ECE 3250 **HOMEWORK ASSIGNMENT VI** **Fall 2015**

1. In each case, h is the impulse response of a LTI system and x is an input signal. Your task is to find $S(x)$.

(a) $h(t) = e^{3t}u(t)$ for all $t \in \mathbb{R}$ and $x(t) = e^{5t}u(t)$ for all $t \in \mathbb{R}$.
(b) $h(t) = e^{-3t}u(t)$ for all $t \in \mathbb{R}$ and $x(t) = e^{5t}u(-t)$ for all $t \in \mathbb{R}$.
(c) $h(t) = e^{-3(t-7)}u(t-7) + \delta(t-7)$ for all $t \in \mathbb{R}$ and $x(t) = e^{3t}u(t)$ for all $t \in \mathbb{R}$.
(d) $h(t) = e^{-5t}u(t)$ for all $t \in \mathbb{R}$ and $x(t) = u(-t)$ for all $t \in \mathbb{R}$.

2. A certain LTI system has impulse response h with specification

$$h(t) = 5(e^{-3t} - e^{-7t})u(t) \text{ for all } t \in \mathbb{R} .$$

(a) Find $S(u) = h * u$, the system's step response. Express $S(u)$ as follows:

$$S(u)(t) = c_o u(t) + (c_1 e^{-3t} + c_2 e^{-7t})u(t) \text{ for all } t \in \mathbb{R} .$$

(b) For what values of $\sigma \in \mathbb{R}$ does signal x with specification $x(t) = e^{\sigma t}$ for all $t \in \mathbb{R}$ belong to $\mathcal{D}_h$? For any such value of σ, find $S(x)$.
(c) For what values of $\sigma \in \mathbb{R}$ does signal x with specification $x(t) = e^{\sigma t}u(t)$ for all $t \in \mathbb{R}$ belong to $\mathcal{D}_h$? For any such value of σ, find $S(x)$. Compare the $S(x)$ you find with the $S(x)$ you found in (b). For example, how do the signals compare for large $t > 0$?

3. A certain LTI system has impulse response h with specification $h(t) = 7e^{-3t}u(t)$ for all $t \in \mathbb{R}$.

(a) Given $\Omega_1 \in \mathbb{R}$, let x be the signal with specification $x(t) = e^{j\Omega_1 t}$ for all $t \in \mathbb{R}$. Find $h * x$ in the form

$$h * x(t) = C_1 e^{j\Omega_1 t} \text{ for all } t \in \mathbb{R} .$$

(b) Observe that the constant C_1 depends on Ω_1. Why do people call the system a "low-pass filter?"
(c) Use Euler's formulas and LTI-ness to find the output of the system when the input is the signal x with specification $x(t) = 13\cos\Omega_1 t$ for all $t \in \mathbb{R}$. Express $S(x)$ in two different forms: (1) the sum of a sine and a cosine; and (2) a phase-shifted cosine of the form (constant) $\times \cos(\Omega_1 t + \phi_1)$. (This isn't difficult, but the formulas for things like ϕ_1 aren't pretty. It helps to remember that $z + \overline{z} = 2\text{Re}\{z\}$ for every complex number z. It also helps to remember that you can write any such z in polar form as $|z|e^{j\phi}$.)

4. In each case I've described the input space X and system mapping S for some I/O system. All of the systems are linear, but none is time-invariant. For each system, find an input signal $x \in \mathbb{F}^{\mathbb{R}}$ and a $t_o \in \mathbb{R}$ for which

$$S(\text{Shift}_{t_o}(x)) \neq \text{Shift}_{t_o}(S(x)) .$$

(a) X is the set of all decent signals and $S(x)$ for every $x \in X$ has specification

$$S(x)(t) = \begin{cases} \int_0^t \tau x(\tau) d\tau & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 . \end{cases}$$

(b) $X = \mathbb{F}^{\mathbb{R}}$ and $S(x)$ for every $x \in X$ has specification

$$S(x)(t) = x(7t) \text{ for all } t \in \mathbb{R} .$$

(c) X is the set of all decent signals and $S(x)$ for every $x \in X$ is the constant signal with specification

$$S(x)(t) = \frac{1}{T}\int_{-T}^{0} x(\tau)d\tau \text{ for all } t \in \mathbb{R} ,$$

where $T > 0$ is given.

5. Explain why each of the systems in the previous problem is causal or not causal.

6. Suppose you have two continuous-time LTI systems with system mappings S_1 and S_2 and respective impulse responses h_1 and h_2. Suppose $h_1 \in X_2$, where X_2 is the input space of the system with system mapping S_2. Let X be the set of all $x \in X_1$ such that $S_1(x) \in X_2$, where, X_1 is the input space of the system with system mapping S_1.

(a) Show that X contains all the finite-duration signals decent signals in $\mathbb{F}^{\mathbb{R}}$.

(b) Show that the continuous-time system with input space X and system mapping $S : X \to \mathbb{F}^{\mathbb{R}}$ defined by

$$S(x) = S_2(S_1(x)) \text{ for all } x \in X$$

is LTI.

(c) Show that the impulse response of the system in (b) is the convolution of h_1 with h_2.

(d) Show that if the original two systems are both BIBO stable, then so is the system in (b).

7. The transmitter is in downtown Boston and you, the receiver, are across the river in Cambridge. The signal from the transmitter gets attenuated by a factor of $\alpha \in (0, 1)$ every time it travels a mile and takes time $T > 0$ to travel a mile (for example, if the signal x travels five miles, it arrives as $\alpha^5\mathrm{Shift}_{5T}(x)$). The line-of-sight distance from the transmitter to you is one mile. The signal, however, also bounces off the Prudential Center, so an attenuated replica of the line-of-sight signal arrives late — assume that the replica signal travels two miles altogether from transmitter to you, and that the signal you receive is the sum of the line-of-sight signal and the replica.

(a) You pass the received signal through a LTI system with decent impulse response h_0. Find the impulse response of the overall system that has as its input the transmitter's output signal and has as its output the output of your h_0-system.

(b) Explain why, for $N \in \mathbb{N}$ large, the LTI system with input space $X = \mathbb{R}^{\mathbb{R}}$ and system mapping

$$S(x) = \sum_{k=0}^{N} (-1)^k \alpha^k \mathrm{Shift}_{kT}(x) \ \text{ for all } \ x \in X$$

does a decent job of "undoing" the multipath interference in the sense that if you use the signal you receive as input to this system, the system's output will be reasonably close to the signal you would have received had the Prudential Center not been there.