ECE 3250 **PRELIM I** **Fall 2015**

Problems 1 through 11 are worth 8 points each. Problem 12 is worth 12 points. Throughout, $*$ denotes convolution.

1. A is a set, B is a proper subset of A (i.e. $B \subset A$ and $B \neq A$), and $f : A \to B$ is a mapping. You can be certain that
(i) If B is finite, the f is surjective
(ii) For every $a \in A$ there's a unique $b \in B$ such that $b = f(a)$
(iii) If f is injective, then $\text{card}(A) = \text{card}(B)$
(iv) If $A = \mathbb{R}$ and $B = \mathbb{Z}$, then f is not injective

2. Which of the following sets are countably infinite?
(i) The set of all irrational numbers between -3 and 3
(ii) The set of all numbers of the form π^k, where $k \in \mathbb{Z}$
(iii) The set of all pairs (p, q), where p and q are prime numbers
(iv) The set of all numbers of the form $\sin(n\pi/2)$ where $n \in \mathbb{Z}$

3. As usual, for a positive integer $a > 1$, let $\mathbb{Z}_a = \{0, 1, 2, \ldots, a-1\}$ and let $\mathbb{Z}_a^*$ be the set of all $k \in \mathbb{Z}_a$ such that k and a are coprime, i.e. have no factors other than 1 in common. You can be certain that
(i) If a is even, then $\mathbb{Z}_a^*$ contains exactly $a/2$ numbers
(ii) If $m \in \mathbb{Z}_a^*$, then there exist integers k and l such that $km + la = 1$
(iii) If a is prime and $k \in \mathbb{Z}_a^*$ and $k^3 < a$, then $\langle\langle k^{a+2} \rangle\rangle_a = k^3$
(iv) $\langle\langle m^{a-1} \rangle\rangle_a = 1$ for all $m \in \mathbb{Z}_a^*$ if and only if a is prime

4. Suppose p is a prime number bigger than 2, e and d are in $\mathbb{Z}_{p-1}^*$, and $\langle\langle ed \rangle\rangle_{p-1} = 1$. You can be certain that
(i) Whenever m is a positive integer greater than 1 and less than p, $\langle\langle m^{ed} \rangle\rangle_p = 1$
(ii) There exist integers k and l such that $ke + ld = 1$
(iii) p is a divisor of $\left(2^{ed}\right)^p$
(iv) p is not a divisor of ed

5. Let A be a bounded set of real numbers. You can be certain that
(i) If $\min(A)$ doesn't exist, then $\inf(A) < a$ for every $a \in A$
(ii) B is a bounded set, where $B = \{e^{-at} : a \in A \text{ and } t \in \mathbb{R}\}$
(iii) $\inf(A) < \sup(A)$
(iv) If $C \subset A$ is finite, then $\max(C)$ exists

6. Let $\{q_n : n \in \mathbb{N}\}$ be a sequence of nonzero rational numbers. Let $s_n = \sum_{m=0}^{n} |q_m|$ for each $n \in \mathbb{N}$. Suppose $3 \leq s_n \leq 7$ for every $n \in \mathbb{N}$. You can be certain that
(i) The sequence $\{q_n\}$ is a Cauchy sequence
(ii) The sequence $\{q_n\}$ is summable
(iii) $\lim_{n\to\infty} s_n$ exists
(iv) The sequence $\{s_n\}$ is absolutely summable

7. Let $\{x_n : n \in \mathbb{N}\}$ be a sequence of rational numbers between $-\pi$ and π You can be certain that
(i) If the sequence $\{x_n\}$ is increasing, then $\lim_{n\to\infty} x_n = \pi$
(ii) If the sequence $\{x_n\}$ converges to a limit $\overline{x}$, then $\overline{x} > -\pi$

(iii) If the sequence $\{x_n\}$ converges to a limit $\overline{x}$, you can find $N > 0$ so that $|x_n - \overline{x}|$ is less than 10^{-137} for all $n > N$
(iv) If the sequence $\{x_n\}$ converges, then so does the sequence $\{y_n\}$ defined by $y_n = x_n + x_{n+1}$ for all $n \in \mathbb{N}$

8. Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$ and let x_1 and x_2 be discrete-time signals with values in $\mathbb{F}$ (i.e. x_1 and x_2 are in $\mathbb{F}^{\mathbb{Z}}$.) You can be certain that
(i) If x_1 is right-sided and x_2 is left-sided, then $x_1 * x_2$ exists
(ii) If $x_1 * x_2$ exists and x_1 is absolutely summable, then x_2 is bounded
(ii) If $x_1 * x_2$ exists, then so does $\mathrm{Shift}_{-3}(x_1) * \mathrm{Shift}_7(x_2)$
(iv) If $x_1(n) = x_2(n) = 0$ for $n > 0$, then $x_1 * x_2$ exists and $x_1 * x_2(n) = 0$ for $n < 0$

9. Which of the following statements apply to every discrete-time FIR LTI system over $\mathbb{F}$? (S denotes the system mapping and h denotes the system's impulse response.)
(i) If $x \in \mathbb{F}^{\mathbb{Z}}$ is right-sided, then so is $S(x)$
(ii) $h * x$ exists for every left-sided signal $x \in \mathbb{F}^{\mathbb{Z}}$
(iii) If $S(x)$ has finite duration, then x has finite duration
(iv) $S(h)$ has finite duration

10. In each case, S is the system mapping of a discrete-time LTI system with input space X. Indicate which systems are causal.
(i) $S(x)(n) = \sum_{k=-\infty}^{n+3} \sin(n-k)x(k)$ for every $x \in X$ and $n \in \mathbb{Z}$
(ii) $S(x) = \mathrm{Shift}_7(x)$ for every $x \in X$
(iii) $S(x) = h * x$ for every $x \in X$, where h is the signal with specification $h(n) = 3^{n+7}u(n+7)$ for every $n \in \mathbb{Z}$.
(iv) $S(x) = h * x$ for every $x \in X$, where h is a nonzero signal satisfying $h(n)u(n) = 0$ for all $n \in \mathbb{Z}$

11. S is the system mapping and h is the impulse response of a discrete-time LTI system over $\mathbb{F}$. You can be certain that
(i) If $S(\delta)$ is a square-summable signal, then the system is BIBO stable
(ii) If the system is BIBO stable, then $S(\delta)(n) \to 0$ as $n \to \infty$
(iii) If the system is BIBO stable and $|x(n)| \le R$ for every $n \in \mathbb{Z}$, then $\{|S(x)(n)| : n \in \mathbb{Z}\}$ is a bounded set
(iv) If $h(n) = 0$ for $|n| > 7$, then the system is BIBO stable

12. The system mappings S_1 and S_2 for two discrete-time LTI systems with respective input spaces X_1 and X_2 are

$$S_1(x)(n) = \frac{1}{4}\sum_{k=1}^{4} x(n-k) \ \text{ for all } \ x \in X_1 \ \text{ and } \ k \in \mathbb{Z}\ ,$$

where $X_1 = \mathbb{F}^{\mathbb{Z}}$, and

$$S_2(x) = h_2 * x \ \text{ for all } \ x \in X_2\ ,$$

where $h_2(n) = 7^{-n}u(n)$ for every $n \in \mathbb{Z}$ and $X_2 = \mathcal{D}_{h_2}$ is the set of all signals $x \in \mathbb{F}^{\mathbb{Z}}$ for which $h_2 * x$ exists.

(a) Find h_1, the impulse response of the first system.
(b) Find $S_2(x)(n)$ for all $n \in \mathbb{Z}$ when $x = u$.
(c) Find $S_2(S_1(x))(n)$ for all $n \in \mathbb{Z}$ when $x(n) = (-1)^n$ for all $n \in \mathbb{Z}$.