Department of Electrical and Computer Engineering, Cornell University

ECE 5310: Applied Quantum Optics for Photonics and Optoelectronics

Fall 2013

Final Exam (Due on Monday, Dec. 16 by 5:00 PM)

1) All questions do not carry equal points

First problem: 40 points
Second problem: 30 points
Third problem: 35 points
Fourth problem: 15 points
Fifth problem: 20 points

Total: 140 points

2) No points will be awarded if the work that supports the answer is not shown.

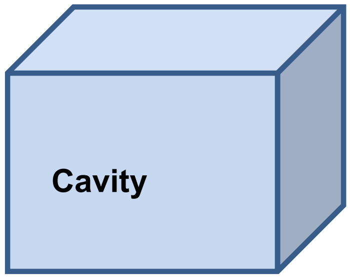
3) No points will be awarded if the answer is right but the reasoning is wrong. Points may also be deducted if the reasoning is weak.

Exam Rules:

1) **You cannot discuss with anybody the exam questions or even the course material/concepts during the exam period (from the start time to the time when you hand in the exam)**
2) **You have to work on the exam completely by yourself**
3) **You cannot consult any other material other than the course material (which includes handouts and homeworks/solutions only). You cannot consult even the recommended texts.**
4) **If you have question, feel free to email me**
5) **The exam is due by 5:00 PM on Monday, Dec. 16 in PH 316.**

Problem 1 (Gain and Photon Statistics – 40 Points)

Consider a single-mode closed cavity in which the destruction operator for the cavity mode with frequency ω_o is given by $\hat{a}$. The cavity contains a gain medium and is otherwise lossless.



For this problem the following definitions might or might not prove helpful.

i) Suppose $P(n)$ is the probability for finding n photons in the cavity mode. The z-transform $\tilde{P}(z)$ of the probability distribution $P(n)$ is:

$$\tilde{P}(z) = \sum_{n=0}^{\infty} z^n \, P(n)$$

Some common distributions and their z-transforms are:

Bose-Einstein or thermal distribution: $P(m) = \dfrac{1}{1+\langle n\rangle}\left(\dfrac{\langle n\rangle}{1+\langle n\rangle}\right)^m \quad \Leftrightarrow \quad \tilde{P}(z) = \dfrac{1}{1-\langle n\rangle(z-1)}$

Poisson distribution: $P(m) = e^{-\langle n\rangle}\dfrac{\langle n\rangle^m}{m!} \quad \Leftrightarrow \quad \tilde{P}(z) = e^{\langle n\rangle(z-1)}$

ii) The H-function for a quantum state is defined as (in the Schrodinger picture):

$$H(s) = \sum_{r=0}^{\infty}\frac{s^r}{r!}\left\langle (\hat{a})^r \left(\hat{a}^+\right)^r \right\rangle = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{s^r}{r!}P(n)\frac{(n+r)!}{n!} = \frac{1}{1-s}\sum_{n=0}^{\infty}P(n)\left(\frac{1}{1-s}\right)^n = \frac{1}{1-s}\tilde{P}\left(\frac{1}{1-s}\right)$$

iii) The G-function for a quantum state is defined as (in the Schrodinger picture):

$$G(s) = \sum_{r=0}^{\infty}\frac{s^r}{r!}\left\langle \left(\hat{a}^+\right)^r (\hat{a})^r \right\rangle = \sum_{n=0}^{\infty}\sum_{r=0}^{n}\frac{s^r}{r!}P(n)\frac{n!}{(n-r)!} = \sum_{n=0}^{\infty}P(n)\,(1+s)^n = \tilde{P}(1+s)$$

a) Suppose $|\psi(t=0)\rangle = |0\rangle$ (i.e. initial state of the radiation is a vacuum state). At time $t=0$ optical gain in the cavity is "switched on", and at time $t=T$ the gain is "switched off". Assume that the gain involved complete population inversion and the gain is completely linear. At time $t=T$, the destruction operator for the cavity mode can be written as:

$$\hat{a}(T) = \sqrt{G}\,\hat{a}(0) + \hat{F}_G$$

Find the probability $P(N)$ of finding N photons in the cavity at time $t=T$. Exact and complete derivation is required. Suggestive reasoning will carry no points. Does the distribution look familiar? **(10 points)**

b)) Suppose $|\psi(t=0)\rangle = |m\rangle$ (i.e. initial state of the radiation is a number state). At time $t=0$ optical gain in the cavity is "switched on", and at time $t=T$ the gain is "switched off". Assume that the gain involved complete population inversion and the gain is completely linear. At time $t=T$, the destruction operator for the cavity mode can be written as:

$$\hat{a}(T) = \sqrt{G}\ \hat{a}(0) + \hat{F}_G$$

Find the probability $P(N)$ of finding N photons in the cavity at time $t = T$. Obviously, $P(N) = 0$ for $N < m$. Exact and complete derivation is required. Suggestive reasoning will carry no points.
(10 points)

(c) Suppose the initial state of the radiation is a statistical mixture and is given by,

$$\hat{\rho}(t=0) = \sum_{m=0}^{\infty} \frac{1}{1+\langle n\rangle}\left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^m |m\rangle\langle m|$$

At time $t = 0$ optical gain in the cavity is "switched on", and at time $t = T$ the gain is "switched off". Assume that the gain involved complete population inversion and the gain is completely linear. At time $t = T$, the destruction operator for the cavity mode can be written as:

$$\hat{a}(T) = \sqrt{G}\ \hat{a}(0) + \hat{F}_G$$

Find the probability $P(N)$ of finding N photons in the cavity at time $t = T$. Exact and complete derivation is required. Suggestive reasoning will carry no points.
(10 points)

d) Now consider a cavity containing an extremely large number of modes with the same frequency ω_o. Let the total number of these degenerate modes be g. The probability of finding m photons in any one of these modes is given by the thermal (or Bose-Einstein) distribution,

$$P(m) = \frac{1}{1+\langle n\rangle}\left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^m \qquad \Leftrightarrow \qquad \tilde{P}(z) = \frac{1}{1-\langle n\rangle(z-1)}$$

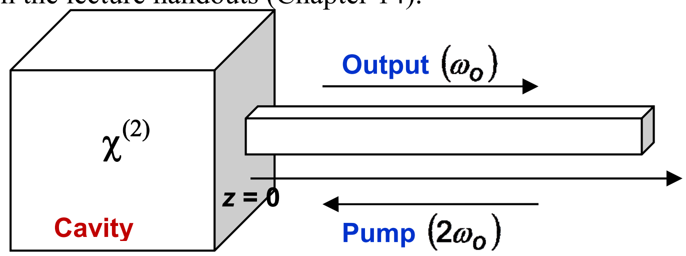
The density matrix of each mode is as in part (c). We assume that $\langle n\rangle << 1$, $g\langle n\rangle \sim 1$, and $g >> 1$. Suppose one now performs a measurement of the photon number inside the cavity but with no regard to the mode. So photons belonging to any mode can contribute to the measurement. For example, if there were only two degenerate modes in the cavity, i.e. $g = 2$, the probability of finding N photons upon measurement would equal the convolution,

$$\sum_{r=0}^{N} P(r)P(N-r)$$

But, of course, in our actual problem $g >> 1$. You need to find the probability of finding N photons in the cavity upon measurement in the limit that g is very large. You will notice that although each mode has a thermal distribution, the answer will look very different (and perhaps even familiar). Exact and complete derivation is required. Suggestive reasoning will carry no points.
(10 points)

Problem 2 (Optical Parametric Oscillator – 30 Points)

Consider an OPO as discussed in the lecture handouts (Chapter 14).



a) Consider the OPO operating below threshold. Find the average photon flux coming out of the cavity at frequency ω_o as a function of the pumping rate r_p.

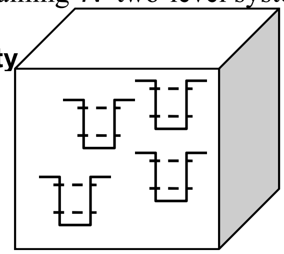
(15 points)

b) Consider an OPO operating below threshold. The spectral density of the radiation coming out of the cavity at frequency ω_o is given by the Fourier transform of the first order coherence function. Find the spectral density of the radiation coming out of the cavity at frequency ω_o, and from your result specify the radiation linewidth and also specify what happens to the radiation linewidth as the pumping rate r_p is increased towards threshold.

(15 points)

Problem 3 (Cavity Quantum Optics – 35 Points)

Parts (a) and (b):
Consider an optical cavity containing N two-level systems, as shown below.



The Hamiltonian is,

$$\hat{H} = \sum_{j=1}^{N} \left\{ \varepsilon_1 \hat{N}_{1j} + \varepsilon_2 \hat{N}_{2j} \right\} + \hbar\omega_o \hat{a}^+ \hat{a} + \sum_{j=1}^{N} \left\{ k\, \hat{\sigma}_{+j}\, \hat{a} + k^* \hat{a}^+ \hat{\sigma}_{-j} \right\}$$

$$\hat{N}_{1j} = |e_1\rangle_j \, {}_j\langle e_1| \qquad \hat{N}_{2j} = |e_2\rangle_j \, {}_j\langle e_2|$$

$$\hat{\sigma}_{+j} = |e_2\rangle_j \, {}_j\langle e_1| \qquad \hat{\sigma}_{-j} = |e_1\rangle_j \, {}_j\langle e_2|$$

Assume zero detuning, $\varepsilon_2 = \varepsilon_1 + \hbar\omega_o$.

a) Suppose the initial state of the N two-level systems and the cavity mode is,

$$|\psi(t=0)\rangle = \left\{ |e_2\rangle_1 \otimes |e_1\rangle_2 \otimes |e_1\rangle_3 \ldots\ldots\ldots |e_1\rangle_N \right\} \otimes |0\rangle$$

In the initial state, one of the two-level systems is in the upper state and the rest are in the lower state and the cavity contains no photons. Find the state of the system for time $t > 0$. There should be nothing undetermined in your answer.
(15 points)

b) Consider the following state:

$$|E\rangle = \frac{1}{\sqrt{N}} \Big[\left\{ |e_2\rangle_1 \otimes |e_1\rangle_2 \otimes |e_1\rangle_3 \ldots\ldots\ldots |e_1\rangle_N \right\} + \left\{ |e_1\rangle_1 \otimes |e_2\rangle_2 \otimes |e_1\rangle_3 \ldots\ldots\ldots |e_1\rangle_N \right\}$$
$$+ \left\{ |e_1\rangle_1 \otimes |e_1\rangle_2 \otimes |e_2\rangle_3 \ldots\ldots\ldots |e_1\rangle_N \right\} + \ldots\ldots\ldots\ldots + \left\{ |e_2\rangle_1 \otimes |e_1\rangle_2 \otimes |e_1\rangle_3 \ldots\ldots\ldots |e_2\rangle_N \right\} \Big] \otimes |0\rangle$$

which is a superposition of states in which one only one two-level system is in the upper state and the rest are in the lower state and the radiation mode has no photons. Suppose,

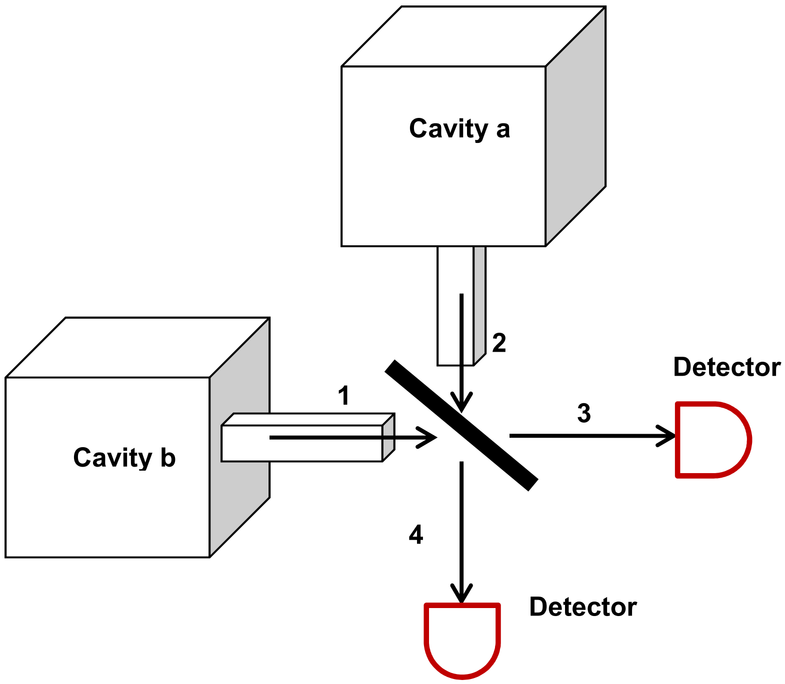
$$|\psi(t=0)\rangle = |E\rangle$$

Find the probability $P(t)$ that at time t there will be a photon inside the cavity. If the probability is a periodic function of time then find the Rabi frequency. Does the Rabi frequency depend on the number of two-level systems in the cavity?
(10 points)

Part (c):
c) Now consider two cavities, as shown below. The output of each cavity is directed at a 50-50 beam splitter with the relation,

$$\begin{bmatrix} \hat{b}_3(z_o,t) \\ b_4(z_o,t) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \hat{b}_1(z_o,t) \\ \hat{b}_2(z_o,t) \end{bmatrix}$$

There are detectors placed at the output of each beam splitter.



Suppose at time $t = 0$, the state of radiation in the cavities is described by the state:

$$|\psi(t=0)\rangle = |n\rangle_a \otimes |m\rangle_b$$

$$\Rightarrow \hat{\rho}(t=0) = |\psi(t=0)\rangle\langle\psi(t=0)|$$

Suppose that the photon lifetimes in the cavities are very long (which means that once in a long time a photon escapes from the cavity and gets detected).

The system, prepared as above, is allowed to evolve in time. At time T, a single photon is detected by the detector placed in channel 4 of the beam splitter. Specify the state of the radiation in the cavities at time T after the photon has been detected.

(10 points)

Problem 4 (Nonlinear Propagation and Quantum Noise – 15 Points)

Consider wave propagation in a waveguide with a Kerr-like nonlinearity and also two-photon absorption. In the case of Kerr-nonlinearity the refractive index of the medium becomes intensity dependent. In the case of two-photon absorption, two photons are simultaneously absorbed by the medium through a nonlinear process and therefore the loss is intensity dependent. Both these effects need to be included when modeling short pulse propagation in modelocked lasers. The quantum equation for the field operator in the presence of these nonlinearities can be written as:

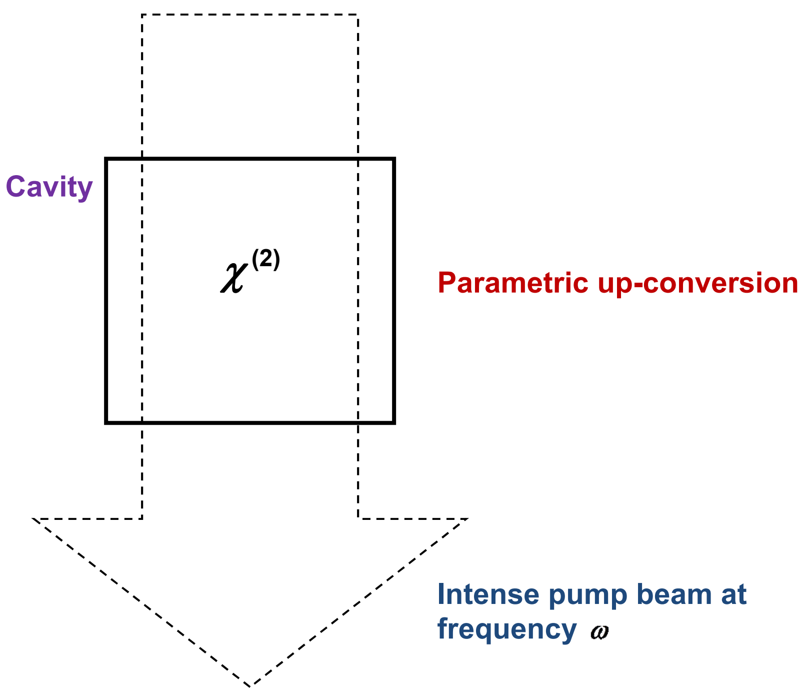
$$\left(\frac{\partial}{\partial z}+\frac{1}{v_g}\frac{\partial}{\partial t}\right)\hat{b}(z,t)=i\,\kappa\,\hat{b}^{+}(z,t)\hat{b}(z,t)\hat{b}(z,t)-\gamma\,\hat{b}^{+}(z,t)\hat{b}(z,t)\hat{b}(z,t)$$

The first term on the RHS stands for the Kerr nonlinearity and the second term models the loss due to the two-photon absorption.

a) Is the above equation quantum mechanically consistent? Explain and explicitly show if it is or is not consistent.
(5 points)

b) If your answer to part (a) is negative and you choose to add noise sources on the RHS to make the equation consistent, give the commutation relations as well as the relevant correlation functions of all the noise sources.
(10 points)

Problem 5 (Non-linear Frequency Up-conversion in a Cavity – 20 Points)
Parametric up-conversion is a process in which a pump photon of frequency ω combines with an idler photon of frequency ω_1 to produce a signal photon of frequency ω_2, where $\omega + \omega_1 = \omega_2$. The figure below shows a cartoon diagram of the setup. The cavity is assumed to support only two field modes at frequencies ω_1 and ω_2. The cavity is filled with a medium that has a non-zero second order nonlinear susceptibility $\chi^{(2)}$, and the cavity is irradiated with an intense pump field at frequency ω. The pump beam will be treated as being classical in this problem. The amplitude of the pump beam everywhere inside the cavity is assumed to be $E(t) = \mathrm{Re}\{ E_o \exp(-i\omega t + i\theta_o) \}$. The pump beam frequency is not supported by the cavity and therefore the pump beam photons just "pass" through the cavity (i.e. they don't stay for long in the cavity). For the modes at frequencies ω_1 and ω_2, the cavity is assumed to be lossless (i.e. the photons belonging to these modes do not escape from the cavity – the cavity is closed). The basic physics is as follows. The photons in the pump beam at frequency ω will combine with photons of frequency ω_1 that are assumed to be already present in the cavity, to produce photons of frequency ω_2 and as time goes on, photon population at frequency ω_2 will build up in the cavity. We would like to know the quantum state of the photons that get generated as a result of the up-conversion process and the associated dynamics.



The Hamiltonian of the system (including the non-linearity) is given by the expression,

$$\hat{H} = \hbar\omega_1\, \hat{a}_1^+ \hat{a}_1 \; + \; \hbar\omega_2\, \hat{a}_2^+ \hat{a}_2 \; + \; \hbar\kappa\left[\exp(i\omega t - i\theta_o)\, \hat{a}_1^+ \hat{a}_2 \; + \; \exp(-i\omega t + i\theta_o)\, \hat{a}_2^+ \hat{a}_1\right]$$

where the positive real constant κ is proportional to the product of the pump field amplitude E_o and the nonlinear susceptibility $\chi^{(2)}$.

a) Suppose the initial quantum state of the system prior to turning on the pump field is given by the expression,

$$|\psi(t=0)\rangle = |n\rangle_1 \, |0\rangle_2$$

which implies exactly n photons in mode 1 and 0 photons in mode 2. Find the quantum state of the system at some later time t and find the probability of finding m photons in mode 2 at time t.
(10 points)

b) Suppose the initial quantum state of the system prior to turning on the pump field is given by the expression,

$$|\psi(t=0)\rangle = |\alpha\rangle_1 \, |\beta\rangle_2$$

which implies coherent state α for mode 1 and coherent state β for mode 2 (where α and β are complex numbers). Find the quantum state of the system at some later time t and indicate whether the state is a squeezed state, or coherent state, or a two-photon coherent state.
(5 points)

c) Suppose the initial quantum state of the system prior to turning on the pump field is given by the expression,

$$|\psi(t=0)\rangle = |n\rangle_1 \, |m\rangle_2$$

which implies exactly n photons in mode 1 and m photons in mode 2. Find the average number of photons in mode 2 at some later time t.
(5 points)