ECE 5310: Exam 1 Solutions
(Farhan Rana)

a) $i\hbar \dfrac{d\hat{n}(t)}{dt} = [\hat{n}(t), \hat{H}(t)] = 0 \quad \Rightarrow \quad \hat{n}(t) = \hat{n}$

b) $\langle\alpha|\,\hat{n}(t)\,|\alpha\rangle = \langle\alpha|\,\hat{n}\,|\alpha\rangle = |\alpha|^2$

c) Consider $\hat{H}\,|n\rangle = \left\{ n\hbar\omega_0 - \dfrac{\hbar k}{2}\, n(n-1) \right\} |n\rangle \Rightarrow$ states $|n\rangle$ are eigenstates of $\hat{H}$ with energy eigenvalues $= n\hbar\omega_0 - \dfrac{\hbar k}{2}\, n(n-1)$

d) $i\hbar \dfrac{d\hat{a}(t)}{dt} = [\hat{a}(t), \hat{H}(t)] = \hbar\omega_0\, \hat{a}(t) - \hbar k\, \hat{a}^+(t)\,\hat{a}(t)\,\hat{a}(t)$

$\Rightarrow \dfrac{d\hat{a}(t)}{dt} = -i\omega_0\, \hat{a}(t) + ik\, \hat{a}^+(t)\,\hat{a}(t)\,\hat{a}(t) = -i\omega_0\hat{a}(t) + ik\, \hat{n}(t)\,\hat{a}(t)$

$= -i\omega_0 \hat{a}(t) + ik\,\hat{n}\,\hat{a}(t) = \left[-i\omega_0 + ik\,\hat{n}\right]\hat{a}(t)$

$\Rightarrow \hat{a}(t) = \exp\left[-i\omega_0 t + ikt\,\hat{n}\right] \hat{a}$

e) Need to find $\langle\alpha|\,\hat{a}(t)\,|\alpha\rangle = \langle\alpha|\, e^{-i\omega_0 t}\, e^{ikt\hat{n}}\, \hat{a}\,|\alpha\rangle$

$= \alpha e^{-i\omega_0 t} \langle\alpha|\, e^{ikt\hat{n}}\, |\alpha\rangle = \alpha e^{-i\omega_0 t} \langle\alpha|\,\alpha e^{ikt}\rangle$

$= \alpha e^{-i\omega_0 t}\, e^{-|\alpha|^2 + |\alpha|^2 e^{ikt}} = \alpha e^{-i\omega_0 t} \exp\left[-|\alpha|^2(1-\cos kt) + i|\alpha|^2 \sin kt\right]$

f) Energy lost from the cavity $= \left[n\hbar\omega_0 - \dfrac{\hbar k}{2}\, n(n-1)\right] - \left[(n-1)\hbar\omega_0 - \dfrac{\hbar k}{2}(n-1)(n-2)\right]$
$= \hbar\omega_0 - \hbar k(n-1)$. This energy must be taken away by the photon $\Rightarrow$ freq. of photon that escaped $= \omega_0 - k(n-1)$. The spectrometer will measure this with probability 1.

g) A coherent state is a linear superposition of number states with probability $\dfrac{|\alpha|^{2n}}{n!}\, e^{-|\alpha|^2}$ for number state $|n\rangle$. Spectrometer will measure frequency $\omega_0 - k(n-1)$ with probability $\dfrac{|\alpha|^{2n}}{n!}\, e^{-|\alpha|^2}$ for $n = 1, 2, 3 \ldots \infty$.

h) We write a state as $|n_1, n_2\rangle$. Since $n_1+n_2=2$, the three possible states are: $|2,0\rangle$, $|1,1\rangle$, $|0,2\rangle$. Suppose the eigenstate is $|\psi\rangle = a|2,0\rangle + b|1,1\rangle + c|0,2\rangle$. We have $\hat{H}|\psi\rangle = E|\psi\rangle$. Taking the product of this equation with $\langle 2,0|$, then $\langle 1,1|$, and then $\langle 0,2|$ gives:

$$\begin{bmatrix} 2\hbar\omega_0 - \hbar K & -U & 0 \\ -U & 2\hbar\omega_0 & -U \\ 0 & -U & 2\hbar\omega_0 - \hbar K \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = E \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$E_1 = 2\hbar\omega_0 - \hbar K$$

$$E_2 = 2\hbar\omega_0 - \frac{\hbar K}{2} - \frac{\sqrt{8U^2 + (\hbar K)^2}}{2}$$

$$E_3 = 2\hbar\omega_0 - \frac{\hbar K}{2} + \frac{\sqrt{8U^2 + (\hbar K)^2}}{2}$$

$$V_1 \propto \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad V_2 \propto \begin{bmatrix} 1 \\ -\frac{\hbar K}{2U} + \frac{\sqrt{8U^2 + (\hbar K)^2}}{2U} \\ 1 \end{bmatrix} \quad V_3 \propto \begin{bmatrix} 1 \\ -\frac{\hbar K}{2U} - \frac{\sqrt{8U^2 + (\hbar K)^2}}{2U} \\ 1 \end{bmatrix}$$

i) $$|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}\left\{|v_2\rangle - |v_3\rangle\right\} \Rightarrow |\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{-iE_2 t/\hbar}|v_2\rangle - e^{-iE_3 t/\hbar}|v_3\rangle\right].$$

2.

a) Since $i(t) = f(t) + r(t) \Rightarrow n_i(t) = n_f(t) + n_r(t) = 0 \Rightarrow n_r(t) = -n_f(t)$.

$$\Rightarrow \langle n_f(t+\tau)\, n_r(t)\rangle = -\langle n_f(t+\tau)\, n_f(t)\rangle = -\eta(1-\eta)\langle i(t)\rangle\,\delta(\tau).$$

b) The probability that n out of N particles incident on the splitter will get transmitted is $= \frac{N!}{n!(N-n)!}\eta^n(1-\eta)^{N-n}$. Therefore,

$$P_B(n,T) = \sum_{N=n}^{\infty} \frac{N!}{n!(N-n)!}\eta^n(1-\eta)^{N-n} \quad P_A(n,T) = \sum_{N=n}^{\infty}\frac{N!}{n!(N-n)!}\eta^n(1-\eta)^{N-n}\frac{(\alpha T)^N}{N!}e^{-\alpha T}$$

$$= \frac{(\eta\alpha)^n e^{-\eta\alpha T}}{n!}$$

3.

a) One can first boost the energy of level 1 to get the effective Hamiltonian:

$$\hat{H}_R = (\varepsilon_3 - \Delta)|e_1\rangle\langle e_1| + \varepsilon_3|e_2\rangle\langle e_2| + \varepsilon_3|e_3\rangle\langle e_3| - \frac{\hbar\Omega_R}{2}\left[e^{i\phi}|e_1\rangle\langle e_3| + e^{-i\phi}|e_3\rangle\langle e_1|\right] - U\left[|e_2\rangle\langle e_3| + |e_3\rangle\langle e_2|\right]$$

Now this is equivalent to the EIT hamiltonian of homework 4, problem 1. Therefore:

$$\chi(\omega) = \frac{q^2 N(\vec{d}_{13}\cdot\hat{n})^2}{\varepsilon_0\hbar}\;\frac{-\Delta/\hbar}{i\gamma_{13}\frac{\Delta}{\hbar} - \left(\frac{\Delta}{\hbar}\right)^2 + \left(\frac{U}{\hbar}\right)^2}$$

b) When $\Delta = 0$, $\mathrm{Im}\{\chi(\omega)\} = 0$.

c) The EIT condition is req'd.:

$$\varepsilon_3 - \varepsilon_2 = \varepsilon_3 - (\varepsilon_1 + \hbar\omega) = \Delta$$