otimes |0\rangle_2$$

The correlator computes the current correlation $\left\langle \hat{i}_3(t_1)\hat{i}_4(t_2)\right\rangle$. The beam splitter relation is,

$$\begin{bmatrix} \hat{b}_3(z_o,t)\, e^{i\beta_o z_o - i\omega_o t} \\ b_4(z_o,t)\, e^{i\beta_o z_o - i\omega_o t} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \hat{b}_1(z_o,t)\, e^{i\beta_o z_o - i\omega_o t} \\ \hat{b}_2(z_o,t)\, e^{i\beta_o z_o - i\omega_o t} \end{bmatrix}$$

which implies,

$$\hat{b}_3(z_o,t) = \frac{1}{\sqrt{2}}\left[\hat{b}_1(z_o,t) + \hat{b}_2(z_o,t)\right]$$
$$\hat{b}_4(z_o,t) = \frac{1}{\sqrt{2}}\left[-\hat{b}_1(z_o,t) + \hat{b}_2(z_o,t)\right]$$

And,

$$\hat{i}_3(t) = qv_g\ \hat{b}_3^+(z_d,t)\,\hat{b}_3(z_d,t)$$
$$\hat{i}_4(t) = qv_g\ \hat{b}_4^+(z_d,t)\,\hat{b}_4(z_d,t)$$

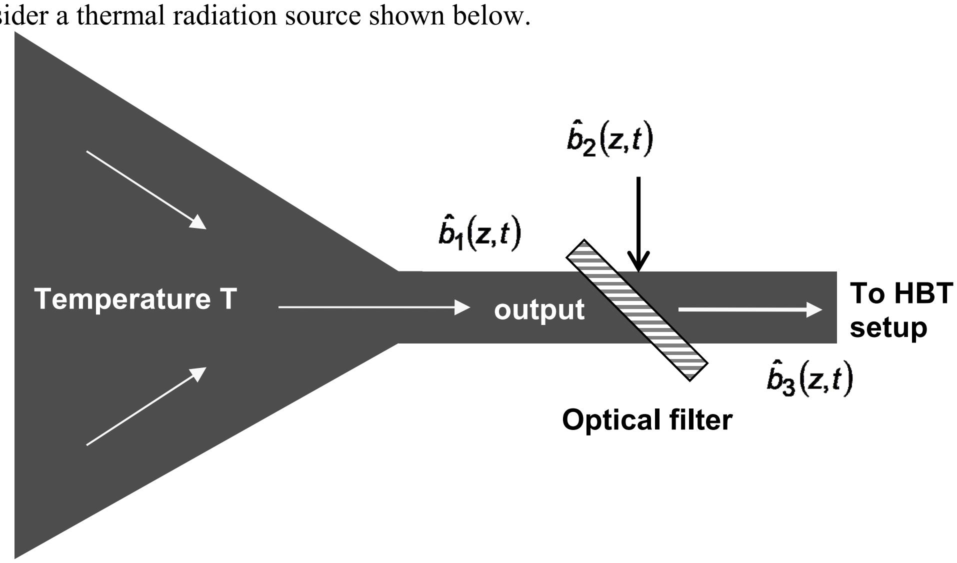
The current correlation function is then,

$$\frac{\langle\psi(t=0)|\hat{i}_3\left(t_1 + \frac{z_d - z_o}{v_g}\right)\hat{i}_4\left(t_2 + \frac{z_d - z_o}{v_g}\right)|\psi(t=0)\rangle}{(qv_g/2)^2}$$
$$= {}_1\langle\phi|\hat{b}_1^+(z_o,t_1)\,\hat{b}_1(z_o,t_1)\hat{b}_1^+(z_o,t_2)\,\hat{b}_1(z_o,t_2)|\phi\rangle_1$$
$$\quad - {}_1\langle\phi|\otimes{}_2\langle 0|\hat{b}_1^+(z_o,t_1)\,\hat{b}_2(z_o,t_1)\hat{b}_2^+(z_o,t_2)\,\hat{b}_1(z_o,t_1)|\phi\rangle_1 \otimes |0\rangle_2$$
$$= {}_1\langle\phi|\hat{b}_1^+(z_o,t_1)\,\hat{b}_1(z_o,t_1)\hat{b}_1^+(z_o,t_2)\,\hat{b}_1(z_o,t_2)|\phi\rangle_1$$
$$\quad - {}_1\langle\phi|\hat{b}_1^+(z_o,t_1)\,\hat{b}_1(z_o,t_2)|\phi\rangle_1 \frac{\delta(t_1 - t_2)}{v_g}$$
$$= {}_1\langle\phi|\hat{b}_1^+(z_o,t_2)\,\hat{b}_1^+(z_o,t_1)\hat{b}_1(z_o,t_1)\,\hat{b}_1(z_o,t_2)|\phi\rangle_1$$
$$= {}_1\langle\phi|\hat{b}_1^+(z_o,t_1)\hat{b}_1(z_o,t_1)|\phi\rangle_1\langle\phi|\hat{b}_1^+(z_o,t_2)\hat{b}_1(z_o,t_2)|\phi\rangle_1\ g_2(z_o : t_1,t_2)$$

Therefore, the current correlation is proportional to the second order coherence function of the input radiation.

### 10.2.7 Hanburry Brown and Twiss Technique for Thermal Radiation

Consider a thermal radiation source shown below.




The optical filter narrows the bandwidth of the thermal radiation. The power transmission through the filter can be written as,

$$|t(\beta)|^2 = e^{-\frac{(\beta-\beta_o)^2 L_c^2}{2}} = e^{-\frac{(\omega(\beta)-\omega_o)^2 L_c^2}{2 v_g^2}} = |t(\omega)|^2 = e^{-\frac{(\omega(\beta)-\omega_o)^2 \tau_c^2}{2}}$$

The quantities $L_c$ and $\tau_c$, as we will see, will also turn out to be the thermal correlation length and time, respectively, of the thermal source. Recall that,

$$\hat{b}_1(z,t) = L \int_{\beta_o-\Delta\beta/2}^{\beta_o+\Delta\beta/2} \frac{d\beta}{2\pi}\; \hat{b}_1(\beta) \frac{\exp[i(\beta-\beta_o) z]}{\sqrt{L}} \exp[-i(\omega(\beta)-\omega(\beta_o))\, t]$$

For thermal radiation,

$$\left\langle \hat{b}^+(\beta)\,\hat{b}(\beta')\right\rangle = n_{th}(\omega(\beta))\,\delta_{\beta\beta'} = n_{th}(\omega_o)\,\delta_{\beta\beta'}$$

$$\left\langle \hat{b}^+(\beta)\,\hat{b}^+(\beta')\,\hat{b}(\beta'')\hat{b}(\beta''')\right\rangle = (n_{th}(\omega_o))^2\left(\delta_{\beta\beta'''}\delta_{\beta'\beta''} + \delta_{\beta\beta''}\delta_{\beta'\beta'''}\right)$$

One can model the filter as a beam splitter with frequency, or equivalently, wavevector dependent transmission,

$$\hat{b}_3(\beta) = t(\beta)\hat{b}_1(\beta) + r(\beta)\hat{b}_2(\beta)$$

Putting it all together, the average flux, the first order coherence function, and the second order coherence function for the radiation going to the Hanbury Brown and Twiss (HBT) setup are,

$$\left\langle \hat{F}_3(z,t)\right\rangle = v_g n_{th}(\omega_o) \int_{\beta_o+\Delta\beta/2}^{\beta_o+\Delta\beta/2} \frac{d\beta}{2\pi} \left|t(\beta)^2\right| = \frac{v_g}{\sqrt{2\pi}\; L_c} n_{th}(\omega_o) = \frac{n_{th}(\omega_o)}{\sqrt{2\pi}\;\tau_c}$$

$$g_1(\tau) = e^{-(\tau/\tau_c)^2/2} e^{-i\omega_o \tau}$$

$$g_2(\tau) = 1 + e^{-(\tau/\tau_c)^2} = 1 + |g_1(\tau)|^2$$

The second order coherence function shows that the photons arrive bunched and the temporal duration of the bunches is $\tau_c$. The first order coherence function shows that the spectrum of the radiation, as expected, is,

$$S(\omega) = \int_{-\infty}^{\infty} d\tau \quad g_1(\tau)\;\; e^{i\omega\tau} = e^{-\frac{(\omega-\omega_o)^2 \tau_c^2}{2}}$$

The interesting property,

$$g_2(\tau) = 1 + |g_1(\tau)|^2$$

generally holds for radiation states for which the following factorization can be performed,

$$\left\langle \hat{b}^+(\beta)\,\hat{b}^+(\beta')\,\hat{b}(\beta'')\hat{b}(\beta''')\right\rangle = \left\langle \hat{b}^+(\beta')\,\hat{b}(\beta'')\right\rangle\left\langle \hat{b}^+(\beta)\,\hat{b}(\beta''')\right\rangle + \left\langle \hat{b}^+(\beta')\,\hat{b}(\beta''')\right\rangle\left\langle \hat{b}^+(\beta)\,\hat{b}(\beta'')\right\rangle$$

and is the basis for the practical measurement of the spectra of extremely narrowband radiation signals. For example, consider a radiation state with a coherence time, as given by $g_1(\tau)$, of ~10 nanoseconds. An interferometer capable of path length differences of more than 10 meters is required to measure $g_1(\tau)$. A more practical method is the measurement of the second order coherence function $g_2(\tau)$, using HBT setup, from which $g_1(\tau)$ can be extracted.

## 10.3 Coherent Detection

In the first Section of this Chapter, we discussed direct photodetection. In Direct detection, the photon flux or the photon number is detected. Coherent detection is used when one or both quadratures of a quantum state need to be measured. Coherent detection comes in two varieties: homodyne detection and heterodyne detection.

### 10.3.1 Balanced Homodyne Detection

In homodyne detection, the signal $|\phi\rangle$ whose quadrature is to be measured is mixed with a very strong local oscillator signal $|\beta\rangle$ using a 50/50 beam splitter, as shown below. The difference current from the two balanced photodetectors is taken as the output.

We assume that the input signal is some continuous wave quantum state and the local oscillator signal is a continuous wave coherent state with a constant complex amplitude $\beta$, where $\beta = |\beta|\, e^{i\theta}$. We assume that $|\beta|$ is very large ($|\beta| >> \Delta\beta$). The complete input state $|\psi(t=0)\rangle$ can be written as,

$$|\psi(t=0)\rangle = |\phi\rangle_1 \otimes |\beta\rangle_2$$

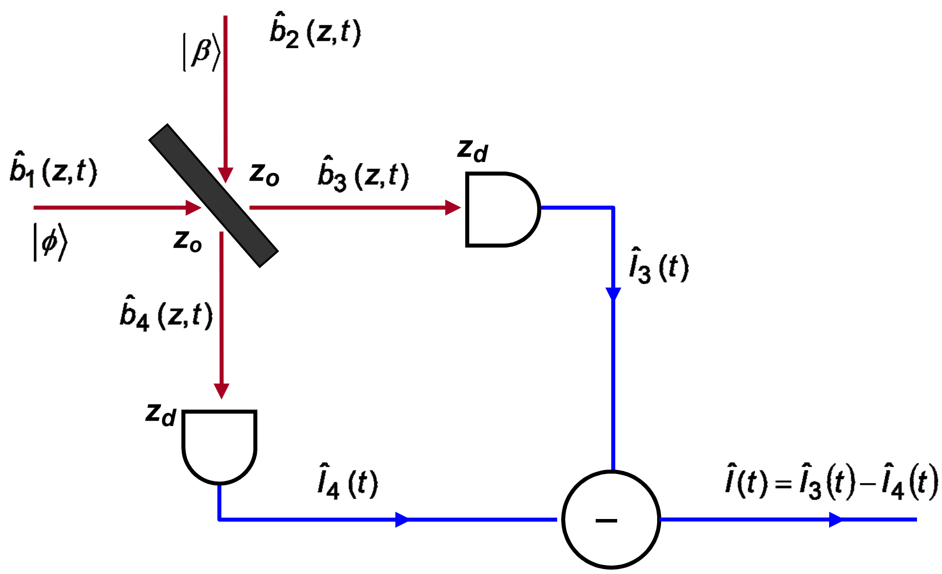
The beam splitter relation is,

$$\begin{bmatrix} \hat{b}_3(z_o,t)\, e^{i\beta_o z_o - i\omega_o t} \\ b_4(z_o,t)\, e^{i\beta_o z_o - i\omega_o t} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \hat{b}_1(z_o,t)\, e^{i\beta_o z_o - i\omega_o t} \\ \hat{b}_2(z_o,t)\, e^{i\beta_o z_o - i\omega_o t} \end{bmatrix}$$

which implies,

$$\hat{b}_3(z_o,t) = \frac{1}{\sqrt{2}}\left[\hat{b}_1(z_o,t) + \hat{b}_2(z_o,t)\right]$$

$$\hat{b}_4(z_o,t) = \frac{1}{\sqrt{2}}\left[-\hat{b}_1(z_o,t) + \hat{b}_2(z_o,t)\right]$$




The difference current obtained by subtracting the currents from the two photodetectors is (assuming $v_g t > z_d - z_o$),

$$\hat{i}(t) = \hat{i}_3(t) - \hat{i}_4(t) = q v_g \left[\hat{b}_3^+(z_d,t)\,\hat{b}_3(z_d,t) - \hat{b}_4^+(z_d,t)\,\hat{b}_4(z_d,t)\right]$$

$$= q v_g \left[\hat{b}_2^+(z_d - v_g t,0)\,\hat{b}_1(z_d - v_g t,0) + \hat{b}_1^+(z_d - v_g t,0)\,\hat{b}_2(z_d - v_g t,0)\right]$$

The average difference current is,

$$\left\langle \hat{i}(t) \right\rangle = \langle \psi(t=0) | \, \hat{i}(t) \, | \psi(t=0) \rangle$$
$$= q v_g \langle \psi(t=0) | \, \hat{b}_2^+(z_d - v_g t, 0) \, \hat{b}_1(z_d - v_g t, 0) + \hat{b}_1^+(z_d - v_g t, 0) \, \hat{b}_2(z_d - v_g t, 0) \, | \psi(t=0) \rangle$$
$$= q v_g \, |\beta| \; {}_1\langle \phi | \, e^{-i\theta} \; \hat{b}_1(z_d - v_g t, 0) + e^{i\theta} \; \hat{b}_1^+(z_d - v_g t, 0) \, | \phi \rangle_1$$
$$= 2 q v_g \, |\beta| \; {}_1\langle \phi | \, \hat{x}_\theta(z_d - v_g t, 0) \, | \phi \rangle_1$$

The above expression shows that the difference photocurrent is proportional to the average value of the $\theta$ quadrature of the input state. The signal quadrature to be measured is therefore selected by adjusting the phase of the local oscillator. Note that the difference photocurrent is proportional to the strength of the local oscillator signal. Usually a low-noise high-power laser is used as the local oscillator.

The current correlation can be obtained similarly (ignoring terms that are not proportional to $|\beta|^2$),

$$\left\langle \hat{i}(t_1) \, \hat{i}(t_2) \right\rangle = \langle \psi(t=0) | \, \hat{i}(t_1) \, \hat{i}(t_2) \, | \psi(t=0) \rangle$$
$$= \left(2 q v_g \, |\beta|\right)^2 \; {}_1\langle \phi | \, \hat{x}_\theta(z_d - v_g t_1, 0) \, \hat{x}_\theta(z_d - v_g t_2, 0) \, | \phi \rangle_1$$

The current noise correlation function is then,

$$\left\langle \Delta\hat{i}(t_1) \, \Delta\hat{i}(t_2) \right\rangle = \langle \psi(t=0) | \, \hat{i}(t_1) \, \hat{i}(t_2) \, | \psi(t=0) \rangle$$
$$= \left(2 q v_g \, |\beta|\right)^2 \; {}_1\langle \phi | \, \Delta\hat{x}_\theta(z_d - v_g t_1, 0) \, \Delta\hat{x}_\theta(z_d - v_g t_2, 0) \, | \phi \rangle_1$$

The current noise correlation is directly related to the quadrature noise correlation function of the input signal. Therefore, current noise measurements can be used to characterize quadrature noise in the input signal. Note that there is no shot noise in the current even though the local oscillator is a strong coherent state and its photon flux has shot noise. The balanced detection scheme has resulted in the complete cancellation of the local oscillator shot noise. This cancellation happened when the currents from the two photodetectors were subtracted to obtain the output current.

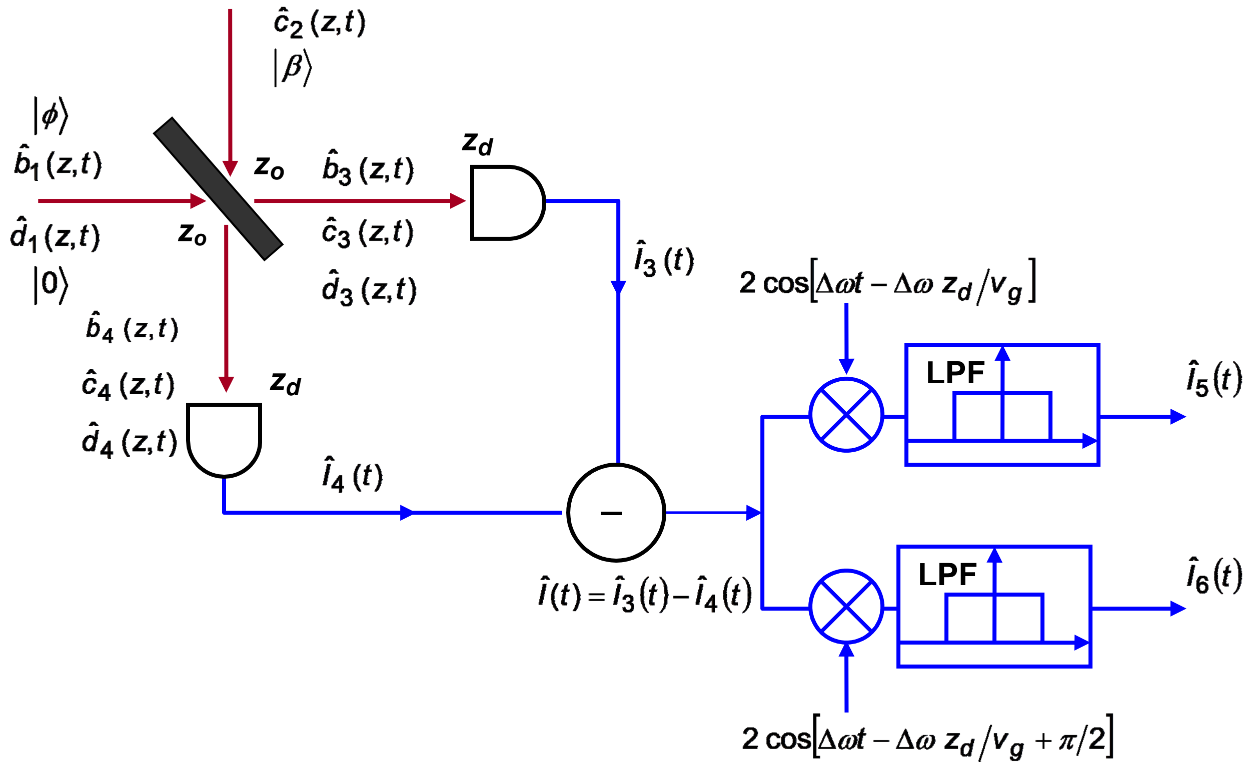
### 10.3.2 Balanced Heterodyne Detection

Balanced heterodyne detection is similar to balanced homodyne detection except that the signal $|\phi\rangle$ whose quadrature is to be measured is mixed with a local oscillator signal $|\beta\rangle$ that has a slightly different frequency than the center frequency of the signal. If the center frequency of the signal is $\omega_o$ and that of the local oscillator is $\omega_o + \Delta\omega$, then the difference current from the two detectors contains components at RF frequency $\Delta\omega$ and the strengths of these components are proportional to the two quadratures of the input signal. Any two desired orthogonal quadratures can be measured via electrical mixing and filtering, as shown below.

The vacuum, with center frequency $\omega_o + 2\Delta\omega$ and entering the beam splitter from the same port as the signal, also mixes with the local oscillator and contributes noise at RF frequency $\Delta\omega$ and must therefore be taken into account. We assume that the field operator for radiation with center frequency $\omega_o$ is $\hat{b}_k(z,t)$, for radiation with center frequency $\omega_o + \Delta\omega$ is $\hat{c}_k(z,t)$ and for radiation with center frequency $\omega_o + 2\Delta\omega$ is $\hat{d}_k(z,t)$.

The beam splitter relation for operators corresponding to frequency $\omega_o$ is,

$$\begin{bmatrix} \hat{b}_3(z_o,t) \\ b_4(z_o,t) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \hat{b}_1(z_o,t) \\ \hat{b}_2(z_o,t) \end{bmatrix}$$




Similar relations hold for operators corresponding to frequencies $\omega_o + \Delta\omega$ and $\omega_o + 2\Delta\omega$. We get (assuming $z_d - v_g t < z_o$),

$$\hat{b}_3(z_d,t) = \frac{1}{\sqrt{2}}\left[\hat{b}_1(z_d - v_g t,0) + \hat{b}_2(z_d - v_g t,0)\right]$$

$$\hat{b}_4(z_d,t) = \frac{1}{\sqrt{2}}\left[-\hat{b}_1(z_d - v_g t,0) + \hat{b}_2(z_d - v_g t,0)\right]$$

The photon flux in the output port 3 is proportional to the square of the total electric field. The total electric field in turn is proportional to the sum of the fields corresponding to the three frequencies,

$$\hat{E}(z,t) \propto \hat{b}_3(z_d,t)e^{i\beta_o z_d - i\omega_o t} + \hat{c}_3(z_d,t)e^{i\left(\beta_o + \frac{\Delta\omega}{v_g}\right)z_d - i(\omega_o + \Delta\omega)t}$$
$$+ \hat{d}_3(z_d,t)e^{i\left(\beta_o + \frac{2\Delta\omega}{v_g}\right)z_d - i(\omega_o + 2\Delta\omega)t}$$

We assume that the local oscillator signal is a continuous wave coherent state with a constant complex amplitude $\beta$, where $\beta = |\beta|\, e^{i\theta}$. We assume that $|\beta|$ is very large ($|\beta| >> \Delta\beta$). The complete input state $|\psi(t=0)\rangle$ can be written as,

$$|\psi(t=0)\rangle = |\phi\rangle_1^{\omega_o} \otimes |0\rangle_1^{\omega_o + 2\Delta\omega} \otimes |\beta\rangle_2^{\omega_o + \Delta\omega}$$

The operator for the current $\hat{i}_5(t)$ is,

$$\hat{I}_5(t) = qv_g\left[\hat{c}_2^+(z_d - v_g t,0)\,\hat{b}_1(z_d - v_g t,0) + \hat{b}_1^+(z_d - v_g t,0)\,\hat{c}_2(z_d - v_g t,0)\right] + qv_g\left[\hat{c}_2^+(z_d - v_g t,0)\,\hat{d}_1(z_d - v_g t,0) + \hat{d}_1^+(z_d - v_g t,0)\,\hat{c}_2(z_d - v_g t,0)\right]$$

The average value of the current $\hat{I}_5(t)$ is,

$$\begin{aligned}\left\langle \hat{I}_5(t)\right\rangle &= \langle\psi(t=0)|\,\hat{I}_5(t)\,|\psi(t=0)\rangle \\ &= qv_g\,\langle\psi(t=0)|\,\hat{c}_2^+(z_d - v_g t,0)\,\hat{b}_1(z_d - v_g t,0) + \hat{b}_1^+(z_d - v_g t,0)\,\hat{c}_2(z_d - v_g t,0)\,|\psi(t=0)\rangle \\ &= qv_g\,|\beta|\;{}_1^{\omega_o}\langle\phi|\,e^{-i\theta}\,\hat{b}_1(z_d - v_g t,0) + e^{i\theta}\,\hat{b}_1^+(z_d - v_g t,0)\,|\phi\rangle_1^{\omega_o} \\ &= 2qv_g\,|\beta|\;{}_1^{\omega_o}\langle\phi|\,\hat{x}_\theta(z_d - v_g t,0)\,|\phi\rangle_1^{\omega_o}\end{aligned}$$

The above expression shows that the average current is proportional to the average value of the $\theta$ quadrature of the input signal. Although, the signal quadrature to be measured can be selected by adjusting the phase of the local oscillator, a much more convenient scheme is to vary the phase of the RF signal that multiplies the difference current.

Similarly, the operator for the current $\hat{I}_6(t)$ is,

$$\hat{I}_6(t) = -iqv_g\left[\hat{c}_2^+(z_d - v_g t,0)\,\hat{b}_1(z_d - v_g t,0) - \hat{b}_1^+(z_d - v_g t,0)\,\hat{c}_2(z_d - v_g t,0)\right] + iqv_g\left[\hat{c}_2^+(z_d - v_g t,0)\,\hat{d}_1(z_d - v_g t,0) - \hat{d}_1^+(z_d - v_g t,0)\,\hat{c}_2(z_d - v_g t,0)\right]$$

The average value of the current $\hat{I}_6(t)$ is,

$$\begin{aligned}\left\langle \hat{I}_6(t)\right\rangle &= \langle\psi(t=0)|\,\hat{I}_6(t)\,|\psi(t=0)\rangle \\ &= 2qv_g\,|\beta|\;{}_1^{\omega_o}\langle\phi|\,\hat{x}_{\theta+\pi/2}(z_d - v_g t,0)\,|\phi\rangle_1^{\omega_o}\end{aligned}$$

The average current is proportional to the average value of the $\theta + \pi/2$ quadrature of the input signal.

Note that the balanced heterodyne scheme, in contrast to the balanced homodyne scheme, can be used to measure the values of both the quadratures of the input signal simultaneously. Heisenberg uncertainty relation tells us that simultaneous measurement of non-commuting physical observables cannot be done with arbitrary high accuracy. The lack of measurement accuracy in the present case is due to the vacuum noise that comes in from the input port 1 at center frequency $\omega_o + 2\Delta\omega$. To see this explicitly we look at the noise in the currents $\hat{I}_5(t)$ and $\hat{I}_6(t)$. The current noise correlation function for $\hat{I}_5(t)$ is,

$$\begin{aligned}\left\langle \Delta\hat{I}_5(t_1)\,\Delta\hat{I}_5(t_2)\right\rangle &= \langle\psi(t=0)|\,\Delta\hat{I}_5(t_1)\,\Delta\hat{I}_5(t_2)\,|\psi(t=0)\rangle \\ &= (2qv_g\,|\beta|)^2\;{}_1^{\omega_o}\langle\phi|\,\Delta\hat{x}_\theta(z_d - v_g t_1,0)\,\Delta\hat{x}_\theta(z_d - v_g t_2,0)\,|\phi\rangle_1^{\omega_o} \\ &\quad + (2qv_g\,|\beta|)^2\;{}_1^{\omega_o+2\Delta\omega}\langle\phi|\,\Delta\hat{x}_\theta(z_d - v_g t_1,0)\,\Delta\hat{x}_\theta(z_d - v_g t_2,0)\,|\phi\rangle_1^{\omega_o+2\Delta\omega}\end{aligned}$$

The first term on the right hand side is proportional to the noise in the input signal quadrature and the second term is proportional to the quadrature noise in the vacuum that comes in from the input port 1 at center frequency $\omega_o + 2\Delta\omega$. The second term represents the extra noise introduced due to the fact that the non-commuting quadratures are being measured simultaneously. Thus, while balanced heterodyne detection is able to measure both the quadratures simultaneously, it incurs a noise penalty that is not present in the case of balanced homodyne detection. We finally have,

$$\left\langle \Delta\hat{I}_5(t_1)\,\Delta\hat{I}_5(t_2)\right\rangle = \left(2qv_g\,|\beta|\right)^2 \left[ {}_1^{\omega_o}\langle\phi|\,\Delta\hat{x}_\theta\left(z_d - v_g t_1,0\right)\Delta\hat{x}_\theta\left(z_d - v_g t_2,0\right)|\phi\rangle_1^{\omega_o} + \frac{\delta(t_1-t_2)}{4v_g}\right]$$

Similarly,

$$\left\langle \Delta\hat{I}_6(t_1)\,\Delta\hat{I}_6(t_2)\right\rangle = \left(2qv_g\,|\beta|\right)^2 \left[ {}_1^{\omega_o}\langle\phi|\,\Delta\hat{x}_{\theta+\pi/2}\left(z_d - v_g t_1,0\right)\Delta\hat{x}_{\theta+\pi/2}\left(z_d - v_g t_2,0\right)|\phi\rangle_1^{\omega_o} + \frac{\delta(t_1-t_2)}{4v_g}\right]$$