1. In this problem we settle an "annoying borderline case" that we've dodged a few times. Let $x \in \mathbb{C}^{\mathbb{Z}}$ have specification $x(n) = (-1)^n$ for all $n \in \mathbb{Z}$. Note that x arises when you sample the continuous-time signal x_c with specification $x_c(t) = e^{j\Omega_o t}$ every T seconds and $\Omega_o T$ is an odd multiple of π. This last situation arises in the movie example when the wheel spins 12 revs per second. The DTFT of x has specification

$$\widehat{X}(\omega) = \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + (2k+1)\pi) .$$

Note that $\widehat{X}$ features impulses at $\omega = \pm\pi$. Your job is to verify that

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \widehat{X}(\omega)e^{jn\omega}d\omega \text{ for all } n \in \mathbb{Z} ,$$

i.e. that equation $\mathcal{DTFT}^{-1}$ holds for x. Please do this by using the rigorous interpretation of an integral featuring an impulse in the integrand, namely

$$\int_{\omega_1}^{\omega_2} \delta(\omega)d\omega = \lim_{a\to 0} \frac{1}{a}\int_{\omega_1}^{\omega_2} p_a(\omega)d\omega .$$

The basic idea: the integral captures only half of each impulse at $\omega = \pm\pi$.

2. This problem is about Haar wavelets. First, some key pieces of notation.

- For $x \in \mathbb{C}^{\mathbb{R}}$, $a > 0$, and $n \in \mathbb{Z}$, $\text{Scale}_a\text{Shift}_n(x)$ has specification

  $$\text{Scale}_a\text{Shift}_n(x)(t) = x(at - n) \text{ for all } t \in \mathbb{R} ,$$

  and $\text{Scale}_{2^k}\text{Shift}_n(x) = \text{Shift}_{n2^{-k}}\text{Scale}_k(x)$.
- The Haar wavelet scaling function ϕ has specification

  $$\phi(t) = \begin{cases} 1 & \text{when } 0 \le t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

  The Haar mother wavelet ψ has specification

  $$\psi(t) = \begin{cases} 1 & \text{when } 0 \le t < 1/2 \\ -1 & \text{when } 1/2 \le t < 1 \\ 0 & \text{otherwise.} \end{cases}$$
- For each $k \in \mathbb{Z}$, V_k is the set of L^2-signals constant on all intervals of the form $\left[l2^{-k}, (l+1)2^{-k}\right)$. For each k, $\left\{2^{k/2}\text{Scale}_{2^k}\text{Shift}_n(\phi) : n \in \mathbb{Z}\right\}$ is a complete orthonormal set in V_k. Essentially, V_k is a set of L^2 step functions whose steps have width at least 2^{-k}.
- For integers k and n, the Haar wavelet $\psi_{n,k}$ is given by

  $$\psi_{n,k} = 2^{k/2}\text{Scale}_{2^k}\text{Shift}_n(\psi) .$$
- Given $x \in \mathbb{L}^2$, denote by $\text{Proj}(x|V_k)$ the orthogonal projection of x onto the subspace V_k.

Let x be the signal with specification

$$x(t) = \begin{cases} 1 & \text{when } -1/3 \le t < 1/3 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find and graph $\text{Proj}(x|V_0)$ and $\text{Proj}(x|V_4)$.
(b) Find the wavelet coefficients $\langle x, \psi_{n,k}\rangle$ for all $n \in \mathbb{Z}$ and $0 \le k \le 3$. (For each k, only a couple of these will be nonzero.)

(c) Verify that

$$\mathrm{Proj}(x|V_4) - \mathrm{Proj}(x|V_0) = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{3} \langle x, \psi_{n,k} \rangle \; \psi_{n,k} \; .$$

3. Recall that for an integer $N > 0$, $\psi_N = e^{j\frac{2\pi}{N}}$.

(a) Show that $\{\psi_N^k : 0 \leq k < N\}$ is the set of so-called *Nth roots of unity.* I.e., this is the complete set of solutions to the polynomial equation $z^N - 1 = 0$.

(b) Indicate (a diagram is okay) where these numbers lie in the complex plane.

4. You want to send a bunch of telephone conversations simultaneously over the same pair of copper wires using time-division multiplexing. Each conversation signal is band limited to within 3400 Hz. Your channel specifications are as follows:

- The channel will pass without excessive distortion a pulse 1 microsecond long.
- The switch that switches from conversation to conversation to set up the TDM takes .5 microseconds to travel from one conversation terminal to the next.

How many conversations can you multiplex at once given these constraints and signal properties?

5. Consider the N-point signal x with specification

$$x(n) = e^{jn\omega_o} \;\text{ for }\; 0 \leq n < N \;,$$

where ω_o is a given real number.

(a) Find the DTFT $\widehat{X}$ of x in closed form (i.e., no summation signs in your answer).

(b) Find a closed-form expression for the N-point DFT $\{\widehat{X}_k : 0 \leq k < N\}$ for x.

6. Two 5-point signals x_1 and x_2 have specifications

$$x_1(n) = 5\cos(n\frac{\pi}{2}) \;\text{ and }\; x_2(n) = 5^n \;\text{ for }\; 0 \leq n < 5 \;.$$

(a) Find the 5-point DFT $\{(\widehat{X}_1)_k : 0 \leq k < 5\}$ of x_1. (Leave your answer in terms of powers ψ_5^k, $0 \leq k < 5$.)

(b) Find the 5-point DFT $\{(\widehat{X}_2)_k : 0 \leq k < 5\}$ of x_2. (Leave your answer in terms of powers ψ_5^k, $0 \leq k < 5$.)

(c) Calculate directly the 5-point circular convolution $y = \mathrm{CConv}(x_1, x_2)$.

(d) Calculate y from part (c) using DFTs. (Suggestion: use the expressions you got in (a) along with the Circular Convolution Rule for DFTs. Keep in mind the fundamental property $\psi_N^N = 1$. Note that you can "read a signal off" from its N-point DFT if you've got the DFT expressed as a sum of powers ψ_N^{-k} for $0 \leq k < N$ — the coefficient of ψ_N^{-nk} in such an expression is the value of the signal at time n.)

7. This problem is about using DFTs to compute convolutions.

(a) Let x_1 be as in the previous problem and let x_2 have specification

$$x_2(n) = \begin{cases} 7 & 0 \leq n < 4 \\ 0 & \text{otherwise.} \end{cases}$$

What is the duration of $y_1 = x_1 * x_2$, the "usual" convolution of x_1 and x_2?

(b) Use zero-padding and N-point DFTs to compute y_1.
(c) Use block convolution and your answer to (b) to compute $y_2 = x_1 * x_3$, where $x_3 = 7u$.

8. Let x_1 be a 1024-point signal and x_2 be a 1025-point signal. Note that $x_1 * x_2$ is a 2048-point signal.

(a) Show that it takes a worst-case 512.5×2^{11} multiplications to compute $x_1 * x_2$ directly. (Suggestion: it takes one multiplication to compute $x_1 * x_2(0)$, two multiplications to compute $x_1 * x_2(1)$, etc. You may use the fact that $\sum_{k=1}^{n} k = n(n+1)/2$.)
(b) Suppose you use N-point DFTs to compute $x_1 * x_2$. First of all, what is N? Second, assuming you use the standard decimation-in-time FFT to compute the DFTs, find how many multiplications it takes to compute $x_1 * x_2$. You may assume that it takes the same number of multiplications to compute an inverse DFT as it takes to compute a DFT when you use the FFT.

9. In the monograph discussion of DFTs I mention Vandermonde matrices. I'd like to show you how they make Shamir's so-called *polynomial secret-sharing* scheme work. I heard about this originally from Joe Halpern in the Cornell CS department. Here's how it goes. You're the supervisor and you have an important piece of information that you encode as a number S. You have a large number of agents numbered 1 through M. You want to give each agent a piece of information so that

- No single agent knows the value of S based on his or her piece of information alone, but
- Any set of n agents can figure out the value of S by pooling their information. (We assume of course that $n < M$ here.)

You proceed as follows: you pick numbers $c_1, c_2, \ldots, c_{n-1}$ and construct a polynomial

$$f(z) = S + c_1 z + c_2 z^2 + \cdots + c_{n-1} z^{n-1} .$$

So, S is the value of $f(z)$ at $z = 0$ — i.e., $S = f(0)$. To agent k you give the value $f(k)$ — you do this for all agents, $1 \leq k \leq M$.

In this way, none of the agents knows S by himself or herself. However, it turns out that any n of the agents can figure out S by pooling their information. Explain how they can do this. (Suggestion: let c be the column vector with S as its first entry and the rest of the c_j in order as the rest of the entries. If a set of n agents can figure out c, they can figure out S since S is just an entry in c. Suppose, for example, that $n = 5$, and agents numbered 2, 3, 5, 7, and 11 get together. Consider the equation

$$V_5(2, 3, 5, 7, 11)c = b .$$

What is b? How can these five agents figure out c? How does this work for general n?)