Department of Electrical and Computer Engineering, Cornell University

ECE 5310: Applied Quantum Optics for Photonics and Optoelectronics

Fall 2013

Midterm Exam

1) All questions do not carry equal points

First problem: 80 points
Second problem: 20 points
Third problem: 30 points

Total: 130 points

2) No points will be awarded if the work that supports the answer is not shown.

3) No points will be awarded if the answer is right but the reasoning is wrong. Points may also be deducted if the reasoning is weak.

Exam Rules:

1) **You cannot discuss with anybody the exam questions or even the course material/concepts during the exam period (from the start time to the time when you hand in the exam)**

2) **You have to work on the exam completely by yourself**

3) **You cannot consult any other material other than the course material (which includes handouts and homeworks/solutions only). You cannot consult even the recommended texts.**

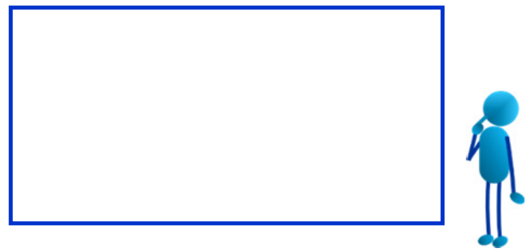
4) **If you have question, feel free to email me**

5) **The exam is due in class on Wednesday, October 23.**

Problem 1 (Optical nonlinearities)

Third order optical non-linearity (described by $\chi^{(3)}$) leads to many interesting quantum optical processes. Consider the optical cavity shown below.

Fig. p1: An optical cavity



The cavity is assumed to consist of a single optical mode. The cavity is filled uniformly with a material that exhibits a third order optical non-linearity. In the presence of the non-linearity the Hamiltonian becomes,

$$\hat{H} = \hbar\omega_o \hat{a}^+\hat{a} \quad - \quad \hbar\frac{\kappa}{2}\hat{a}^+\hat{a}^+\hat{a}\hat{a} \qquad (1)$$

where, κ is a real constant that is proportional to $\chi^{(3)}$. The Hamiltonian above describes photons interacting with each other via the optical non-linearity.

a) Find the Heisenberg time evolution equation for the photon number operator $\hat{n}(t) = \hat{a}^+(t)\hat{a}(t)$, and solve it with appropriate boundary conditions to find $\hat{n}(t)$ for all time $t \geq 0$ in terms of Schrodinger operators. **(5 points)**

b) Suppose at time $t = 0$ the quantum state of the field is given as follows,

$$|\psi(t=0)\rangle = \hat{D}(\alpha)|0\rangle = |\alpha\rangle$$

Using any method of your choice, find the average value of the photon number for all time $t \geq 0$. **(5 points)**

c) Using any method of your choice, find all the eigenstates and the corresponding eigenvalues of the above Hamiltonian. **(10 points)**

d) Find the Heisenberg time evolution equation for the destruction operator $\hat{a}(t)$, and solve it with appropriate boundary conditions to find $\hat{a}(t)$ for all time $t \geq 0$ in terms of Schrodinger operators. **(10 points)**

e) Suppose at time $t = 0$ the quantum state of the field is given as follows,

$$|\psi(t=0)\rangle = \hat{D}(\alpha)|0\rangle = |\alpha\rangle$$

Using any method of your choice, find the average value of the field operator $\hat{a}(t)$ for all time $t \geq 0$ and show that the answer is,

$$\alpha \exp(-i\omega_o t)\exp\left[-|\alpha|^2(1-\cos(\kappa t)) + i|\alpha|^2\sin(\kappa t)\right]$$

(Notice the time dependent magnitude and phase of the average field value). **(10 points)**

f) Suppose at time $t = 0$ the quantum state of the field is given as follows,

$$|\psi(t=0)\rangle = \frac{(\hat{a}^+)^n}{\sqrt{n!}}|0\rangle = |n\rangle$$

At time $T > 0$, a single photon somehow escapes from the cavity and its frequency is measured with a spectrometer. Assuming that the spectrometer can measure the frequency of a single photon very accurately, what should be the result of this frequency measurement? (i.e. what frequencies could be measured by the spectrometer and with what probabilities). Make sure you explain your answer well. **(10 points)**

g) Now suppose at time $t = 0$ the quantum state of the field is given as follows,

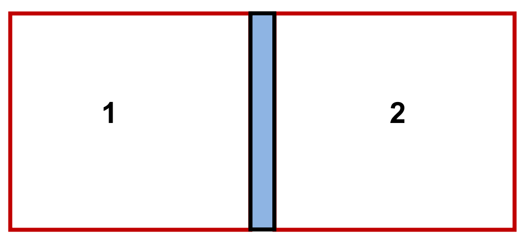
$$|\psi(t=0)\rangle = \hat{D}(\alpha)|0\rangle = |\alpha\rangle$$

At time $T > 0$, a single photon somehow escapes from the cavity and its frequency is measured with a spectrometer. Assuming that the spectrometer can measure the frequency of a single photon very accurately, what should be the result of this frequency measurement? (i.e. what frequencies could be measured by the spectrometer and with what probabilities). Make sure you explain your answer well. **(10 points)**

Now, for parts (h) and (i), consider a coupled cavity system in which each cavity has a nonlinearity, as shown below. The Hamiltonian is,

$$\hat{H} = \hbar\omega_o\left(\hat{a}_1^+\hat{a}_1 + \hat{a}_2^+\hat{a}_2\right) - \hbar\frac{\kappa}{2}\hat{a}_1^+\hat{a}_1^+\hat{a}_1\hat{a}_1 - \hbar\frac{\kappa}{2}\hat{a}_2^+\hat{a}_2^+\hat{a}_2\hat{a}_2 - U\left(\hat{a}_1^+\hat{a}_2 + \hat{a}_2^+\hat{a}_1\right)$$

Coupled optical cavities



h) Suppose we confine ourselves to the case where the total number of photons in the coupled-cavity system equals two. In this subspace of the full Hilbert space, there are three eigenstates of the Hamiltonian. Find these eigenstates and the corresponding eigenenergies. **(15 points)**

i) Suppose the quantum state at time $t = 0$ is,

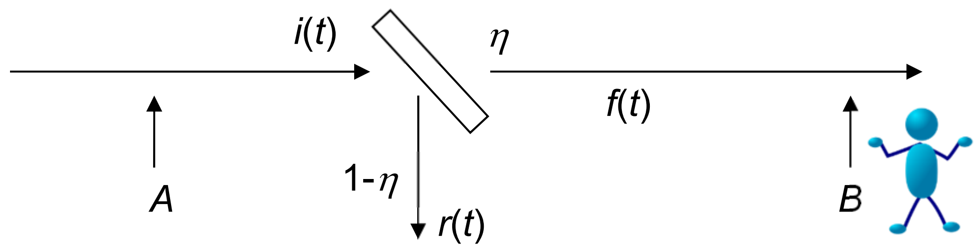
$$|\psi(t=0)\rangle = a_1^+ a_2^+ |0\rangle$$

Find the state for times $t > 0$.

(5 points)

Problem 2 (Particle splitter noise)

Consider a particle splitter (or beam splitter), as shown below:



$n_i(t) = i(t) - \langle i(t) \rangle$
$n_f(t) = f(t) - \langle f(t) \rangle$
$n_r(t) = r(t) - \langle r(t) \rangle$

a) Assume for simplicity that the noise in the incident particle stream is zero (i.e. $n_i(t) = 0$). Calculate the cross-correlation function $R_{n_f\, n_r}(\tau) = \langle\, n_f(t+\tau)\, n_r(t)\,\rangle$ between the noises in the forward and in the reflected particle streams.

10 points

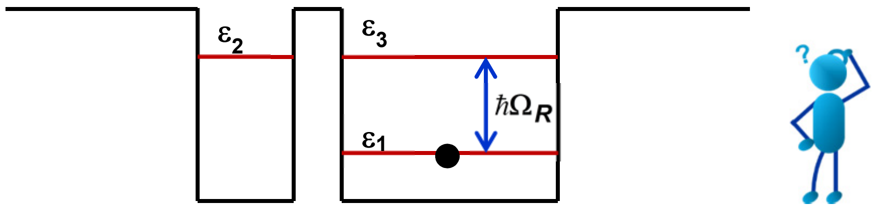
b) Suppose somebody sitting at point A counts the number of particles in the input stream crossing the point A in a given time interval T . Now suppose the person sitting at point A concludes that the probability $P_A(n,T)$ of counting n particles in time interval T is given by the relation,

$$P_A(n,T) = \frac{(\alpha T)^n}{n!} \exp(-\alpha T)$$

Find the probability $P_B(n,T)$ of counting n particles in time interval T in the transmitted stream crossing the location indicated by the point B in the figure above. **10 points**

Problem 3 (A coupled potential well system)

Consider the following coupled potential well system:



The state 2 is coupled via tunneling to state 3 in the adjacent well. The electron is initially sitting in the ground state 1. The states 3 and 1 have a non-zero optical dipole matrix element. A weak probe field, with frequency ω close to the separation between levels 1 and 3, detuning $\Delta = \varepsilon_3 - \varepsilon_1 - \hbar\omega$, and phase ϕ is used to measure the optical absorption in the system. The decoherence rate between levels 1 and 2 is assumed to be zero, i.e. $\gamma_{12} = 0$. The decoherence rate between levels 1 and 3 is γ_{13} and between levels 2 and 3 is γ_{23}. The relaxation time from level 3 to level 1 is T_1. The Hamiltonian is,

$$\hat{H}_R = \varepsilon_1 |e_1\rangle\langle e_1| + \varepsilon_2 |e_2\rangle\langle e_2| + \varepsilon_3 |e_3\rangle\langle e_3|$$
$$-\frac{\hbar\Omega_R}{2}\left[e^{i\omega t + i\phi} |e_1\rangle\langle e_3| + e^{-i\omega t - i\phi} |e_3\rangle\langle e_1| \right] - U\left[|e_2\rangle\langle e_3| + |e_3\rangle\langle e_2| \right]$$

Suppose $\varepsilon_2 = \varepsilon_3$.

a) Using any method that is reasonable, find the linear susceptibility $\chi(\omega)$ as a function of the detuning Δ.

(20 points)

b) What is imaginary part of the susceptibility $\chi(\omega)$ when the detuning $\Delta = \varepsilon_3 - \varepsilon_1 - \hbar\omega$ is zero?

(5 points)

c) Suppose $\varepsilon_2 \neq \varepsilon_3$ but is only slightly different. Is there a value of detuning $\Delta = \varepsilon_3 - \varepsilon_1 - \hbar\omega$ that will make the absorption go to zero?

(5 points)