ECE 4960

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Lecture 6

Local Analysis: Integration

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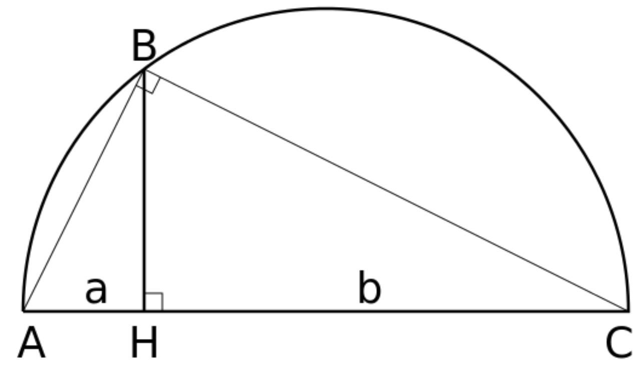
Integration; Interpolation; Quadrature

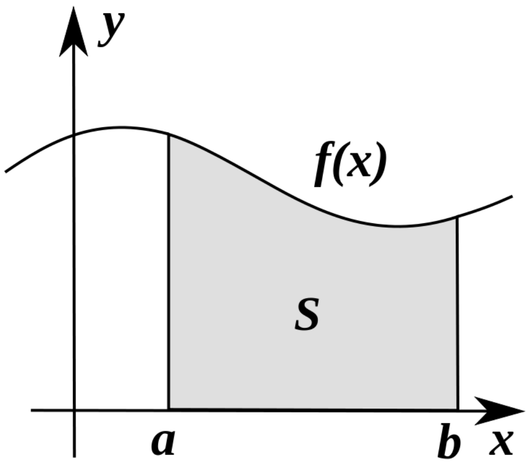
$$I = \int_a^b f(x)dx = \hat{f} \cdot (a-b)$$

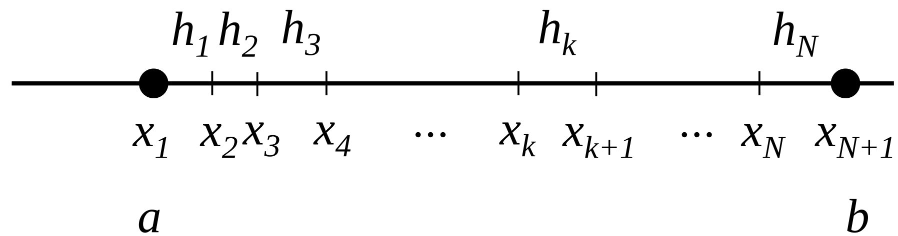


$$BH = \sqrt{ab}$$

- Integration is important to investigate the "average" behavior or the "general trend".
- Mathematically this is very similar to "interpolation" and "quadrature scheme", where numerical quadrature means finding the value of a definite integral.
- In the finite element method, quadrature also refers to how the approximation function approaches the real function within the element.

Discretization of the Finite Integral





$$I_k = \int_{x_k}^{x_{k+1}} f(x)dx; \qquad h_k = x_{k+1} - x_k$$

$$\int_a^b f(x)dx \cong \sum_{k=1}^{N} h_k\left(w_k f(x_k)\right) = \sum_{k=1}^{N} \hat{I}_k = \hat{I}$$

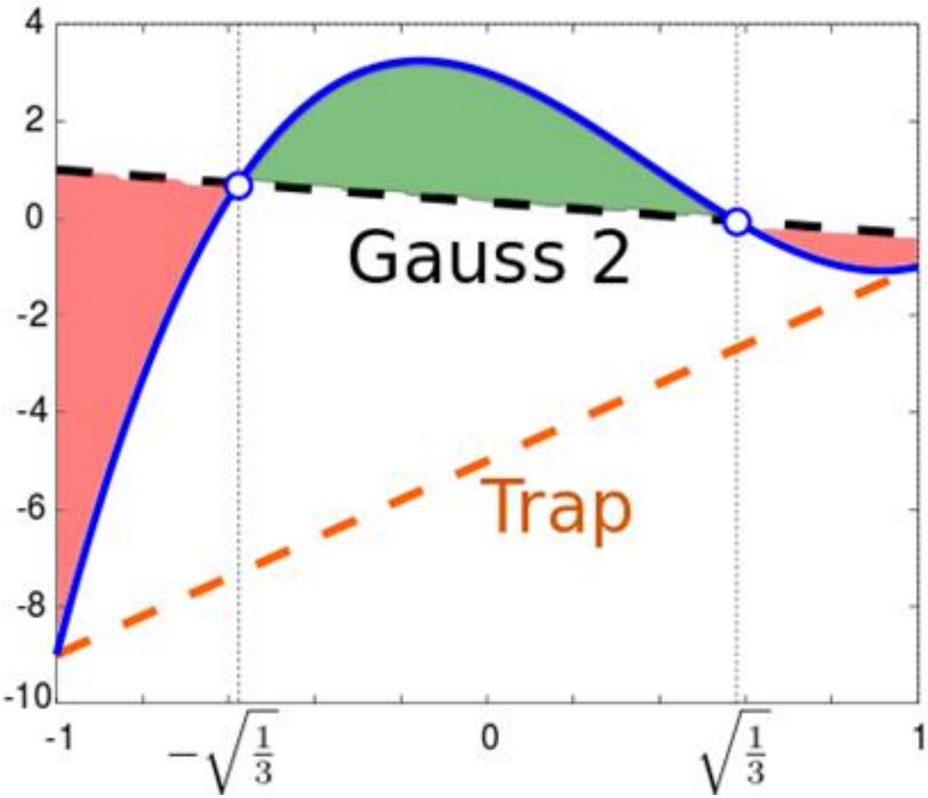
- Assume the integration value of *f(x)* is between *(a*, *b*), which we will discretize to *N* segments by h_k ($k = 1 \ldots N$). Finer *h* resolution will give a better approximation to the analytical integration.
- Without loss of generality, we will **shift and normalize** the coordinates from (x_k, x_{k+1}) to (–1, 1), which will add a constant to the original integral.

Possible Choices of the Quadrature Schemes

Normalized quadrature approximation between (–1, 1)

Type	Approximation	Order of precision
Rectangle	$\hat{I}_k = h_k f\left(x_k\right)$	1st order
Trapezoid	$\hat{I}_k = h_k\left(\frac{1}{2} f\left(x_k\right)+\frac{1}{2} f\left(x_{k+1}\right)\right)$	2nd order
Midpoint	$\hat{I}_k = h_k f\left(x_{k+1/2}\right)$	2nd order
Simpson	$\hat{I}_k = h_k\left(\frac{1}{6} f\left(x_k\right)+\frac{4}{6} f\left(x_{k+1/2}\right)+\frac{1}{6} f\left(x_{k+1}\right)\right)$	4th order
2-point Gaussian	$\hat{I}_k = h_k\left(\frac{1}{2} f\left(x_{k+1/2}-\frac{1}{2\sqrt{3}}h_k\right)+\frac{1}{2} f\left(x_{k+1/2}+\frac{1}{2\sqrt{3}}h_k\right)\right)$	4th order
3-point Gaussian	$\hat{I}_k = h_k\left(\frac{5}{18} f\left(x_{k+1/2}-\frac{\sqrt{3}}{2\sqrt{5}}h_k\right)+\frac{8}{18} f\left(x_{k+1/2}\right)+\frac{5}{18} f\left(x_{k+1/2}+\frac{\sqrt{3}}{2\sqrt{5}}h_k\right)\right)$	6th order

Examples of Quadrature Schemes



- Take $f(x) = 7x^3 - 8x^2 - 3x + 3$, and $\int_{-1}^{1} f(x)dx = \frac{2}{3}$ in blue line
- Trapezoidal rule is the orange dash line and the approximation is –10.
- Both the Simpson rule and the 2-point Gaussian quadrature (in the black dash line) will give the exact 2/3 answer!

Simpson:

$$\hat{I}_k = h_k\left(\frac{1}{6}f(x_k) + \frac{4}{6}f(x_{k+1/2}) + \frac{1}{6}f(x_{k+1})\right) = 2\times\left(\frac{1}{6}\cdot(-9) + \frac{4}{6}\cdot 3 + \frac{1}{6}(-1)\right) = \frac{2}{3}$$

Similar Derivation from Taylor Series

- Assume that we are looking at the normalizd and shifted $f(x)$ is polynomial as $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots$

Zero-order term:

- All quadrature approximations in Table 1 can only estimate the a_0 integration correctly to $2a_0$ between (–1, 1), if:

$$\sum_{i=1}^{N} w_i = 1$$

Quadrature from Taylor Series

Second-order term:

- Integrate $f(x) = a_2 x^2$ correctly to achieve 4th order accuracy (as x and x^3 terms will not contribute to the integrand)

$$\frac{a_2}{3}\left(x_{k+1}^3 - x_k^3\right) = \frac{a_2}{3}\left(\left(x_k + h_k\right)^3 - x_k^3\right) = \frac{a_2}{3}\left(3x_k^2 h_k + 3x_k h_k^2 + h_k^3\right)$$

Simpson quadrature at boundary and midpoint:

$$h_k\left(\frac{1}{6} f\left(x_k\right) + \frac{4}{6} f\left(x_{k+1/2}\right) + \frac{1}{6} f\left(x_{k+1}\right)\right)$$

$$= h_k\left(\frac{1}{6} f\left(x_k\right) + \frac{4}{6}\left(f\left(x_k\right) + \frac{h_k}{2} f'\left(x_k\right) + \frac{h_k^2}{8} f''\left(x_k\right)\right) + \frac{1}{6}\left(f\left(x_k\right) + h_k f'\left(x_k\right) + \frac{h_k^2}{2} f''\left(x_k\right)\right)\right)$$

$$= h_k\left(f\left(x_k\right) + \frac{h_k}{2} f'\left(x_k\right) + \frac{h_k^2}{6} f''\left(x_k\right) + O\left(h_k^3\right)\right) = h_k\left(a_2 x_k^2 + \frac{h_k}{2}\left(2a_2 x_k\right) + \frac{h_k^2}{6}\left(2a_2\right) + O\left(h_k^3\right)\right)$$

$$= \frac{a_2}{3}\left(3x_k^2 h_k + 3x_k h_k^2 + h_k^3\right) + O\left(h_k^4\right) \qquad for \quad f\left(x\right) = a_2 x^2$$

Fourth-Order Two-Point Quadrature

Solve the w_k and x_k so that all three coefficients match:

$$w_1 f\left(x_k + \ell_1\right) + w_2 f\left(x_k + \ell_2\right) = f\left(x_k\right) + \frac{h_k}{2} f'\left(x_k\right) + \frac{h_k^2}{6} f''\left(x_k\right)$$

$$= w_1\left(f\left(x_k\right) + \ell_1 f'\left(x_k\right) + \frac{\ell_1^2}{2} f''\left(x_k\right)\right) + w_2\left(f\left(x_k\right) + \ell_2 f'\left(x_k\right) + \frac{\ell_2^2}{2} f''\left(x_k\right)\right)$$

$$w_1 + w_2 = 1$$

$$w_1 \ell_1 + w_2 \ell_2 = \frac{h_k}{2}$$

$$w_1 \ell_1^2 + w_2 \ell_2^2 = \frac{h_k^2}{3}$$

Four variables and three constraint equations: infinite choices

Two-Point Gaussian Quadrature

Make the sampling symmetric: $\ell_1, \ell_2 = \frac{1}{2}\left(1 \pm \frac{1}{\sqrt{3}}\right)$

Check:

$$h_k\left(\frac{1}{2}f\left(x_{k+1/2} - \frac{1}{2\sqrt{3}}h_k\right) + \frac{1}{2}f\left(x_{k+1/2} + \frac{1}{2\sqrt{3}}h_k\right)\right)$$

$$= h_k\left(\begin{array}{l}\frac{1}{2}\left(f\left(x_k\right) + \left(\frac{h_k}{2} - \frac{h_k}{2\sqrt{3}}\right)f'\left(x_k\right) + \frac{1}{2}\left(\frac{h_k}{2} - \frac{h_k}{2\sqrt{3}}\right)^2 f''\left(x_k\right)\right) \\ + \frac{1}{2}\left(f\left(x_k\right) + \left(\frac{h_k}{2} + \frac{h_k}{2\sqrt{3}}\right)f'\left(x_k\right) + \frac{1}{2}\left(\frac{h_k}{2} + \frac{h_k}{2\sqrt{3}}\right)^2 f''\left(x_k\right)\right)\end{array}\right)$$

$$= h_k\left(f\left(x_k\right) + \frac{h_k}{2}f'\left(x_k\right) + \frac{h_k^2}{6}f''\left(x_k\right) + O\left(h_k^3\right)\right)$$

$$= \frac{a_2}{3}\left(3x_k^2 h_k + 3x_k h_k^2 + h_k^3\right) + O\left(h_k^4\right) \qquad for \quad f\left(x\right) = a_2 x^2$$

hp Adaptivity in Quadrature

- Similar to hp adaptivity in the local analysis for differentiation:
  - Making h smaller, we can have better integration approximation according to $O(h^p)$.
  - Making the order of approximation higher (p adaptivity), we can have better integration approximation on the segment h.
- We can use different choices of h_k to compare the relative accuracy of approximation.
- We can use different choices of order of approximation to compare the relative accuracy of approximation, and can then choose the appropriate h_k!

Hacker Practice

For $f(x) = e^x$, we know the exact solution to the integration as:

$$\int_{-1}^{1} e^x dx = e - e^{-1} \cong 2.3504$$

Notice that this is a monotonic function, so the order of approximation will not cause big deviation as the cubic function before. Fill in the table below:

Quadrature Schemes	Numerical approximation	Error
Retangle		
Trapezoid		
Mid-point		
Simpson		
2-point Gaussian		