1. In Chapter 9 of the monograph I asserted in passing that l^2 is an inner-product space. Here's how to prove that

$$\langle x, y \rangle = \sum_{k=-\infty}^{\infty} x(k)\overline{y(k)}$$

defines an inner product on l^2. First show that show that for any complex numbers a and b we have

$$|a|\,|b| \le \frac{|a|^2 + |b|^2}{2} \; .$$

(Suggestion: $(|a|-|b|)^2 \ge 0$.) Conclude that for any x and y in l^2, the sequence $\{x(k)y(k) : k \in \mathbb{Z}\}$ is absolutely summable, so the series

$$\sum_{k=-\infty}^{\infty} x(k)\overline{y(k)}$$

converges, and l^2 is an inner-product space with inner product $\langle x, y \rangle$ defined by that sum.

2. Verify the identity

$$\langle v, w \rangle = \frac{\|v + w\|^2 - \|v\|^2 - \|w\|^2}{2} + j\,\frac{\|v + jw\|^2 - \|v\|^2 - \|w\|^2}{2} \; ,$$

which holds for all v and w in any complex inner-product space V. Conclude that if $T : V \to V'$ is a linear mapping between inner-product spaces that preserves norms, i.e. $\|T(v)\| = \|v\|$ for all $v \in V$, then T preserves inner products, i.e. $\langle T(v), T(w) \rangle = \langle v, w \rangle$ for all v and w in V.

3. Recall that $\mathbb{C}^4$, the set of all complex 4-vectors, is an inner-product space with inner product

$$\langle v, w \rangle = w^H v \;\text{ for all }\; v, w \in \mathbb{C}^4 \; ,$$

where w^H is the conjugate transpose of w. Find the orthogonal projection of the vector

$$v = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

on the subspace of $\mathbb{C}^4$ spanned by u_1, u_2, and u_3, where

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \; ; \; u_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \; ; \; u_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \end{bmatrix} \; .$$

4. Let $\mathcal{P}_3$ be the set of all polynomials of degree less than or equal to 3 in a real variable t with complex coefficients. Each v in $\mathcal{P}_3$ has specification

$$v(t) = c_3 t^3 + c_2 t^2 + c_1 t + c_0$$

for some complex constants $c_0, \ldots, c_3$.

(a) Show that $\mathcal{P}_3$ is a subspace of $\mathbb{C}^{\mathbb{R}}$ — i.e., it's closed under taking linear combinations.

(b) Verify that

$$\langle v, w\rangle = \int_0^1 v(t)\overline{w}(t)dt \text{ for all } v, w \in \mathcal{P}_3$$

defines an inner product on $\mathcal{P}_3$.

(c) As it happens, the set $\{v_0, v_1, v_2, v_3\}$ of polynomials in $\mathcal{P}_3$ with respective specifications

$$\begin{aligned} v_0(t) &= 1 \text{ for all } t \in \mathbb{R} \\ v_1(t) &= t \text{ for all } t \in \mathbb{R} \\ v_2(t) &= t^2 \text{ for all } t \in \mathbb{R} \\ v_3(t) &= t^3 \text{ for all } t \in \mathbb{R} \end{aligned}$$

is a linearly independent subset of (in fact a basis for) $\mathcal{P}_3$. Apply to this set the Gram-Schmidt procedure described in the proof of Fact 9.9 in the monograph to generate an orthonormal basis $\{w_0, w_1, w_2, w_3\}$ for $\mathcal{P}_3$. Write out each w_k's specification as a polynomial in t. (Note: the computation gets a little hairy. If you don't want to do the normalization to get the final formula for w_3, no big deal.)

5. Show that every continuous-time system with input space X consisting of all decent signals and and system mapping of the form

$$S(x) = h_0 * x + \sum_{k=1}^{N} d_k \text{Shift}_{t_k}(x) ,$$

where h_0 is decent and has finite duration, has a frequency response.

6. Show that every (causal) BIBO stable continuous-time system has a frequency response. Also, find the impulse response h of some causal continuous-time system that doesn't have a frequency response.

7. A certain LTI system has frequency response $\widehat{H}$ with specification

$$\widehat{H}(\Omega) = \frac{1}{5 + j\Omega} \text{ for all } \Omega \in \mathbb{R} .$$

Let x be the signal with Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{1}{k^2+1} e^{jk13t} .$$

Find a Fourier series for the system's output $S(x)$ in response to input x. Why might one say that the system acts as a crude low-pass filter?

8. Consider the averager system whose input space X is the set of all decent $\mathbb{F}$-valued signals and whose system mapping is

$$S(x)(t) = \frac{1}{T_o} \int_{t-T_o}^{t} x(\tau)d\tau \text{ for all } x \in X \text{ and } t \in \mathbb{R} ,$$

where $T_o > 0$ is given.

(a) Show that if x is a zero-mean decent signal that has T_o a period, then $S(x) = 0$.

(b) Find the frequency response $\widehat{H}$ of the system. Then give a frequency-domain explanation of why the assertion in part (a) holds. (Suggestion: expand x from (a) in a Fourier series.)

9. Let $T_o > 0$ be given. Find a finite-duration signal h such that the system with impulse response h takes any periodic input signal x that has T_o as a period and produces an output $S(x)$ whose Fourier series

$$S(x)(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\Omega_o t}$$

satisfies $c_k = 0$ for every nonzero even integer k. I.e., the system filters out all the even harmonics from T_o-periodic inputs. Find the frequency response of the system you've constructed. Is the system causal? Can you come up with a causal system that works in this problem? Suggestion: think carefully about the solution to the previous problem.

10. In class, I told you that a Gaussian signal had a Gaussian Fourier transform. This problem steps you through a derivation of

$$t \mapsto e^{-t^2/2} \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad \Omega \mapsto \sqrt{2\pi} e^{-\Omega^2/2} \; .$$

(a) Begin with equation $\mathcal{F}$ and take the derivative with respect to Ω of both sides. Conclude from this that

$$D\widehat{X}(\Omega) = -\Omega \widehat{X}(\Omega) \; ,$$

where D denotes derivative with respect to Ω. (You'll need to differentiate under the integral sign and then use integration by parts. You'll be dealing with

$$\int_{-\infty}^{\infty} \left(-te^{-t^2/2}\right) \left(je^{-j\Omega t}\right) dt \; .$$

My suggestion is to let $v = je^{-j\Omega t}$ and $dw = -te^{-t^2/2}dt$ when you do integration by parts.)

(b) Conclude from (a) that

$$\widehat{X}(\Omega) = \widehat{X}(0)e^{-\Omega^2/2} \quad \text{for all} \quad \Omega \in \mathbb{R} \; .$$

Find $\widehat{X}(0)$ by using

$$\int_{-\infty}^{\infty} e^{-\tau^2} d\tau = \sqrt{\pi} \; .$$