1. If $\{c_n\}$ is a convergent sequence of real numbers, does there necessarily exist $R > 0$ such that $|c_n| \le R$ for every $n \in \mathbb{N}$? Equivalently, is $\{c_n : n \in \mathbb{N}\}$ necessarily a bounded set of real numbers? Explain why or why not.

2. Find the sup and inf of each of the following bounded sets of real numbers. If the set has a max or a min or both, find it/them as well.

(a) The union of the two intervals $[-3, -1]$ and $[1, 3)$.
(b) The set A of all rational numbers between $-\pi$ and π.
(c) The set of the numbers in the sequence $\{c_n\}$, where

$$c_n = 7\left(1 - 7^{-(n+1)}\right) \quad \text{for all } \; n \in \mathbb{N}\,.$$

(d) $A = \{\sin\theta : -\pi/2 < \theta \le \pi\}$.
(e) $A = \{\sin^2\theta : -\pi/2 < \theta \le \pi\}$.

3. Throughout this problem, A and B are sets of real numbers.

(a) Show that if $B \subset A$ and A is bounded, then B is also bounded.
(b) Show that if $B \subset A$ and A is bounded, then $\sup(B) \le \sup(A)$ and $\inf(B) \ge \inf(A)$.

4. Suppose $\{b_n\}$ is a sequence of real or complex numbers and $\{a_n\}$ is a sequence of real or complex numbers satisfying $|a_n| \le |b_n|$ for all $n \in \mathbb{N}$. Show that if $\{b_n\}$ is absolutely summable, then $\{a_n\}$ is summable, i.e. $\sum_{n=0}^{\infty} a_n$ converges. (Suggestion: you can probably use Facts 3.3 and 3.7 from the Monograph.)

5. In class I stated the following result: if $\sum_{n=0}^{\infty} |c_n|$ converges, then so does $\sum_{n=0}^{\infty} c_n$. In other words, if the sequence $\{c_n\}$ is absolutely summable, then it's summable. Use that fact plus the previous problem plus what you know about the geometric series to deduce that $\sum_{n=0}^{\infty} 7^{-n} \sin\left(n^2 + e^n\right)$ converges. This is an example where you can tell that a series converges without figuring out what it converges to.

6. Suppose $\{c_n\}$ is a summable sequence from $\mathbb{R}$ or $\mathbb{C}$. Explain briefly why each of the following statements is or isn't always true. You can quote any results I've stated either in class or in Chapter 3 of the monograph.

(a) The infinite series $\sum_{n=0}^{\infty} |c_n|$ converges.
(b) If $s_n = \left|\sum_{m=0}^{n} c_m\right|$, then there exists some $R > 0$ such that $s_n \le R$ for every $n \in \mathbb{N}$.
(c) If $\tilde{s}_n = \sum_{m=0}^{n} |c_m|$ and at least one c_m is nonzero, then there exists some $N \in \mathbb{N}$ such that $\inf(\{\tilde{s}_n : n \ge N\}) > 0$.
(d) If every c_n is a positive real number, then $\lim_{n\to\infty} c_n = 0$.

7. We talked in class about sequences of rational numbers converging to irrational numbers (and you can read about it in Chapter 1 of the monograph). This problem addresses one famous example.

(a) It happens to be true that $\sqrt{5}$ is irrational. Given that fact, show that $(1+\sqrt{5})/2$ is also irrational. (Suggestion: sums and products of rational numbers are also rational.)

(b) Show that if $\{x_n\}$ is a sequence of real numbers converging to $\bar{x} \in \mathbb{R}$, then the sequence $\{y_n = x_{n+1}\}$ also converges to $\bar{x}$.

(c) You've doubtless heard of the Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

of natural numbers. If a_n is the nth term in the sequence, we have $a_{n+2} = a_n + a_{n+1}$ for all $n \in \mathbb{N}$. It turns out that the sequence

$$q_n = \frac{a_{n+1}}{a_n}$$

converges. Show that

$$q_{n+1} = 1 + \frac{1}{q_n} \ ,$$

Given that $\{q_n\}$ converges, use (b) to conclude that it converges to to $\rho = (1 + \sqrt{5})/2$, which is the so-called *Golden Ratio.* (You may use the fact that if $\{q_n\}$ converges to a positive limit ρ, , then $1/q_n$ converges to $1/\rho$.)

8. I asserted in class that l^2, the set of all square-summable discrete-time signals, is a subspace of $\mathbb{F}^{\mathbb{Z}}$ — i.e. that it's closed under taking linear combinations. Here's one way to prove it — let's just assume $\mathbb{F} = \mathbb{C}$, so we're dealing with complex-valued signals.

(a) Show that for any complex numbers a and b, we have

$$|a + b|^2 = |a|^2 + |b|^2 + 2\text{Re}\{a\bar{b}\} \ .$$

Conclude from the fact that $|a - b|^2 \geq 0$ that

$$2\text{Re}\{a\bar{b}\} \leq |a|^2 + |b|^2$$

from which it follows that $|a + b|^2 \leq 2|a|^2 + 2|b|^2$.

(b) Let x_1 and x_2 be complex-valued l^2-signals and let c_1 and c_2 be complex numbers. Show using the result of part (a) that $c_1x_1 + c_2x_2 \in l^2$. (Suggestion: apply (a) to $c_1x_1(n) + c_2x_2(n)$ for all n and then add.)