1. Suppose x_c is a continuous-time signal and we find that

$$x_c\left(n\frac{\pi}{7}\right) = \begin{cases} 3(-1)^{\frac{n-1}{2}} & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}$$

I.e., we sample x_c every $T = \frac{\pi}{7}$ seconds and get the indicated results.

(a) Find three different possibilities for x_c). Try to find at least one that's a pure sinusoid with specification $x_c(t) = \cos \Omega_o t$ or $x_c(t) = \sin \Omega_o t$.
(b) Is every different possible x_c (including ones you didn't find in (a)) a pure sinusoid? Explain.

2. We sample the signal x_c with specification $x_c(t) = \cos(\Omega_o t)$ every $T_1 = 10^{-3}$ seconds and obtain the discrete-time signal x with specification

$$x(n) = x_c(nT_1) = \cos(3n) \;\text{ for all }\; n \in \mathbb{Z}\ .$$

(a) Find three possible values for Ω_o.
(b) We put x through a sinc-function T_1-interpolator. What signal x_R emerges?
(c) We put x through a sinc-function T_2-interpolator, with $T_2 = 1/500$ seconds. What signal x_R emerges?

3. A movie shot at the incredibly slow speed of four frames per second shows a tricycle moving pretty much to the right. The tricycle's wheels have diameter $2/\pi$ feet. One of the wheels has a mark on its edge; the frame-by-frame position of the mark is a periodic sequence that looks like the picture in Figure 1. To our parsimonious brains, the wheel appears to be turning the "wrong" way.

(a) Without any further information, what can we say about the way the tricycle was moving when filmed? We're not concerned, e.g., with who's riding the tricycle, how old they are, whatever. Specifically, can we tell whether the tricycle was moving at a uniform speed? Can we even tell whether it was moving always to the right? Explain using the words "intersample behavior."
(b) The documentation for the film insists that the tricycle was moving at a constant speed to the right. We believe it. Can we tell from the given information how fast it was moving? If so, explain why. If not, give two different uniform speeds consistent with what the film shows us.
(c) The tricycle's owner claims that its maximum riding speed is 5 miles per hour. We believe her. Now can we figure out how fast it was moving when filmed? If not, explain why not. If so, how fast (in miles per hour) was it moving?

4. We saw in class that infinitely many T-interpolations of a given discrete-time signal x exist. Some T-interpolations looked a lot more systematic than others. One particularly systematic way of T-interpolating a given x proceeds as follows. For the purposes of this problem, a *T-interpolation function* is any continuous-time signal g that satisfies $g(0) = 1$ and $g(nT) = 0$ for every nonzero integer n. The continuous-time signal y_c given by

$$y_c = \sum_{n=-\infty}^{\infty} x(n)\text{Shift}_{nT}(g)\ ,$$

which has specification

$$y_c(t) = \sum_{n=-\infty}^{\infty} x(n)g(t-nT) \text{ for all } t \in \mathbb{R} ,$$

assuming the last sum converges for all $t \in \mathbb{R}$, is called the *T-interpolation of x arising from the T-interpolation function g.* Note that the sinc-function T-interpolation of x arises in this fashion with T-interpolation function g specified by

$$g(t) = \frac{\sin\left(\frac{\pi}{T}t\right)}{\frac{\pi}{T}t} \text{ for all } t \in \mathbb{R} .$$

(a) Verify that y_c given by the formulas above is indeed a T-interpolation of x for any T-interpolation function g.
(b) Verify that the T-interpolation of the signal $x = \delta$ using T-interpolation function g is g itself. (This is really easy.)
(c) In class we discussed the linear T-interpolation of a discrete-time signal x — that's the continuous-time signal y_c you obtain by connecting the pairs of consecutive "dots" in x with straight lines. It turns out that the linear T-interpolation uses a T-interpolation function g as above. Given this piece of information and given your answer to (b), find g.
(d) Verify that when you use the g you found in (c), the signal y_c with specification

$$y_c(t) = \sum_{n=-\infty}^{\infty} x(n)g(t-nT) \text{ for all } t \in \mathbb{R} ,$$

is indeed the linear T-interpolation of x. (Suggestion: when $t \in (nT, (n+1)T)$, all but at most two terms in the series defining $y_c(t)$ are zero.)

5. Let x_c be the continuous-time signal whose Fourier transform $\widehat{X}_c$ has specification

$$\widehat{X}_c(\Omega) = \begin{cases} 1 - .25\Omega & 0 \le \Omega \le 4 \\ 1 + .25\Omega & -4 \le \Omega < 0 \\ 0 & \text{otherwise.} \end{cases}$$

Let x be the discrete-time signal with specification $x(n) = x_c\left(n\frac{\pi}{3}\right)$ for all $n \in \mathbb{Z}$.

(a) Graph $\widehat{X}_c(\Omega)$ as a function of Ω and $\widehat{X}(\omega)$ as a function of ω.
(b) Let x_R be the sinc-function T-interpolation of x (with $T = \pi/3$). Graph $\widehat{X}_R(\Omega)$ as a function of Ω.

6. Let $x_c(t) = e^{j22\pi t}$ for all $t \in \mathbb{R}$.

(a) What is the Nyquist rate for x_c?
(b) Let $T = \frac{1}{4}$ and let $x(n) = x_c(nT)$ for all $n \in \mathbb{Z}$. Observe that x is a T-sampled version of x_c but T is such that the sampling is slower than the Nyquist rate for x_c. Find x_R, the output of an ideal sinc-function T-interpolator ($T = \frac{1}{4}$ still) driven by x, i.e. x_R comes from equation (R1).

7. Reese and Malcolm would like to sample and reconstruct a continuous-time signal x_c, which is bandlimited to within Ω_m. Unfortunately, they have access only to samplers that sample every $T = 4\pi/3\Omega_m$, which is longer than the Nyquist interval for x_c. Each of them samples x_c every T seconds; Malcolm obtains a discrete-time signal y_1 and Reese gets y_2. Obviously, the DTFTs $\hat{Y}_1$ and $\hat{Y}_2$ will suffer from aliasing because of the too-slow sampling.

Reese is frustrated, but Malcolm notices that Reese started sampling a little later than did Malcolm — in fact, exactly $T/2$ seconds later. So in the notation of the previous paragraph, $y_1(n) = x_c(nT)$ and $y_2(n) = x_c(nT+T/2)$ for all $n \in \mathbb{Z}$. Accordingly, Malcolm reasons, they could interleave their results and assemble a sampled version of x_c —- call it x — sampled every $T/2$ seconds, which is sufficient to recover x_c by the usual means (since $T/2 = 2\pi/3\Omega_m < \pi/\Omega_m$).

Reese is suspicious. "How," he asks, "could we take two badly aliased DTFTs and recover an un-aliased one? Two wrongs can't make a right."

(a) Derive a formula for $\widehat{X}$ in terms of $\widehat{Y}_1$ and $\widehat{Y}_2$. (Suggestion: first define w_1 and w_2 as follows:

$$w_1(n) = \begin{cases} y_1(n/2) & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

and

$$w_2(n) = \begin{cases} y_2(n/2) & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Note that $x = w_1 + \text{Shift}_1(w_2)$. Show that $\widehat{W_1}(\omega) = \widehat{Y}_1(2\omega)$ and $\widehat{W_2}(\omega) = \widehat{Y}_2(2\omega)$ for all $\omega \in \mathbb{R}$. Now apply the time-shift rule for DTFTs.)

(b) Use (R1) to write an explicit formula for x_c in terms of $\hat{Y}_1$ and $\hat{Y}_2$.

8. This is a much cooler version of the multipath problem. Undergraduates at MIT who don't have cable TV endure ghost-infested TV pictures because signals emanating from the top of the Prudential Center in Boston get reflected off the neighboring John Hancock building *en route* to the MIT dorms. Suppose x_c is the signal leaving the Prudential tower; it arrives at a student's TV along with its time-delayed (reflected) signal $\alpha\text{Shift}_\tau(x_c)$, where $|\alpha| < 1$ and $\tau > 0$, so that the student's TV processes the signal

$$x_{\text{TV}} = x_c + \alpha\text{Shift}_\tau(x_c) \ .$$

The processing system inside the TV looks consists of a T-sampler followed by a filter with frequency response $\widehat{H}$ followed by a sinc-function interpolator with interpolation interval T that takes the discrete-time filter output y and processes it through equation (R1) or (R2) to yield output y_R. Is it possible to specify T and $\hat{H}$ so that $y_R = x_c$, i.e., so as to remove the ghost-inducing echo $\alpha\text{Shift}_\tau(x_c)$ from the signal x_{TV}? You may assume that x_c is bandlimited, so $\hat{X}_c(\Omega) = 0$ when $|\Omega| \geq \Omega_m$ for some $\Omega_m > 0$. (Notation-wise, it helps to set $x_1(n) = x_c(nT)$ and $x_2(n) = x_{\text{TV}}(nT)$ for all $n \in \mathbb{Z}$. Keep in mind that $\widehat{H}$, being a DTFT, must be a periodic function of ω with 2π as a period.)