1. A certain LTI system has impulse response h whose specification is

$$h(n) = \begin{cases} 7^{-n} & \text{if } 0 \leq n < 11 \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is $\mathcal{D}_h$, the set of all $x \in \mathbb{F}^{\mathbb{Z}}$ for which $h * x$ exists?
(b) Find the output $S(x)$ of the system when the input is the signal x whose specification is

$$x(n) = 3 \text{ for all } n \in \mathbb{Z} .$$

(c) Find $S(u)$, the *step response* of the system, which is the output of the system when its input is a unit step. What I'd like here are formulas for $S(u)(n)$ for all $n \in \mathbb{Z}$.
(d) Suppose $x \in \mathbb{F}^{\mathbb{Z}}$ has duration 13. What is the maximum possible duration for $S(x)$?

2. Denote by S the system mapping for the causal sliding-window M-fold averager with $M = 5$.

(a) Find a non-constant signal $x \in \mathbb{F}^{\mathbb{Z}}$ for which $S(x)(n) = 13$ for every $n \in \mathbb{Z}$.
(b) Show that $\lim_{n\to\infty} S(x)(n) = 0$ for every left-sided signal $x \in \mathbb{F}^{\mathbb{Z}}$. Show also that $\lim_{n\to-\infty} S(x)(n) = 0$ for every right-sided signal $x \in \mathbb{F}^{\mathbb{Z}}$.
(c) Find a signal $x \in \mathbb{F}^{\mathbb{Z}}$ that doesn't have finite duration but that satisfies

$$\lim_{n\to\pm\infty} S(x)(n) = 0 .$$

3. In each case I've described the input space X and system mapping S for some discrete-time I/O system. All of the systems are linear, but none is time-invariant. For each system, find an input signal $x \in \mathbb{F}^{\mathbb{Z}}$ and a $k_o \in \mathbb{Z}$ for which

$$S(\text{Shift}_{k_o}(x)) \neq \text{Shift}_{k_o}(S(x)) .$$

(This problem is easier than it might appear — how about trying δ?)

(a) $X = \mathbb{F}^{\mathbb{Z}}$ and $S(x)$ for every $x \in X$ has specification

$$S(x)(n) = \begin{cases} \sum_{k=0}^{n}(k+1)x(k) & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 . \end{cases}$$

(b) $X = \mathbb{F}^{\mathbb{Z}}$ and $S(x)$ for every $x \in X$ has specification

$$S(x)(n) = \begin{cases} x(n) & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

(c) $X = \mathbb{F}^{\mathbb{Z}}$ and $S(x)$ for every $x \in X$ has specification

$$S(x)(n) = x(3n) \text{ for all } n \in \mathbb{Z} .$$

(d) $X = \mathbb{F}^{\mathbb{Z}}$ and $S(x)$ for every $x \in X$ is the constant signal with specification

$$S(x)(n) = \frac{1}{M}\sum_{k=0}^{M-1} x(k) \text{ for all } n \in \mathbb{Z} ,$$

where M is some given positive integer.

4. Explain why each of the systems in the previous problem is causal or not causal.

5. Suppose you have two LTI systems with system mappings S_1 and S_2 and respective impulse responses h_1 and h_2. Suppose $h_1 \in X_2$, where $X_2 = \mathcal{D}_{h_2}$ is the input space of the system with system mapping S_2. Let X be the set of all $x \in X_1$ such that $S_1(x) \in X_2$, where $X_1 = \mathcal{D}_{h_1}$ is the input space of the system with system mapping S_1.

(a) Show that X contains all the finite-duration signals in $\mathbb{F}^{\mathbb{Z}}$. (Suggestion: X_1 contains them all; the question is whether $h_1 * x \in X_2$ whenever x has finite duration. Since $h_1 \in X_2$, $h_2 * h_1$ exists. Does $h_2 * h_1 * x$ exist when x has finite duration?)

(b) Show that the discrete-time system with input space X and system mapping $S : X \to \mathbb{F}^{\mathbb{Z}}$ defined by

$$S(x) = S_2(S_1(x)) \text{ for all } x \in X$$

is LTI.

(c) Show that the impulse response of the system in (b) is the convolution of h_1 with h_2.

6. Suppose you take any two FIR systems and concatenate them as in the previous problem. Explain why, without any assumptions other than FIR-ness of the two systems, the concatenated system is an FIR system with input space $X = \mathbb{F}^{\mathbb{Z}}$.

7. Explain why each of the following causal LTI systems is or is not BIBO stable.

(a) The system has impulse response h with specification

$$h(n) = 7^{-n}u(n) \quad \text{for all} \quad n \in \mathbb{Z}\ .$$

(b) The system has system mapping S with specification

$$S(x)(n) = \sum_{k=-\infty}^{n} 5^{(n-k)}x(k) \quad \text{for all} \quad x \in \mathcal{D}_h \text{ and } n \in \mathbb{Z}\ .$$

Here, $\mathcal{D}_h$ is the set of all $x \in \mathbb{F}^{\mathbb{Z}}$ for which $h * x$ exists, where h is the system's impulse response (which I'm not telling you).

(c) The system has input space $X = \mathbb{F}^{\mathbb{Z}}$ and system mapping

$$S(x) = 10^{53}x - 10^{13}\text{Shift}_{17}(x) \quad \text{for all} \quad x \in X\ .$$

8. You can think of BIBO stability as imposing a continuity condition on the system mapping S. Specifically, show that if $\{x_k\}$ is a sequence of bounded signals converging to a bounded signal x in the sense that

$$\lim_{k \to \infty} \|x_k - x\|_\infty = 0\ ,$$

then $S(x_k)$ converges to $S(x)$ as $k \to \infty$ in the sense that

$$\|S(x_k) - S(x)\|_\infty = 0$$

whenever S is the system mapping of a BIBO stable system.